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QUASI HEMI-SLANT SUBMANIFOLDS OF QUASI-SASAKIAN MANIFOLDS

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Abstract. This research article attempts to explain the quasi hemi-slant submanifolds as a generalization of slant submanifolds, semi-slant submanifolds and hemi-slant submanifolds of quasi-Sasakian manifolds. Also, we obtain necessary and sufficient conditions for integrability of distributions of quasi hemi-slant submanifolds of quasi-Sasakian manifolds. In addition, we investigate the necessary and sufficient condition for quasi hemi-slant submanifolds of quasi-Sasakian manifolds to be totally geodesic and study the geometry of foliations determine by the distribution. Furthermore, we illustrate an example of quasi hemi-slant submanifolds of quasi-Sasakian manifolds.

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1. Introduction

The concept of geometry of submanifolds begin with the idea of extrinsic geometry of surface and it is developed for ambient space. The course of time and this theory of submanifolds play an important role in economic modeling, computer design, image processing and in mathematical physics and in mechanics. The extensive application of such topic makes it an active and interesting field of research for all geometers.

The notion of geometry slant submanifolds are the natural generalization of both holomorphic and totally real immersions has been defined and introduced by Chen [6], then after several geometers during last two decades studies this interesting topic ([4, 20]). Lotta [3] investigate the properties of slant immersion of a Riemannian manifolds in to an almost contact metric manifolds. Th concept of semi-slant submanifolds of Kaehlerian manifolds was studies by Papaghuic [17]. Furthermore, the slant submanifolds were generalized as semi-slant submanifolds, pseudo-slant submanifolds, bi-slant submanifolds and quasi-slant submanifolds etc, on different kind

of differentiable manifolds (for more details see [1, 5, 11, 12, 14]).
In 1967, Blair [8], established and explain the geometry of quasi-Sasakian structure. Prasad et al. [18] studied the quasi hemi-slant submanifolds of Sasakian manifolds and Cosymplectic manifolds in [19], and investigated some properties of integrability of distribution and totally geodesic, in 2020.

Motivated enough from the above studies, we will attempt to study quasi hemi-slant submanifolds of hemi-Sasakian manifolds which include the semi-slant and hemi-slant submanifolds.

This present paper is organized as follows. In section 2, we introduce the basic definition and some properties of an almost contact metric manifolds. In section 3, we describe about quasi hemi-slant submanifolds of quasi-Sasakian manifolds and investigate some basic results and provide a non trivial example of quasi hemi-slant submanifolds of quasi-Sasakian manifolds. Section 4, deals with necessary and sufficient conditions for integrability of distribution for quasi hemi-slant submanifolds of quasi-Sasakian manifolds. In section 5, we also drive some necessary and sufficient conditions for totally geodesic for quasi hemi-slant submanifolds of quasi-Sasakian manifolds.

2. Preliminaries

Let $\bar{M}$ be a real $(2n+1)$ dimensional manifold endowed with an almost contact metric structure (ϕ, ξ, η, g), where ϕ is a tensor of type (1,1), ξ is a vector field, η is 1-form and g is a Riemannian metric on $\bar{M}$ such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta o \phi = 0, \quad \eta(\xi) = 1, \tag{1}$$

$$g(X, \xi) = \eta(X), \qquad \phi(\xi) = 0, \tag{2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{3}$$

for any vector field X,Y tangent to $\bar{M}$, where I is the identity on the tangent bundle $\Gamma(\bar{M})$ on $\bar{M}$.

An almost contact metric structure (ϕ, ξ, η, g) on $\bar{M}$ is said to be normal [9] if an almost complex structure J on the product manifold $\bar{M} \times R$ is given by

$$J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt}),$$

where $J^2 = -1$ and f is a differentiable function on $\bar{M} \times R$ has no torsion, i.e. J is integrable. The condition for normality in terms of ϕ, η and ξ is $[\phi, \phi] + 2d\eta \otimes \xi$ on $\bar{M}$, where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ.

An almost contact metric structure (ϕ, ξ, η, g) on $\bar{M}$ is called quasi-Sasakian manifold [8] if

$$(\bar{\nabla}_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{4}$$

where A is a symmetric linear transformation field, $\bar{\nabla}$ denotes the Riemannian connection of g on $\bar{M}$. We have also on quasi-Sasakian manifold $\bar{M}$

$$\bar{\nabla}_X \xi = \phi AX. \tag{5}$$

Now, let M be a Riemannian manifold isometrically immersed in $\bar{M}$ and the induced Riemannian metric on M is denoted by the same symbol g throughout this paper. The Gauss and Weingarten formulas are given by [10]

$$\bar{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \tag{6}$$

$$\bar{\nabla}_X V = -\Lambda_V X + \nabla^{\perp}_X V \tag{7}$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, where ∇ is induced connection on M and $\nabla^{\perp}$ is the connection on the normal bundle $T^{\perp}M$ of M and Λ_V is the shape operator on M with normal vector $V \in \Gamma(T^{\perp}M)$. Moreover, σ is the second fundamental form of A_V is the Weingarten map associated with V as

$$g(\Lambda_V X, Y) = g(\sigma(X, Y), V) \tag{8}$$

The mean curvature vector H of M is defined by

$$H = \frac{1}{n} trace(\sigma) = \frac{1}{n} \sum_{i=1}^{n} \sigma(e_i, e_i) \tag{9}$$

where n is the dimension of M and $\{e_1, e_2, ..., e_n\}$ is a local orthogonal frame of M.

A submanifolds M of an almost contact metric manifold $\bar{M}$ is said to be *totally umbilical* if

$$\sigma(X, Y) = g(X, Y)H \tag{10}$$

where H is the mean curvature vector. A submanifold M is said to be *geodesic* if $\sigma(X, Y) = 0$, for each $X, Y \in \Gamma(TM)$ and M is said to minimal submanifold if $H = 0$.

For any $X \in TM$, we can write

$$\phi X = TX + NT, \tag{11}$$

where TX and NX are tangential and normal component of ϕX on M respectively. Similarly, for any $V \in \Gamma(T^{\perp}M)$, we have

$$\phi V = tV + nV, \tag{12}$$

where tV and nV are the tangential and normal component of ϕV on M respectively.

The covariant derivative of projection morphisms in (11) and (12) are defines as

$$(\bar{\nabla}_X T)Y = \nabla_X TY - T\nabla_X Y, \tag{13}$$

$$(\bar{\nabla}_X N)Y = \nabla^{\perp}_X NY - N\nabla_X Y, \tag{14}$$

$$(\bar{\nabla}_X t)V = \nabla_X tV - t\nabla_X V, \tag{15}$$

$$(\bar{\nabla}_X n)V = \nabla^{\perp}_X nV - n\nabla_X V \tag{16}$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.

Definition 1. *[13] Let M be a Riemannian manifold isometrically immersed in an almost contact metric manifold $\bar{M}$. A submanifold M of an almost contact metric manifold $\bar{M}$ is said to invariant, if $\phi(T_xM) \subseteq T_xM$, for every point $x \in M$.*

Definition 2. *[16] A submanifold M of an almost contact metric manifold $\bar{M}$ is said to be anti-invariant, if $\phi(T_xM) \subseteq T_x^{\perp}M$, for every point $x \in M$.*

Definition 3. *[19] A submanifold M of an almost contact metric manifold $\bar{M}$ is said to be slant, if for each non-zero vector X tangent to M at $x \in M$, linearly independent on ξ, the angle $\theta(K)$ between ϕX and T_xM is constant, i.e. it does not depend on the choice of the point $x \in M$ and $X \in T_xM$. In this case, the angle θ is called the slant angle of the submanifold. A slant submanifold M is called proper slant submanifold if neither $\theta = 0$ nor $\theta = \frac{\pi}{2}$.*

We note that, on a slant submanifold M if $\theta = 0$, then it is an invariant submanifold and $\theta = \frac{\pi}{2}$, then it is an anti-invariant submanifold. This means that the slant submanifold is a generalization of invariant and anti-invariant submanifolds.

Definition 4. *[2] A submanifold M of an almost contact metric manifold $\bar{M}$ is said to be semi-invariant , if there exist two orthogonal complementary distributions D and $D^{\perp}$ on M such that*

$$TM = D \oplus D^{\perp} \oplus < \xi >,$$

where D is invariant and $D^{\perp}$ is anti-invariant.

Definition 5. *[17] A submanifold M of an almost contact metric manifold $\bar{M}$ is said to be semi-slant, if there exist two orthogonal complementary distributions D and D^{θ} on M such that*

$$TM = D \oplus D^{\theta} \oplus < \xi >,$$

where D is invariant and D^{θ} is slant with slant angle θ. In this case, the angle θ is called semi-slant angle.

Definition 6. *[5] A submanifold M of an almost contact metric manifold $\bar{M}$ is said to be hemi-slant , if there exist two orthogonal complementary distributions D^{θ} and $D^{\perp}$ on M such that*

$$TM = D^{\theta} \oplus D^{\perp} \oplus < \xi >,$$

where D^{θ} is slant with slant angle θ and $D^{\perp}$ is anti-invariant. In this case, the angle θ is called hemi-slant angle.

3. Quasi hemi-slant submanifolds of quasi-Sasakian manifolds

In this section of paper, we explain the necessary background of quasi-hemi-slant submanifold of quasi-Sasakian manifolds and also, provide an example of quasi-hemi-slant submanifold of quasi-Sasakian manifolds.

Definition 7. *A submanifold M of quasi-Sasakian manifold $\bar{M}$ is called quasi hemi-slant submanifold if there exist distributions D D^{θ} and $D^{\perp}$ on M such that*

(i) TM admits the orthogonal direct decomposition as

$$TM = D \oplus D^{\theta} \oplus D^{\perp} \oplus < \xi > \tag{17}$$

(ii) The distribution D is ϕ invariant, i.e., $\phi D = D$.

(iii) For any non-zero vector field $X \in (D^{\theta})_p, p \in M$, the angle θ between JX and $(D^{\theta})_p$ is constant and independent of the choice of point p and X in $(D^{\theta})_p$.

(iv) The distribution $D^{\perp}$ is ϕ anti-invariant, i'e', $\phi D^{\perp} \subseteq T^{\perp}M$.

In this case, we call θ quasi-hemi-slant angle of M. Let the dimension of distributions D, D^{θ} and $D^{\perp}$ are n_1, n_2 and n_3 respectively. Then we can easily see the following cases.

(i) If $n_1 = 0$, then M is a hemi-slant submanifold.

(ii) If $n_2 = 0$, then M is a semi-invariant submanifold.

(iii) If $n_3 = 0$, then M is a semi-slant submanifold.

We say that a quasi hemi-slant submanifold M is proper if $D \neq \{0\}, D^{\perp} \neq \{0\}$ and $\theta \neq 0, \frac{\pi}{2}$.

This means that the notion of quasi hemi-slant submanifold is a generalization of invariant, anti-invariant, semi-invariant, slant, hemi-slant, semi-slant submanifolds and also, they are the examples of quasi hemi-slant submanifolds.

Remark 3.2. The definition can be generalized by taking $TM = D \oplus D^{\theta_1} \oplus D^{\theta_2} \cdots \oplus D^{\theta_k} \oplus D^{\perp} \oplus < \xi >$. Hence, we can define multi-slant submanifolds, quasi multi-slant submanifolds, quasi hemi multi-slant submanifolds, etc [14].

Example 1. *Let $\mathbb{R}^7$ be a 7-dimensional Euclidean space with a local coordinate system $(x_1, x_2, x_3, y_1, y_2, y_3, z)$, which admits the following quasi Sasakian structure ϕ and the metric g*

$$\phi = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2y_1 & 2y_2 & 0 & 0 \end{bmatrix}.$$

and

$$g = \begin{bmatrix} 1+4y_1^2 & 4y_1y_2 & 0 & 0 & 0 & 0 & -2y_1 \\ 4y_1y_2 & 1+4y_2^2 & 0 & 0 & 0 & 0 & -2y_2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -2y_1 & -2y_2 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

with $\xi = \frac{\partial}{\partial z}$; $\eta = dz - 2y_1 dx_1 - 2y_2 dx_2$. Let $\left\{E_i = \frac{\partial}{\partial x_i}\right\}_{i=1,2,\ldots,7}$ be the global fields of frames of $\mathbb{R}^7$, where $(x_1, x_2, x_3, x_4, x_5, x_6, z) = (x_1, x_2, x_3, y_1, y_2, y_3, z)$. Then the quintet of $(\mathbb{R}^7, \phi, \xi, \eta, g)$ consist a quasi Sasakian manifold [7].

Now, we define a submanifold N of $\mathbb{R}^7$ by the immersion γ as follows:
$\gamma(u_1, u_2, u_3, u_4, u_5, u_6) = (u_1, (-\tan\theta)u_1 + u_2, u_3, (\sin\theta)u_4, (\cos\theta)u_4, u_5, u_6)$

where $0 < \theta < \frac{\pi}{2}$. It is easy check that tangent bundle $TN = Span\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6\}$, where

$$Z_1 = \frac{\partial}{\partial x_3}, \qquad Z_2 = \frac{\partial}{\partial x_6}, \qquad Z_3 = \frac{\partial}{\partial x_2},$$
$$Z_4 = \sin\theta \frac{\partial}{\partial x_4} + \cos\theta \frac{\partial}{\partial x_5},$$
$$Z_5 = \frac{\partial}{\partial x_1} - \tan\theta \frac{\partial}{\partial x_2}, \qquad Z_6 = \frac{\partial}{\partial x_6}.$$

Using the structure ϕ, we obtain

$$\phi Z_1 = \frac{\partial}{\partial x_6}, \qquad \phi Z_2 = -\frac{\partial}{\partial x_3}, \qquad \phi Z_3 = \frac{\partial}{\partial x_5},$$
$$\phi Z_4 = -\sin\theta \frac{\partial}{\partial x_1} - \cos\theta \frac{\partial}{\partial x_2},$$
$$\phi Z_5 = \frac{\partial}{\partial x_4} - \tan\theta \frac{\partial}{\partial x_5}, \qquad \phi X_6 = 0$$

If we choose the following distributions

$$D = Span\{Z_1, Z_2\}, D^\theta = Span\{Z_3, Z_4\}, D^\perp = Span\{Z_5\}. \tag{18}$$

Then, the distributions D, D^θ and $D^\perp$ are invariant, slant with slant angle θ and anti-invariant distributions, respectively. Thus, in view of (17) N is *quasi hemi-slant* submanifold of $\mathbb{R}^7$.

Let M be a quasi hemi-slant submanifold of quasi-Sasakian manifold $\bar{M}$. We denote the projection of $X \in \Gamma(TM)$ on the distribution D, D^θ and $D^\perp$ by P, Q and R respectively. Then we can write for $X \in \Gamma(TM)$

$$X = PX + QX + RX + \eta(X)\xi. \tag{19}$$

Now we put,

$$\phi X = TX + NX, \tag{20}$$

where TX and NX are tangential and normal component of ϕX on M. Using (19) and (20), we get

$$\phi X = TPX + NPX + TQX + NQX + TRX + NRX. \tag{21}$$

Since $\phi D = D$ and $\phi D^\perp \subseteq T^\perp M$, we have $NPX = 0$ and $TRX = 0$. So, we get

$$\phi X = TPX + TQX + NQX + NRX. \tag{22}$$

Then for any $X \in \Gamma(TM)$, it is easy to write

$$TX = TPX + TQX,$$

and

$$NX = NQX + NRX.$$

So, from (22), we have the following decomposition

$$\phi(TM) = D \oplus TD^\theta \oplus ND^\theta \oplus ND^\perp, \tag{23}$$

where, $'\oplus'$ denotes orthogonal direct sum. Since $ND^\theta \subset T^\perp M$ and $ND^\perp \subset T^\perp M$, we have

$$T^\perp M = ND^\theta \oplus ND^\perp \oplus \mu, \tag{24}$$

where μ is the orthogonal complement of $ND^\theta \oplus ND^\perp$ in $\Gamma(T^\perp M)$ and it is invariant with respect to ϕ. For any non-zero vector field $V \in \Gamma(T^\perp M)$.

We put

$$\phi V = tV + nV, \tag{25}$$

where, $tV \in \Gamma(D^\theta \oplus D^\perp)$ and $nV \in \Gamma(\mu)$.

Proposition 1. *Let M be a quasi hemi-slant submanifold of quasi-Sasakian manifold $\bar{M}$, then for any $X \in \Gamma(TM)$, we have*

$$\nabla_X TY - A_{NY}X - T\nabla_X Y - t\sigma(X,Y) = \eta(Y)AX - g(AX,Y)\xi, \tag{26}$$

$$\sigma(X,TY) + \nabla^\perp_X NY - N\nabla_X Y - n\sigma(X,Y) = 0, \tag{27}$$

and

$$TD = D, TD^\theta = D^\theta, \quad TD^\perp = \{0\}, \quad tND^\theta = D^\theta, \quad tND^\perp = D^\perp. \tag{28}$$

Proof. Using equations (4), (6) and (7) and equating tangential and normal components, we obtain the desired results.

Proposition 2. *Let M be a quasi hemi-slant submanifold of quasi-Sasakian manifold $\bar{M}$, then the endomorphism T and N, t and n in the tangent bundle of M, satisfying the following identities.*

(i) $T^2 + tN = -I + \eta \otimes \xi$ *on* TM,

(ii) $NT + nN = 0$ *on* TM,

(iii) $Nt + n^2 = -I$ *on* $T^{\perp}M$,

(iv) $Tt + tn = 0$ *on* $T^{\perp}M$.
where I *is the identity.*

Proof. Using equations (20) and (25) and using the fact that $\phi^2 = -I + \eta \otimes \xi$, then on computing tangential and normal components.

Lemma 1. *Let* M *be a quasi hemi-slant submanifold of quasi-Sasakian manifold* $\bar{M}$, *then*

(i) $T^2X = -(cos^2\theta)X$,

(ii) $g(TX, TY) = (cos^2\theta)g(X, Y)$,

(iii) $g(NX, NY) = (sin^2\theta)g(X, Y)$,
for any $X, Y \in D^{\theta}$.

Proof. The proof follows similar steps as in Proposition 2.8 of [12].

Proposition 3. *Let* M *be a quasi hemi-slant submanifold of quasi-Sasakian manifold* $\bar{M}$, *then*

$$(\bar{\nabla}_X T)Y = \Lambda_{NY}X + t\sigma(X, Y) + \eta(Y)AX - g(AX, Y)\xi,$$

$$(\bar{\nabla}_X N)Y = n\sigma(X, Y) - \sigma(X, TY),$$

$$(\bar{\nabla}_X t)Y = \Lambda_{nV}X - T\Lambda_V X,$$

$$(\bar{\nabla}_X n)V = -\sigma(X, tV) - N\Lambda_V X,$$

for any $X, Y \in \Gamma(TM)$ *and* $V \in \Gamma(T^{\perp}M)$.

Proof. Using the equations (4), (6), (7), (11), (12) and (13) and equating tangential and normal components.

Proposition 4. *Let* M *be a quasi hemi-slant submanifold of quasi-Sasakian manifold* $\bar{M}$, *then*

$$\sigma(X, \xi) = NAX$$

and

$$\nabla_X \xi = TAX,$$

for any $X \in \Gamma(TM)$.

Proof. Using the equations (5), (6) and (11) and equating tangential and normal components.

Lemma 2. *Let M be a quasi hemi-slant submanifold of quasi-Sasakian manifold $\bar{M}$, then*

$$\Lambda_{\phi Z}W = \Lambda_{\phi W}Z + T([Z,W]) + g(AW,Z)\xi - g(AZ,W)\xi,$$
$$\nabla^{\perp}_{Z}\phi W = \nabla^{\perp}_{W}\phi Z + N([Z,W])$$

for any $Z, W \in D^{\perp}$.

Proof. For all $Z, W \in D^{\perp}$ and using covariant differentiation in (4), we have

$$\bar{\nabla}_Z \phi W - \phi(\bar{\nabla}_Z W) = \eta(W)AZ - g(AZ,W)\xi$$

Using (6), (7), (11) and (12), we have

$$-\Lambda_{\phi W}Z + \nabla^{\perp}_{Z}\phi W - T(\nabla_Z W) - N(\nabla_Z W) - t\sigma(Z,W) - n\sigma(Z,W) = -g(AZ,W)\xi$$

Equating tangential and normal part, we have

$$\Lambda_{\phi W}Z + T(\nabla_Z W) + t\sigma(Z,W) = g(AZ,W)\xi, \tag{29}$$

$$\nabla^{\perp}_{Z}\phi W = N(\nabla_Z W) + n\sigma(Z,W). \tag{30}$$

Interchanging Z and W in (29) and (30), we can easily get the desired results.

Lemma 3. *Let M be a quasi hemi-slant submanifold of quasi-Sasakian manifold $\bar{M}$, then*

(*i*) $g([X,Y]\xi) = 2g(TAX,Y)$,

(*ii*) $g(\bar{\nabla}_X Y, \xi) = g(TAX,Y)$,
for any $X, Y \in \Gamma(D \oplus D^{\theta} \oplus D^{\perp})$.

Proof. Using the equations (3), (5) and (11), calculation just straight forward.

4. Integrability of Distributions

In this segment, we examine the integrability conditions of distributions involved in the definition of quasi hemi-slant submanifolds of quasi-Sasakian manifolds.

Theorem 4. *Let M be a proper quasi hemi-slant submanifold of quasi-Sasakian manifold $\bar{M}$, then the distribution $D\oplus <\xi>$ is integrable if and only if*

$$g(\nabla_X TY - \nabla_Y TX, TQZ) = g(\sigma(Y,TX) - \sigma(X,TY), NQZ + NRZ) \tag{31}$$

for all $X, Y \in \Gamma(D\oplus <\xi>)$ and $Z \in \Gamma(D^{\theta} \oplus D^{\perp})$.

Proof. For all $X, Y \in \Gamma(D\oplus < \xi >)$ and $Z \in \Gamma(D^\theta \oplus D^\perp)$, we know that

$$g([X,Y],Z) = g(\bar{\nabla}_X Y) - g(\bar{\nabla}_Y X, Z)$$

Using (3) in above equation, we have

$$g([X,Y],Z) = g(\phi\bar{\nabla}_X Y, \phi Z) - g(\phi\bar{\nabla}_Y X, \phi Z). \tag{32}$$

Using (4) and concept of covariant differentiation, we have

$$g(\phi(\bar{\nabla}_X Y), \phi Z) = g(\bar{\nabla}_X \phi Y, \phi Z)$$

Therefore from (32), we have

$$g([X,Y],Z) = g(\bar{\nabla}_X \phi Y, \phi Z) - g(\bar{\nabla}_Y \phi X, \phi Z). \tag{33}$$

Using (6), (11) and (20) in above equation, we have

$$\begin{aligned} g([X,Y],Z) = g(\nabla_X TY + \sigma(X,TY), TQZ + TRZ + NQZ + NRZ) \\ -g(\nabla_Y TX + \sigma(Y,TX), TQZ + TRZ + NQZ + NRZ) \end{aligned}$$

As $\phi D^\perp \in T^\perp M$, so $TRZ = 0$, therefore from above, we have

$$g([X,Y],Z) = g(\nabla_X TY - \nabla_Y TX, TQZ) + g(\sigma(X,TY) - \sigma(Y,TX), NQZ + NRZ)$$

As $D\oplus < \xi >$ is integrable, so we have the required result.

Theorem 5. *Let M be a proper quasi hemi-slant submanifold of quasi-Sasakian manifold $\bar{M}$, then the slant distribution $D^\theta \oplus < \xi >$ is integrable if and only if*

$$\begin{aligned} g(\Lambda_{NTZ}Y - \Lambda_{NTY}Z, W) = g(\Lambda_{NZ}Y - \Lambda_{NY}Z, TPW) \\ + g(\nabla^\perp_Z NY - \nabla^\perp_Y NZ, NRW) \end{aligned} \tag{34}$$

for all $Y, Z \in \Gamma(D^\theta \oplus < \xi >)$ and $W \in \Gamma(D \oplus D^\perp)$.

Proof. For all $Y, Z \in \Gamma(D^\theta \oplus < \xi >)$ and $W = PW + RW \in \Gamma(D \oplus D^\perp)$ and using (3) and (4), we have

$$g([Y,Z],W) = g(\bar{\nabla}_Y \phi Z, \phi W) - g(\bar{\nabla}_Z \phi Y, \phi W). \tag{35}$$

Using (7) and (11) in (35), we have

$$g([Y,Z],W) = g(\Lambda_{NY}Z - \Lambda_{NZ}Y, \phi W) + g(\nabla^\perp_Y NZ - \nabla^\perp_Z NY, \phi W)$$

$$-g(\bar{\nabla}_Y T^2 Z - \bar{\nabla}_Z T^2 Y, W) + g(\Lambda_{NTZ} Y - \Lambda_{NTY} Z, W).$$

Using (20) and Lemma 1 in above equation, we have

$$(1 - cos^2\theta) g([Y, Z], W) = g(\Lambda_{NY} Z - \Lambda_{NZ} Y, TPW) + g(\nabla^{\perp}_Y NZ - \nabla^{\perp}_Z NY, NRW)$$
$$+ g(\Lambda_{NTZ} Y - \Lambda_{NTY} Z, W).$$

As distribution $D^\theta \oplus < \xi >$ is integrable, so we have required result.

Theorem 6. *Let M be a proper quasi hemi-slant submanifold of quasi-Sasakian manifold $\bar{M}$, if*

$$\nabla^{\perp}_Z NY - \nabla^{\perp}_Y NZ \in ND^\theta \oplus \mu,$$
$$\Lambda_{NTZ} Y - \Lambda_{NTY} Z \in D^\theta,$$

and

$$\Lambda_{NZ} Y - \Lambda_{NY} Z \in D^\perp \oplus D^\theta.$$

for all $Y, Z \in \Gamma(D^\theta \oplus < \xi >)$, then the slant distribution $D^\theta \oplus < \xi >$ is integrable.

Theorem 7. *Let M be a proper quasi hemi-slant submanifold of quasi-Sasakian manifold $\bar{M}$, then the anti-derivative $D^\perp$ is integrable if and only if*

$$g(T[Z, W], TY) = g(N[W, Z], NQY) \tag{36}$$

for all $Z, W \in \Gamma(D^\perp)$ and $Y \in \Gamma(D \oplus D^\theta)$.

Proof. For all $Z, W \in \Gamma(D^\perp)$ and $Y = PY + QY \in \Gamma(D \oplus D^\theta)$ and using (3), (4), (7) and (20), we have

$$g([Z, W], Y) = g(\Lambda_{\phi Z} W - \Lambda_{\phi W} Z, TPY + TQY) - g(\nabla^{\perp}_W \phi Z - \nabla^{\perp}_Z \phi W, NQY).$$

Using Lemma 2 in above equation, we have

$$g([Z, W], Y) = g(T[Z, W], TY) - g(N[W, Z], NQY).$$

As anti-derivative $D^\perp$ is integrable, so we have desired result.

5. Totally Geodesic Foliations

Geodesicness and foliations are significant geometric characteristics. In this section, we determine the geometry of foliations of quasi hemi-slant submanifolds, also, some conditions are given for the totally Geodesicness.

Theorem 8. *Let M be a proper quasi hemi-slant submanifold of quasi-Sasakian manifold $\bar{M}$, then M is totally geodesic if and only if*

$$g(\sigma(X,PY)+cos^2\theta\sigma(X,QY),V)=g(\nabla^{\perp}_{X}NTQY,V)-g(\nabla^{\perp}_{NY},nV) \qquad (37)$$
$$+\,g(\Lambda_{NQY}X+\Lambda_{NRY}X,tV)$$

for all $X,Y\in\Gamma(TM)$ and $V\in\Gamma(T^{\perp}M)$.

Proof. For all $X,Y\in\Gamma(TM),V\in\Gamma(T^{\perp}M)$ and using (19), we have

$$g(\bar{\nabla}_XY,V)=g(\bar{\nabla}_XPY,V)+g(\bar{\nabla}_XQY,V)+g(\bar{\nabla}_XRY,V)$$

Using (3), (4), (6) and (11) in above equation, we have

$$g(\bar{\nabla}_XY,V)=g(\sigma(X,PY),V)-g(\bar{\nabla}_XT^2QY,V)-g(\bar{\nabla}_XNTQY,V)$$
$$+g(\bar{\nabla}_XNQY,\phi V)+g(\bar{\nabla}_XNRY,\phi V)$$

Using (7), (12) and Lemma 1 in above equation, we have

$$g(\bar{\nabla}_XY,V)=g(\sigma(X,PY)+cos^2\theta\sigma(X,QY),V)-g(\Lambda_{NQY}X+\Lambda_{NRY}X,tV+nV)$$
$$+g(\nabla^{\perp}_XNQY+\nabla^{\perp}_XNRY,tV+nV)-g(\bar{\nabla}_XNTQY,V)$$

As $NY=NPY+NQY+NRY$ and $NPY=0$, thus we have

$$g(\bar{\nabla}_XY,V)=g(\sigma(X,PY)+cos^2\theta\sigma(X,QY),V)-g(\Lambda_{NQY}X+\Lambda_{NRY}X,tV)$$
$$+g(\nabla^{\perp}_XNY,nV)-g(\bar{\nabla}_XNTQY,V)$$

Hence the proof follows.

Theorem 9. *Let M be a proper quasi hemi-slant submanifold of quasi-Sasakian manifold $\bar{M}$, then the distribution $D\oplus<\xi>$ defines a totally geodesic foliation on M if and only if*

$$g(\nabla_XTY,TQZ)=-g(\sigma(X,TY),NQZ+NRZ). \qquad (38)$$

$$g(\nabla_XTY,tV)=-g(\sigma(X,TY),nV) \qquad (39)$$

for any $X,Y\in\Gamma(D\oplus<\xi>),Z=QZ+RZ\in\Gamma(D^{\theta}\oplus D^{\perp})$ and $V\in\Gamma(T^{\perp}M)$.

Proof. For any $X, Y \in \Gamma(D\oplus <\xi>), Z = QZ + RZ \in \Gamma(D^\theta \oplus D^\perp)$ and using (3) and (4), we have

$$g(\bar{\nabla}_X Y, Z) = g(\bar{\nabla}_X \phi Y, \phi Z)$$

Using (6), (11), (20) and $NY = 0$, in above equation, we have

$$g(\bar{\nabla}_X Y, Z) = g(\nabla_X TY, TQZ) + g(\sigma(X, TY), NQZ + NRZ). \tag{40}$$

Now, for any $X, Y \in \Gamma(D\oplus <\xi>), V \in \Gamma(T^\perp M)$ and using (3), (4), (20) and $NY = 0$, we have

$$g(\bar{\nabla}_X Y, V) = g(\bar{\nabla}_X TY, \phi V)$$

Using (6) and (12), we have

$$g(\bar{\nabla}_X Y, V) = g(\nabla_X TY, tV) + g(\sigma(X, TY), nV). \tag{41}$$

As distribution $(D\oplus <\xi>)$ defines a totally geodesic foliation on M, So from (40) and (41), we have desired results.

Theorem 10. *Let M be a proper quasi hemi-slant submanifold of quasi-Sasakian manifold $\bar{M}$, then the distribution $D^\perp$ defines a totally geodesic foliation on M if and only if*

$$g(\Lambda_{NZ} Y, TPW + TQW) = g(\nabla^\perp_Y NZ, NQW), \tag{42}$$

$$g(\Lambda_{NZ} Y, tV) = g(\nabla^\perp_Y NZ, nV) \tag{43}$$

for any $Y, Z \in \Gamma(D^\perp), W \in \Gamma(D \oplus D^\theta)$ and $V \in \Gamma(T^\perp M)$.

Proof. For any $Y, Z \in \Gamma(D^\perp), W = PW + QW \in \Gamma(D \oplus D^\theta)$ and using (3), (4) and (11), we have

$$g(\bar{\nabla}_Y Z, W) = g(\bar{\nabla}_Y NZ, \phi W)$$

Using (7) and (20), we have

$$g(\bar{\nabla}_Y Z, W) = -g(\Lambda_{NZ} Y, TPW + TQW) + g(\nabla^\perp_Y NZ, NQW). \tag{44}$$

Now, for any $Y, Z \in \Gamma(D^\perp)$, $V \in \Gamma(T^\perp M)$ and using (3), (4) and (11), we have

$$g(\bar{\nabla}_Y Z, V) = g(\bar{\nabla}_Y NZ, \phi V)$$

Using (7) and (12), we have

$$g(\bar{\nabla}_Y Z, V) = -g(\Lambda_{NZ} Y, tV) + g(\nabla^\perp_Y NZ, nV). \tag{45}$$

As distribution $(D^\perp)$ defines a totally geodesic foliation on M, So from (44) and (45), we have desired results.

Theorem 11. *Let M be a proper quasi hemi-slant submanifold of quasi-Sasakian manifold $\bar{M}$, then the distribution $D^\theta \oplus < \xi >$ defines a totally geodesic foliation on M if and only if*

$$g(\nabla^\perp_Z NW, NRX) = g(\Lambda_{NW} Z, TPX) - g(\Lambda_{NTW} Z, X), \tag{46}$$

$$g(\Lambda_{NW} Z, tV) = g(\nabla^\perp_Z NW, nV) - g(\nabla^\perp_Z NTW, V) \tag{47}$$

for any $Z, W \in \Gamma(D^\theta \oplus < \xi >), X \in \Gamma(D \oplus D^\perp)$ and $V \in \Gamma(T^\perp M)$.

Proof. For any $Z, W \in \Gamma(D^\theta \oplus < \xi >), X = PX + RX \in \Gamma(D \oplus D^\perp)$ and using (3), (4), (7) and (11), we have

$$g(\bar{\nabla}_Z W, X) = -g(\bar{\nabla}_Z \phi TW, X) + g(\bar{\nabla}_Z NW, \phi X)$$

$$g(\bar{\nabla}_Z W, X) = -g(\bar{\nabla}_Z T^2 W, X) - g(\bar{\nabla}_Z NTW, X) - g(\Lambda_{NW} Z, \phi X) + g(\nabla^\perp_Z NW, \phi X)$$

Using (7) and Lemma 1 in above equation, we have

$$(1 - cos^2\theta) g(\bar{\nabla}_Z W, X) = g(\Lambda_{NTW} Z, X) - g(\Lambda_{NW} Z, \phi X) + g(\nabla^\perp_Z NW, \phi X)$$

Using (20) and using the fact that $NPX = 0$ in above, we have

$$(1 - cos^2\theta) g(\bar{\nabla}_Z W, X) = g(\Lambda_{NTW} Z, X) - g(\Lambda_{NW} Z, TPX) + g(\nabla^\perp_Z NW, NRX). \tag{48}$$

Now, for any $Z, W \in \Gamma(D^\theta \oplus < \xi >), V \in \Gamma(T^\perp M)$ and using (3), (4) and (11), we have

$$g(\bar{\nabla}_Z W, V) = -g(\bar{\nabla}_Z \phi TW, V) + g(\bar{\nabla}_Z NW, \phi V)$$

Using 97) and (11), we have

$$g(\bar{\nabla}_Z W, V) = -g(\bar{\nabla}_Z T^2 W, V) - g(\bar{\nabla}_Z NTW, V) - g(\Lambda_{NW} Z, \phi V) + g(\nabla^\perp_Z NW, \phi V)$$

Using (2.7), (2.12)and Lemma 3.5 in above equation, we have

(5.13)(49)
$$(1 - cos^2\theta) g(\bar{\nabla}_Z W, V) = -g(\nabla^\perp_Z NTW, V) - g(\Lambda_{NW} Z, tV) + g(\nabla^\perp_Z NW, nV).$$

As distribution $D^\theta \oplus < \xi >$ defines a totally geodesic foliation on M, So from (48) and (49), we have desired results.

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