By $L > G$ this refines Carlson's inequality (1), since by $L + G < 2L$ one has $2(L - G)/(L + G) > 1 - G/L$, so by (11) one can derive

$$0 < \frac{(L-G)^2}{L(L+G)} < \frac{A+2G}{L} - 3 \tag{12}$$

3. Inequality (6) and (1) improves also the right side of (4). This follows by

$$I > \frac{2A+G}{3} > 2L - G \tag{13}$$

where the second relation is exactly (1). We note that $\sqrt{G \cdot I} > \sqrt[3]{G^2 \cdot A}$ follows by (5), so from the left side of (4) one can write:

$$L > \sqrt{G \cdot I} > \sqrt[3]{G^2 \cdot A} \tag{14}$$

improving inequality (2). The left side of (4) can be sharpened also, if we use the second inequality of (3). Indeed, by the identity $ln\,\frac{I}{G} = \frac{A-L}{L}$ and $ln\,x < (x-1)/\sqrt{x}$ applied with $x = \frac{I}{G} > 1$ one can deduce

$$\sqrt{IG} < \frac{I-G}{A-L} \cdot L < L \tag{15}$$

Remark. Identity $ln\,\frac{I}{G} = \frac{A-L}{L}$ is due to H.-J. Seiffert [9]. For this and similar identities with applications, see [8].

4. Inequality (6) with (1) can be written also as

$$I > \frac{2A+G}{3} > \frac{A+L}{2} \tag{16}$$

Relation

$$I > \frac{A+L}{2} \tag{17}$$

appears also in [6], inequality (9). Since $\frac{A+L}{2} > \sqrt{AL}$, one has

$$I > \sqrt{AL} \tag{18}$$

For a simple method of proof of (18), see [7]. As an application of (18) we note that in a recent paper M. K. Vamanamurthy and M. Vuorinen [11] have proved, among other results, that for the arithmetic-geometric men M of Gauss we have

$$M < \sqrt{AL} \tag{19}$$

$$M < I \tag{20}$$

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