ON SOME CLASSES OF BERNSTEIN TYPE OPERATORS WHICH PRESERVE THE GLOBAL SMOOTHNESS IN THE CASE OF UNIVARIATE FUNCTIONS

by
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Abstract. The aim of this paper is to describe the linear and positive operators who have the property of preservation of the global smoothness. We insist on the Bernstein and Stancu operators. We consider the case of univariate function.
In order to show the preservation of global smoothness, we use the notions of modulus of continuity and K- functionals.

1. Introduction

First of all we give some notation:

- Let be the space
$C(I) := \{ f : I \to R \;/\; f \quad continuous \,\&\, \quad bounded \quad on I, \quad I \neq \theta \}.$
- Let be $f \in C(I)$, I a real interval and $\delta \in I$. The application
$\omega : [0,\infty) \to R, \quad \omega(\delta) := \sup\{ |f(x) - f(x')| : x, x' \in I, |x - x'| \le \delta \}$

is the modulus of continuity.

- If $f \in C(I)$, $I \subset R$ and $k \in N$, $\delta \in I$ then we can define the modulus of smoothness of order k

$$\omega_f = \omega_k(f,\delta) := \sup\{ |\Delta_h^k f(x)| : x, x + kh \in I, |h| \le \delta \}$$ where

$$\Delta_h^k f(x) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f(x + ih)$$ is the divided differences.

- The Pochhammer symbol is defined by $(n)_s := \prod_{i=0}^{s-1} (n-i)$,

$\prod_{i=0}^{-1} (n-i) := 1, \quad \prod_{-1} := \{0\}$.

- $K_s(f;\delta) := K(f;\delta;C[0,1],C^s[0,1]) := \inf\{ \|f - g\| + \delta \|g^{(s)}\| : g \in C^s[0,1] \}$

where $f \in C[0,1]$ and $\delta \ge 0$, are K- functionals of order s, $s \ge 1$.

- Let be the compacts intervals I and I', I'⊆I. For r≥1 an operator $L : C\ (I) \longrightarrow C(I')$ is almost convex of order r-1, if the following holds:

Let $K_{I,i} := \{f \in C(I)$: $[x_0, x_1, \ldots, x_i; f] \geq 0$ for any $x_0 < x_1 < \ldots < x_i \in I\}$. There exist p≥0, integers i_{j}, $1 \leq i \leq p$, satisfying $0 \leq i_1 < \ldots < i_p < r$ such that

$$f \in \left(\bigcap_{j=1}^{p} K_{I,i_j} \right) \bigcap K_{I,r} \to Lf \in K_{I',r}.$$

For p=0 we put $\bigcap_{j=1}^{p} K_{I,i_j} := C(I)$. In this case $K_{I,r}$ is mapped by L into $K_{I',r}$ and L is called "convex" of order r-1≥0.

2. The operators

We introduce here the well-known Bernstein and Stancu operators.

- The **Bernstein operators** are defined for *f*: [0,1]→R as

$$(B_m f)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k}{m}\right); \quad p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k}, \quad k = \overline{0,m}, \quad x \in [0,1]$$

- The **Stancu operators**.

In paper [13] D.D. Stancu has introduced and investigated a linear operator $S_m^{(\alpha)}: C[0,1] \to C[0,1]$ defined by

$$\left(S_m^{(\alpha)} f\right)(x) = \sum_{k=0}^{m} \omega_{m,k}(x;\alpha) f\left(\frac{k}{m}\right),$$

$$\omega_{m,k}(x;\alpha) = \binom{m}{k} \frac{\prod_{\nu=0}^{k-1} (x + \nu\alpha) \prod_{\mu=0}^{m-k-1} (1 - x + \mu\alpha)}{(1+\alpha)(1+2\alpha)\ldots(1+(m-1)\alpha)},$$

α being a parameter which may depend only on the natural number m. If α≥0, then these operators preserve the positivity of the function *f*. For α=0, $P_m^{(\alpha)}$ coincides with the Bernstein operator.

3. The preservation of the global smoothness

Even since 1951 the mathematicians have been showing their interest for the problem of the preservation of the global smoothness preservation by linear operators. In this year, in his paper [14], S.B. Steckin give as the following

Theorem 1. For fixed s,n∈N and $f \in C_{2\pi}$ let t_n be a trigonometric polynomial of degree ≤n such that $\|f - t_n\| \le c_1 \cdot \omega_s\left(f; \frac{1}{n}\right)$.

Then for all $\delta > 0$ one has

$$\omega_s(t_n, \delta) \le \left(\sin\frac{1}{2}\right)^{-s} \left(1 + 2^s \cdot c_1\right)\omega_s(f, \delta).$$

On 1965 a result on smoothness preservation by the Bernstein operators B_m on C[0,1] was given by Hajek [8] :

Theorem 2. Let $f \in Lip_M(1;[0,1])$. Then $B_m f \in Lip_M(1;[0,1])$.

A few years later, this result was generalized by Lindvall [9] and Brown, Elliott & Paget [3]. They showed that we could replace the statement $Lip_M(1;[0,1])$ by $Lip_M(\alpha;[0,1])$, $\alpha \in [0,1]$. This means that, if global smoothness of a function $f \in C[0,1]$ is represented by stating that it satisfied a certain Lipschitz condition, then the same is true for its approximant $B_n f$.

Regarding Stancu operators, in 1987, in paper [5] B. Della Vecchia was proved:

Theorem 3. If $f \in Lip_M(1;[0,1])$ then $S_m^{(\alpha)} f \in Lip_M(1;[0,1])$ for m∈N, α≥0.

Anastassiou, Cottin and Gonska was generalized the second theorem in 1991 [2], so that we have:

Theorem 4. For $\forall f \in C[0,1]$ and $\delta \ge 0$, for the Bernstein operator B_m

$$\omega_1(B_n f;\delta) \le 1 \cdot \varpi_1(f;\delta) \le 2 \cdot \omega_1(f;\delta)$$

Here $\varpi_1(f;\cdot)$ denotes the least concave majorant of $\omega_1(f;\cdot)$. The constants 1 and 2 are best possible.

Another important fact is that Theorem 4 can be generalized by express global smoothness preservation in terms of K- functionals of order s, as Cottin & Gonska showed in [4]:

Theorem 5. For the operators B_m, one has for $\forall s \in N, f \in C[0,1], \delta \geq 0$ the inequalities

$$K_s(B_n f;s) \leq 1 \cdot K_s\left(f;\frac{(n)_s}{n^s} \cdot \delta\right) \leq K_s(f;\delta).$$

For the case s=1, Theorem 5 implies Theorem 4. Next we have a generalized version of Theorem 5 for a certain class of operators which includes those of Bernstein as special cases:

Theorem 6. Let k≥0, s∈N [*] and let I:=[a,b] and I':=[c,d]⊂[a,b] be compact intervals with non-empty interior (a≠b, c≠d). If L is a linear operator satisfying

$$L: C^k(I) \longrightarrow C^k(I') \quad such \quad that \quad \left\|(Lf)^{(k)}\right\|_{I'} \leq a_{k,l} \cdot \left\|f^{(k)}\right\|_I, \quad a_{k,l} \neq 0$$
$$for \quad all\ f \in C^k(I)$$

as well as

$$L: C^{k+s}(I) \longrightarrow C^{k+s}(I') \quad such \quad that \quad \left\|(Lg)^{(k+s)}\right\|_{I'} \leq b_{k,s,l} \cdot \left\|g^{(k+s)}\right\|_I,$$
$$for \quad all\ g \in C^{k+s}(I)$$

then for all $f \in C^k[I]$ and $\delta > 0$ one has

$$K_s\left((Lf)^{(k)};\delta\right)_{I'} \leq a_{k,l} \cdot K_s\left(f^{(k)};\frac{b_{k,s,l}}{a_{k,l}}\delta\right)_I.$$

Because we introduced the notion of almost convexity the next theorem prove the preservation of global smoothness by operators being almost convex of appropriate orders.

Theorem 7. [4] Let $k \geq 0$, $s \in N^*$ and I, I' be given as above. Let $L : C^k(I) \longrightarrow C^k(I')$ be a linear operator having the following properties:

(1) L is almost convex of orders k-1 and k+s-1
(2) L maps $C^{k+s}(I)$ into $C^{k+s}(I')$
(3) $L(\Pi_{k-1}) \subseteq \Pi_{k-1}$ and $L(\Pi_{k+s-1}) \subseteq \Pi_{k+s-1}$
(4) $L(C^k(I)) \not\subset \Pi_{k-1}$

Then for $\forall f \in C^k[I]$ and $\delta > 0$ we have

$$K_s\left((Lf)^{(k)};\delta\right)_{I'} \leq \frac{1}{k!}\cdot\left\|(Le_k)^{(k)}\right\|\cdot K_s\left(f^{(k)};\frac{1}{(k+s)_s}\cdot\frac{\left\|(Le_{k+s})^{(k+s)}\right\|}{\left\|(Le_k)^{(k)}\right\|}\delta\right)_I$$

with $e_k = x^k$.

Proof. We show that the assumptions of Theorem 6 are satisfied. Let $l \in \{k, k+s\}$ and $L : C^p(I) \longrightarrow C^p(I')$ which is almost convex of order l-1, satisfying $L(\Pi_{l-1}) \subseteq \Pi_{l-1}$.

- For l=0, L is positive, maps $C^0(I)$ into $C^0(I')$, and the third assumption is satisfied. For such operators we know that $\|Lf\|_{I'} \leq \|Le_0\|_{I'}\cdot\|f\|_I$.
- For $l \geq 1$, define $I_l : C(I) \longrightarrow C^l(I)$ by

$$I_l(f;x) = \int_a^x \frac{(x-t)^{l-1}}{(l-1)!}\cdot f(t)dt .$$

Since L is almost convex of order l-1, the operator Q_l given by $Q_l := (L\, I_l)^{(l)}$ is linear and positive. The assumption $L(\Pi_{l-1}) \subseteq \Pi_{l-1}$ implies $Q_l f^{(l)} = (Lf)^{(l)}$ for $\forall f \in C^l[I]$. Hence

$$\left\|(Lf)^{(l)}\right\| = \left\|Q_l f^{(l)}\right\| \leq \|Q_l\|\cdot\left\|f^{(l)}\right\| \qquad for \forall f \in C^l(I)$$

Since Q_l is positive, we have $\|Q_l\| = \|Q_l e_0\| = \left\|\frac{1}{l!}\cdot(Le_l)^{(l)}\right\|$.

Putting now

$a_{k,l} := \left\| \frac{1}{k!} (Le_k)^{(k)} \right\|$ *and* $b_{k,s,l} := \left\| \frac{1}{(k+s)!} (Le_{k+s})^{(k+s)} \right\|$ yields two nonnegative constants for which the assumption of Theorem 6 are satisfied.

All that we still have to do is to proof that $a_{k,l} \neq 0$. Suppose that $\left\| \frac{1}{k!} (Le_k)^{(k)} \right\| = \|Q_l\| = 0 \quad \Rightarrow \quad (Lf)^{(k)} = 0 \quad for\, \forall f \in C^k(I)$ or $L : C^p(I) \longrightarrow \prod_{k-1}$. But this contradicts condition (4) and the proof is complete.

Now, we apply the general result to the classical Bernstein operators $B_n : C[0,1] \longrightarrow \prod_n$.

Proposition 8. [4] Let $k \geq 0$ and $s \in N^*$ be fixed. Then for $\forall$ $n \geq k+s$, $\forall f \in C^k[0,1]$ and $\forall\ \delta \geq 0$ the following inequality holds:

$$K_s\left((B_n f)^{(k)}; \delta\right)_{[0,1]} \leq \frac{(n)_k}{n^k} K_s\left(f^{(k)}; \frac{(n-k)_s}{n^s} \cdot \delta\right)_{[0,1]}$$

Proof.

- We have the next representation

$$(B_n f)^{(l)}(x) = \frac{(n)_l}{n^l} \cdot l! \cdot \sum_{k=0}^{n-l} \left[\frac{k}{n}, \ldots, \frac{k+l}{n}; f\right] \cdot C_{n-l}^k \cdot x^k \cdot (1-x)^{n-l-k}$$

due to Lorentz [10] which show as the fact that condition (1) is satisfied (B_n is almost convex for $\forall$ $l-1 \geq -1$).

- Since B_n is a polynomial operator, the general assumption and condition (2) from Theorem 7 are satisfied.
- Since B_n maps a polynomial of degree l into a polynomial of degree min{n,l}, condition (3) is also satisfied for $\forall n \in N$.
- We consider the k-th monomial $e_k \in C^k[0,1]$. From the assumption that $n \geq k+s$ it follows that $B_n e_k \in \Pi_k \setminus \Pi_{k-1}$ so that condition (4) is also verified.

Gonska gives the representation $\left(B_n e_l\right)^{(l)} = \frac{(n)_l}{n^l} \cdot l!$, in paper [6] and plugging into the inequality of Theorem 7 yields our claim.

The main result of this paper regards the case of Stancu operator.

Proposition 9. Let k≥0 and s∈N [*] be fixed. Then for ∀ n≥k+s, $\forall f \in C^k[0,1]$ and ∀ $\delta \geq 0$ the following inequality holds:

$$K_s\left(\left(S_n^{(\alpha)} f\right)^{(k)}; \delta\right)_{[0,1]} \leq \beta_{m,k}^{\alpha} \cdot K_s\left(f^{(k)}; \frac{\beta_{m,k+s}^{\alpha}}{\beta_{m,k}^{\alpha}} \cdot \delta\right)_{[0,1]}$$

Proof. The four condition and general assumption from Theorem 7 are again satisfied.

The representation of $(B_m^{\alpha} f)^{(l)}$ is due to the Mastroianni and Occorsio [11]:

$$\left(S_m^{(\alpha)} f\right)^{(l)} = l! \prod_{\nu=1}^{l} \frac{1 - \frac{\nu-1}{m}}{1 + (m-\nu)\alpha} \sum_{i=0}^{m-l} \omega_{m-l,i}(\alpha, x)\left[\frac{i}{m}, \frac{i+1}{m}, \ldots, \frac{i+l}{m}; f\right] +$$

$$+ \sum_{h \in I_p - \{0\}} a_h \sum_{i=0}^{\nu_l} \omega_{\nu_l, i}(\alpha, x) \prod_{j=1}^{l} \frac{1}{u_j} \Delta_{u_j} f\left(\frac{i}{m}\right)$$

Given the notation :

$$\beta_{m,l}^{\alpha} = \prod_{\nu=1}^{l} \frac{1 - \frac{\nu-1}{m}}{1 + (m-\nu)\alpha}$$

$$\lambda_{h,l}(\alpha, x) = a_h \sum_{i=0}^{\nu_l} \omega_{\nu_l, i}(\alpha, x) \prod_{j=1}^{l} \frac{1}{u_j} \Delta_{u_j} f\left(\frac{i}{m}\right)$$

$$\varphi_{m,l}(\alpha, x) = \sum_{h \in I_p - \{0\}} \lambda_{h,l}(\alpha, x)$$

we can write

$$\left(S_m^{(\alpha)} f\right)^{(l)} = l! \beta_{m,l}^{\alpha} \sum_{i=0}^{m-l} \omega_{m-l,i}(\alpha, x)\left[\frac{i}{m}, \frac{i+1}{m}, \ldots, \frac{i+l}{m}; f\right] + \varphi_{m,k}(\alpha, x).$$

Finally we have the representation of the quantities $(S_m^{(\alpha)}e_l)^{(l)}$, $l\in\{k,k+s\}$ as $\left(S_m^{(\alpha)}e_l\right)^{(l)} = l!\,\beta_{m,l}^{\alpha}$ and plugging into the inequality of Theorem 8 our claim yields.

We now consider two special cases of s≥1, which are of particular interest.

- The first case is s=1:

Proposition 10. Let k≥0 be a fixed integer. Then for ∀ n≥k+1, $\forall f \in C^k[0,1]$ and $\delta \ge 0$ we have:

$$\omega_1\left((B_n f)^{(k)};\delta\right) \le \frac{(n)_k}{n^k}\cdot \varpi_1\left(f^{(k)};\frac{n-k}{n}\cdot\delta\right) \le 1\cdot \varpi_1\left(f;\delta\right).$$

The left inequality is best possible , means that for e_{k+1} both sides are equal .

Proof. Proposition 8 gives the particular case

$$K_1\left((B_n f)^{(k)};\delta\right)_{[0,1]} \le \frac{(n)_k}{n^k}\cdot K_1\left(f^{(k)};\frac{n-k}{n}\delta\right)_{[0,1]} .$$

For K-functionals K_1 we have Brudnyî’ representation: $K_1\left(f;\delta\right)_{[0,1]} = \frac{1}{2}\varpi_1\left(f;2\delta\right)$. Using this in both sides of the inequalities which involves K_l leads to the first condition fron Proposition 10. Furthermore, for the function $e_{k+1}(x)=x^{k+1}$ it can easily be verified that, for n≥k+1, both sides in the left part of the inequality in Proposition 10 equal $\frac{(n)_{k+1}}{n^{k+1}}\cdot(k+1)!\delta \ge 0, \quad for \delta > 0.$

- The case s=2.

As an immediate consequence of Proposition 8 we get:

Consequence 11. For a fixed integer k≥0 and ∀n≥k+2 one has that

$$K_2\left((B_n f)^{(k)};\delta\right) \le \frac{(n)_k}{n^k} \cdot K_2\left(f^{(k)}; \frac{(n-k)(n-k-1)}{n^2} \cdot \delta \right) \le K_2\left(f^{(k)};\delta\right).$$

Proposition 10 was derived from Proposition 8 by an representation of K-functionals K_1 using modulus of continuity ω_1. We present only the case s=2 because only in this situation we know some constants involving in this relations. We use, for $\delta\in[0,1/2]$:

$$\frac{1}{4}\omega_2\left(f;\delta\right) \le K_2\left(f;\frac{1}{4}\delta^2 \right) \le \frac{9}{8}\omega_2\left(f;\delta\right).$$

Similar statements involving $\omega_2(f^{(k)};\delta)$ are obtained if one starts with $\omega_2((B_n f)^{(k)};\delta)$. As the result we have:

Proposition 12. For fixed integer $k\ge 0$, $f\in C^k[0,1]$ and all $\delta\ge 0$ the B_m operator satisfy the inequality

$$\omega_2\left((B_n f)^{(k)};\delta\right) \le 3\frac{(n)_k}{n^k}\left[1+\frac{(n-k)(n-k-1)}{2n^2}\right]\cdot\omega_2\left(f^{(k)};\delta\right).$$

In particular, for k=0 we have

$$\omega_2\left(B_n f;\delta\right) \le 3\left[1+\frac{n-1}{2n}\right]\omega_2\left(f\ ;\delta\right) \le 4{,}5\,\omega_2\left(f;\delta\right).$$

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