Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
24 (2008), 209–214
www.emis.de/journals
ISSN 1786-0091

MAXIMAL (C, α, β) OPERATORS OF TWO-DIMENSIONAL WALSH-FOURIER SERIES

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Abstract. The main aim of this paper is to prove that for the boundedness of the maximal operator $\sigma_*^{\alpha,\beta}$ from the Hardy space $H_p\left(I^2\right)$ to the space $L_p\left(I^2\right)$ the assumption $p > \max\{1/(\alpha+1), 1/(\beta+1)\}$ is essential.

We denote the set of non-negative integers by $\mathbf{N}$. For a set $X \neq \emptyset$ let X^2 be its Cartesian product $X \times X$ taken with itself. By a dyadic interval in $I := [0,1)$ we mean one of the form $\left[l2^{-k}, (l+1)2^{-k}\right)$ for some $k \in \mathbf{N}$, $0 \leq l < 2^k$. Given $k \in \mathbf{N}$ and $x \in [0,1)$, let $I_k(x)$ denote the dyadic interval of length 2^{-k} which contains the point x. The Cartesian product of two dyadic intervals is said to be a rectangle. Clearly, the dyadic rectangle of area $2^{-n} \times 2^{-m}$ containing $(x^1, x^2) \in I^2$ is given by $I_{n,m}(x^1, x^2) := I_n(x^1) \times I_m(x^2)$. We also use the notation $\operatorname{mes}(A)$ for the Lebesgue measure of any measurable set A.

Let $r_0(x)$ be a function defined by

$$r_0(x) = \begin{cases} 1, & \text{if } x \in [0, 1/2), \\ -1, & \text{if } x \in [1/2, 1), \end{cases}$$
$$r_0(x+1) = r_0(x).$$

The Rademacher system is defined by

$$r_n(x) = r_0(2^n x), \; n \geq 1 \text{ and } x \in [0,1).$$

Let $w_0, w_1, \dots$ represent the Walsh functions, i.e. $w_0(x) = 1$ and if

$$n = 2^{n_1} + \cdots + 2^{n_r}$$

is a positive integer with $n_1 > n_2 > \cdots > n_r$ then

$$w_n(x) = r_{n_1}(x) \cdots r_{n_r}(x).$$

2000 *Mathematics Subject Classification.* 42C10.
Key words and phrases. Walsh function, Hardy space, Maximal operator.

The Walsh-Dirichlet kernel is defined by

$$D_n(x) = \sum_{k=0}^{n-1} w_k(x).$$

Recall that

$$D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in [0, 2^{-n}), \\ 0, & \text{if } x \in [2^{-n}, 1). \end{cases}$$

The Kronecker product $(w_{n,m} : n, m \in \mathbf{N})$ of two Walsh systems is said to be the two-dimensional Walsh system. Thus

$$w_{n,m}\left(x^1, x^2\right) := w_n\left(x^1\right) w_m\left(x^2\right).$$

The partial sums of the two-dimensional Walsh-Fourier series are defined as follows:

$$S_{n,m} f\left(x^1, x^2\right) = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \widehat{f}(i,j)\, w_{i,j}\left(x^1, x^2\right),$$

where the number

$$\widehat{f}(i,j) = \int_I f\left(u^1, u^2\right) w_{i,j}\left(u^1, u^2\right) du^1 du^2$$

is said to be the (i,j)th Walsh-Fourier coefficient of the function f.

The norm (or quasinorm) of the space $L_p(I^2)$ is defined by

$$\|f\|_p := \left(\int_{I^2} \left| f\left(x^1, x^2\right) \right|^p dx^1 dx^2 \right)^{1/p} \qquad (0 < p < +\infty).$$

The σ-algebra generated by the dyadic rectangles $\{I_{n,m}(x^1, x^2) : x, y \in I\}$ will be denoted by $F_{n,m}\,(n, m \in \mathbf{N})$, more precisely,

$$F_{n,m} = \sigma\left\{ \left[k2^{-n}, (k+1)\, 2^{-n}\right) \times \left[l2^{-m}, (l+1)\, 2^{-m}\right) : 0 \le k < 2^n, 0 \le l < 2^m \right\},$$

where $\sigma(A)$ denotes the σ-algebra generated by an arbitrary set system A.

Denote by $f = \left(f^{(n,m)}, n \in \mathbf{N}\right)$ two-parameter martingale with respect to $(F_{n,m}, n, m \in \mathbf{N})$ (for details see, e.g. [6, 9]). The maximal function of a martingale f is defined by

$$f^* = \sup_{n,m \in \mathbf{N}} \left| f^{(n,m)} \right|.$$

In case $f \in L_1(I^2)$, the maximal function can also be given by

$$f^*\left(x^1, x^2\right) = \sup_{n,m \in N} \frac{1}{\operatorname{mes}\left(I_n(x^1) \times I_m(x^2)\right)} \left| \int_{I_n(x^1) \times I_m(x^2)} f(u,v)\, du dv \right|,$$

$(x^1, x^2) \in I^2$.

For $0 < p < \infty$ the Hardy martingale space $H_p(I^2)$ consists all martingales for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

If $f \in L_1(I^2)$ then it is easy to show that the sequence $(S_{2^n,2^m}(f) : n, m \in \mathbf{N})$ is a martingale. If f is a martingale, that is $f = (f^{(n,m)} : n, m \in \mathbf{N})$ then the Walsh-Fourier coefficients must be defined in a little bit different way:

$$\widehat{f}(i,j) = \lim_{k,l\to\infty} \int\limits_{I^2} f^{(k,l)}\left(x^1, x^2\right) w_i(x^1) w_j(x^2) dx^1 dx^2.$$

The Walsh-Fourier coefficients of the function $f \in L_1(I^2)$ are the same as the ones of the martingale $(S_{2^n,2^m}(f) : n, m \in \mathbf{N})$ obtained from the function f.

The (C, α, β) means of the two-dimensional Walsh-Fourier series of the martingale f is given by

$$\sigma_{n,m}^{\alpha,\beta}(f, x^1, x^2) = \frac{1}{A_{n-1}^{\alpha}} \frac{1}{A_{m-1}^{\beta}} \sum_{i=1}^{n} \sum_{j=1}^{m} A_{n-i}^{\alpha-1} A_{m-j}^{\beta-1} S_{i,j} f\left(x^1, x^2\right),$$

where

$$A_n^{\alpha} := \frac{(1+\alpha)\dots(n+\alpha)}{n!}$$

for any $n \in \mathbf{N}, \alpha \neq -1, -2, \dots$. It is known ([10]) that $A_n^{\alpha} \sim n^{\alpha}$.

For the martingale f we consider the maximal operator

$$\sigma_*^{\alpha,\beta} f = \sup_{n,m} |\sigma_{n,m}^{\alpha,\beta}(f, x^1, x^2)|.$$

The (C, α) kernel defined by

$$K_n^{\alpha}(x) := \frac{1}{A_{n-1}^{\alpha}} \sum_{k=1}^{n} A_{n-j}^{\alpha-1} D_k(x).$$

In the one-dimensional case, Fine [1] proved that the (C, α) means $\sigma_n^{\alpha} f$ of a function $f \in L(I)$ converge a.e. to f as $n \to \infty$. The maximal operator $\sigma_*^{\alpha} f := \sup_n |\sigma_n^{\alpha} f|$ $(0 < \alpha < 1)$ of the (C, α) means of the Walsh-Paley Fourier series was investigated by Weisz [8]. In his paper Weisz proved the boundedness of $\sigma_*^{\alpha} : H_p \to L_p$ when $p > 1/(1+\alpha)$. The author [3] showed that in Theorem of Weisz the assumption $p > 1/(\alpha+1)$ is essential. In particular, we proved that the maximal operator σ_*^{α} of the (C, α) means of the Walsh-Paley Fourier series is not bounded from the Hardy space $H_{1/(\alpha+1)}(I)$ to the space $L_{1/(\alpha+1)}(I)$.

For double Walsh-Fourier series it is known [5] that the (C, α, β) means $\sigma_{n,m}^{\alpha,\beta} f \to f$ in L_p norm as $n, m \to \infty$ whenever $f \in L_p(I^2)$ for some $1 \leq p < \infty$.

On the other hand, in 1992 Móricz, Schipp and Wade [4] proved with respect to the Walsh-Paley system that

$$\sigma_{n,m} f=\frac{1}{nm}\sum_{i=1}^{n}\sum_{k=1}^{m} S_{i,k}(f)\to f$$

a.e. for each $f\in L\log^{+}L([0,1)^2)$, when $\min\{n,m\}\to\infty$. In 2000 Gát proved [2] that the theorem of Móricz, Schipp and Wade above can not be improved. Namely, let $\delta:[0,+\infty)\to[0,+\infty)$ be a measurable function with property $\lim_{t\to\infty}\delta(t)=0$. Gát proved [2] the existence of a function $f\in L^1(I^2)$ such that $f\in L\log^{+}L\delta(L)$, and $\sigma_{n,m}f$ does not converge to f a.e. as $\min\{n,m\}\to\infty$. That is, the maximal convergence space for the $(C,1)$ means of two-dimensional partial sums is $L\log^{+}L(I^2)$. Weisz [7] investigated the maximal operator of (C,α,β) means of two-dimensional Walsh-Fourier series and proved that the maximal operator $\sigma_*^{\alpha,\beta}f$ is bounded from $H_p(I^2)$ to $L_p(I^2)$ if $1/(1+\alpha),1/(1+\beta)<p<\infty$. In [7] Weisz conjectured that for the boundedness of the maximal operator $\sigma_*^{\alpha,\beta}$ from the Hardy space $H_p(I)$ to the space $L_p(I)$ the assumption $p>1/(\alpha+1),1/(1+\beta)$ is essential. We give answer to the question and prove that the maximal operator $\sigma_*^{\alpha,\beta}$ of the (C,α,β) $(0<\alpha\le\beta\le1)$ means of the two-dimensional Walsh-Fourier series is not bounded from the Hardy space $H_{1/(\alpha+1)}(I^2)$ to the space $L_{1/(\alpha+1)}(I^2)$. The following is true.

Theorem 1. *Let $0<\alpha\le\beta\le1$. Then the maximal operator $\sigma_*^{\alpha,\beta}$ of the (C,α,β) means of the two-dimensional Walsh-Fourier series is not bounded from the Hardy space $H_{1/(\alpha+1)}(I^2)$ to the space $L_{1/(\alpha+1)}(I^2)$.*

In order to prove Theorem 1 we need the following lemma.

Lemma 1. *([3]) Let $n\in\mathbf{N}$ and $0<\alpha\le1$. Then*

$$\int_I \max_{1\le N\le 2^n}\left(A_{N-1}^{\alpha}\left|K_N^{\alpha}(x)\right|\right)^{1/(\alpha+1)}dx\ge c(\alpha)\frac{n}{\log(n+2)}.$$

Proof of Theorem 1. Let

$$f_n\left(x^1,x^2\right):=\left[D_{2^{n+1}}\left(x^1\right)-D_{2^n}\left(x^1\right)\right]w_{2^n-1}\left(x^2\right).$$

Since

$$\begin{aligned}\widehat{f}_n(\nu,\mu)&=\int_I\left[D_{2^{n+1}}\left(u^1\right)-D_{2^n}\left(u^1\right)\right]w_\nu\left(u^1\right)du^1\int_I w_{2^n-1}\left(u^2\right)w_\mu\left(u^2\right)du^2\\&=\begin{cases}1, & \text{if } \nu=2^n,\dots,2^{n+1}-1,\mu=2^n-1,\\ 0, & \text{otherwise,}\end{cases}\end{aligned}$$

we can write

$$S_{i,j}f_n\left(x^1,x^2\right)=\sum_{\nu=0}^{i-1}\widehat{f}_n\left(\nu,2^n-1\right)w_\nu\left(x^1\right)w_{2^n-1}\left(x^2\right) \tag{1}$$

$$= \begin{cases} \left[D_i\left(x^1\right)-D_{2^n}\left(x^1\right)\right] w_{2^n-1}\left(x^2\right), & \text{if } i=2^n+1,\ldots,2^{n+1}-1, j \geq 2^n, \\ f_n\left(x^1,x^2\right), & \text{if } i \geq 2^{n+1},\ j \geq 2^n, \\ 0, & \text{otherwise.} \end{cases}$$

We have

$$f_n^*\left(x^1,x^2\right)=\sup_{i,j}\left|S_{2^i,2^j} f_n\left(x^1,x^2\right)\right|=\left|f_n\left(x^1,x^2\right)\right|,$$

$$\|f_n\|_{H_p}=\|f_n^*\|_p=\|D_{2^n}\|_p=2^{n(1-1/p)}. \tag{2}$$

Let $1 \leq N < 2^n$. Then from (1) we obtain

$$\begin{aligned}
&\sigma^{\alpha}_{2^n+N,2^{n+1}} f_n\left(x^1,x^2\right)= \\
&=\frac{1}{A^{\alpha}_{2^n+N-1}}\frac{1}{A^{\beta}_{2^{n+1}-1}}\left|\sum_{i=1}^{2^n+N}\sum_{j=1}^{2^{n+1}} A^{\alpha-1}_{2^n+N-i}A^{\beta-1}_{2^{n+1}-j} S_{i,j} f_n\left(x^1,x^2\right)\right| \\
&=\frac{1}{A^{\alpha}_{2^n+N-1}}\frac{1}{A^{\beta}_{2^{n+1}-1}}\left|\sum_{i=2^n+1}^{2^n+N}\sum_{j=2^n}^{2^{n+1}} A^{\alpha-1}_{2^n+N-i}A^{\beta-1}_{2^{n+1}-j} S_{i,j} f_n\left(x^1,x^2\right)\right| \\
&=\frac{1}{A^{\alpha}_{2^n+N-1}}\frac{1}{A^{\beta}_{2^{n+1}-1}}\times \\
&\quad\times\left|\sum_{i=2^n+1}^{2^n+N}\sum_{j=2^n}^{2^{n+1}} A^{\alpha-1}_{2^n+N-i}A^{\beta-1}_{2^{n+1}-j}\left[D_i\left(x^1\right)-D_{2^n}\left(x^1\right)\right] w_{2^n-1}\left(x^2\right)\right| \\
&\geq\frac{c(\alpha,\beta)}{2^{n\alpha}2^{n\beta}}\left|\sum_{i=1}^{N} A^{\alpha-1}_{N-i}\left[D_{i+2^n}\left(x^1\right)-D_{2^n}\left(x^1\right)\right]\right|\left|\sum_{j=0}^{2^n} A^{\beta-1}_{2^n-j}\right| \\
&\geq\frac{c(\alpha,\beta)}{2^{n\alpha}}\left|\sum_{i=1}^{N} A^{\alpha-1}_{N-i}\left[D_{i+2^n}\left(x^1\right)-D_{2^n}\left(x^1\right)\right]\right| \\
&=\frac{c(\alpha,\beta)}{2^{n\alpha}}\left|\sum_{i=1}^{N} A^{\alpha-1}_{N-i} D_i\left(x^1\right)\right|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sigma^{\alpha,\beta}_* f_n\left(x^1,x^2\right) &\geq \sup_{1\leq N\leq 2^n}\left|\sigma^{\alpha}_{2^n+N,2^{n+1}} f_n\left(x^1,x^2\right)\right| \\
&\geq \frac{c(\alpha,\beta)}{2^{n\alpha}}\sup_{1\leq N\leq 2^n}\left|\sum_{i=1}^{N} A^{\alpha-1}_{N-i} D_i\left(x^1\right)\right|.
\end{aligned}$$

Then from Lemma 1 and (2) we get

$$\frac{\left\|\sigma^{\alpha,\beta}_* f_n\right\|_{1/(\alpha+1)}}{\|f_n\|_{H_{1/(\alpha+1)}}} \geq \frac{c(\alpha,\beta)}{2^{n\alpha}2^{-n\alpha}}\left(\int_I \sup_{1\leq N\leq 2^n}\left(A^{\alpha}_{N-1}\left|K^{\alpha}_N(x)\right|\right)^{1/(\alpha+1)} dx\right)^{\alpha+1}$$

$$\geq c(\alpha,\beta)\left(\frac{n}{\log(n+2)}\right)^{\alpha+1} \to \infty \text{ as } n \to \infty.$$

Theorem 1 is proved. □

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Received August 28, 2007.

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