Acta Universitatis Apulensis No. 38/2014
ISSN: 1582-5329 pp. 131-141

OBJECT ORIENTED CONCEPT LATTICES CONSTRAINED BY HIERARCHICALLY ORDERED ATTRIBUTES

H. Mao

Abstract. Suggested by the hierarchically ordered attributes of formal concept lattices produced by R. Bělohlávek, V. Sklenář and J. Zacpal in 2004, we discuss the analysis of input data with a predefined hierarchy on attributes extending thus the basic approach of object oriented concept lattices. We define the notion of an object oriented concept respecting the attribute hierarchy. Object oriented concepts which do not respect the hierarchy are considered not relevant. Elimination of the non-relevant object oriented concepts leads to a reduced set of extracted object oriented concepts making the discovered structure of hidden object oriented concepts more comprehensible. We present basic formal results on our approach. In the end, with hierarchy on attributes, we compare the differences and similarities between constrained formal concept lattices and constrained object oriented concept lattices.

2000 *Mathematics Subject Classification*: 68P05; 68R01.

Keywords: object oriented concept lattice, hierarchy of attributes, formal context, formal concept analysis.

1. Introduction

Formal concept analysis is an example of a method for finding patterns and dependencies in data which can be run automatically (see [4]). The patterns looked for are called concepts. The authors [1] present modal operator approach which offers a complementary view of data with respect to derivation operator of formal concept analysis. One type of approximation operators is $^\diamond$ and $^\Box$. It could form a type of concept lattice–object oriented concept lattice (cf. [1-3]). Object oriented concepts are different from formal concept lattices which are discussed by [4]. It derives mainly from the fact that it is interpretable as natural concepts well-understood by humans.

The researchers indicate [4-7] that formal concept analysis treats both the individual objects and the individual attributes as distinct entities for which there is no further information available except for the relationship R saying which objects have which attributes. However, more often than not, both the set of objects and the set of attributes are supplied by an additional information. Further processing of the input data (formal context) should therefore take the additional information into account in such a way that only those concepts which are in an appropriate sense compatible with the additional information, are considered relevant. In the end, this may result in a simplification of the overall processing. Therefore, it seems reasonable to assume that instead of (U, V, R) (a formal context), the input data consists of a richer structure $(U, V, R, \ldots)$ where $\ldots$ may contain further sets, relations, functions, and so on.

This paper considers one particular kind of additional information which has the form of a hierarchy of attributes expressing their relation importance. The hierarchy enables us to eliminate object oriented concepts which are not compatible with the hierarchy. An important effect of the elimination is a natural reduction of the size of the resulting conceptual structure making the structure more comprehensible. The similar discussion for formal concept lattices could be found from [5]. Because the difference between formal concept lattices and object oriented concept lattices makes this paper be valuable to be considered and read. Section 2 gives the definitions and notation needed in this paper. A formal treatment of our approach is presented in Section 3. The fourth section is discussion.

2. Definitions and notation

Throughout this paper, U and V are nonempty finite sets, and $R \subseteq U \times V$. All the knowledge about formal contexts come from [4], we only review some of them. All the knowledge about operators $^{\diamond}$ and $^{\Box}$, and object oriented concept lattices are mainly from [10], and also could be found from [1-3]. For unexplained notation and notions in lattices and order theory, we refer the reader to [8] and [9].

We assume that a data set is given in terms of a formal context.

Elements of U are called *objects* and elements of V are called *attributes*. The relationships between objects and attributes are described by a binary relation R between U and V, which is a subset of the Cartesian product $U \times V$. For a pair of elements $x \in U$ and $y \in V$, if $(x, y) \in R$ written as xRy, we say that x has the attribute y, or the attribute y is possessed by object x. (U, V, R) is called *a formal context*. In general, every information table can be represented by a formal context.

An object $x \in U$ has the set of attributes:

$$xR = \{y \in V \mid xRy\} \subseteq V.$$

The set of attributes xR can be viewed as a description of the object x. In other words, object x is described by the set of attributes xR. Similarly, an attribute y is possessed by the set of objects:

$$Ry = \{x \in U \mid xRy\} \subseteq U.$$

For a set of objects $A \subseteq U$ and a set of attributes $B \subseteq V$, we can define a pair data operators, ${}^\square : 2^U \to 2^V$ and ${}^\square : 2^V \to 2^U$, as follows:

$$A^\square = \{y \in V \mid Ry \subseteq A\},$$
$$B^\square = \{x \in U \mid xR \subseteq B\}.$$

We can define a pair data operators, ${}^\diamond : 2^U \to 2^V$ and ${}^\diamond : 2^V \to 2^U$ as follows:

$$A^\diamond = \{y \in V \mid Ry \cap A \neq \emptyset\} = \bigcup_{x \in A} xR,$$
$$B^\diamond = \{x \in U \mid xR \cap B \neq \emptyset\} = \bigcup_{y \in B} Ry.$$

A pair $(A, B), A \subseteq U, B \subseteq V$, is called an *object oriented concept* if $A = B^\diamond$ and $B = A^\square$. The set of objects A is referred to as the *extension* of the concept (A, B), and the set of attributes B is referred to as the *intension.* If an object has an attribute in B, then the object belongs to A. Moreover, only objects in A possess attributes in B.

For two object oriented concepts (A_1, B_1) and (A_2, B_2), we say that (A_1, B_1) is a *sub-concept* of (A_2, B_2), and (A_2, B_2) is a *super-concept* of (A_1, B_1), in notation, $(A_1, B_1) \leq (A_2, B_2)$, if and only if

$$A_1 \subseteq A_2,$$

or equivalently, if and only if

$$B_1 \subseteq B_2.$$

The family of all object oriented concepts is a complete lattice called *object oriented concept lattice*, in notation $\mathcal{B}(U, V, R)$. The meet $\wedge$ and the join $\vee$ of the object oriented concept lattice are defined by

$$(A_1, B_1) \wedge (A_2, B_2) = ((A_1 \cap A_2)^{\square\diamond}, (B_1 \cap B_2)),$$
$$(A_1, B_1) \vee (A_2, B_2) = ((A_1 \cup A_2), (B_1 \cup B_2)^{\diamond\square}).$$

Let $(U,^{\square\diamond})$ denote the family of sets $(U,^{\square\diamond}) = \{A^{\square\diamond} \mid A \subseteq U\}$. It contains the universe U and the empty set $\emptyset$ and is the family of the extension of all object oriented concepts.

3. Object oriented concept lattices of contexts with hierarchically ordered attributes

To easily compare the difference and similarity between formal concept lattices and object oriented concept lattices for formal concept analysis with hierarchically ordered attributes, in what follows, we select some expressive sentences from [5]. Even though, the different methods to deal with a formal context between formal concept lattices and object oriented concept lattices bring about the different views and proofs, and further, the internal value of all the results here.

When considering a set of attributes that can be observed on certain objects, people naturally consider some attributes more relevant than others, this leads to the following treatment.

Definition 1. *([5]) A formal context with hierarchically ordered attributes (an HA-context, for short) is a structure* $(U, V, R, \trianglelefteq)$ *(written also* $(U, (V, \trianglelefteq), R)$*), where* $(U, (V, \trianglelefteq), R)$ *is a formal context and* $\trianglelefteq$ *is a binary relation on* V*.*

Our primary interpretation of $\trianglelefteq$ is the following: $y_1 \trianglelefteq y_2$ means that y_2 is at least as important as y_1 in the sense discussed above. Further to say, $\trianglelefteq$ represents a restriction on how categories of objects can be formed–only categories which are compatible with $\trianglelefteq$ are considered relevant. This justifies the following definition analogously to the Definition 3 presented by [5].

Definition 2. *For an HA-context* $(U, (V, \trianglelefteq), R)$*, an object oriented concept* $(A, B) \in \mathcal{B}(U, V, R)$ *is called compatible with* $\trianglelefteq$ *if for each* $y_1, y_2 \in V, y_1 \in B$ *and* $y_1 \trianglelefteq y_2$ *implies* $y_2 \in B$*.*

The set of all object oriented concepts from $\mathcal{B}(U, V, R)$ which are compatible with $\trianglelefteq$ will be denoted by $\mathcal{B}(U, (V, \trianglelefteq), R)$, i.e.

$$\mathcal{B}(U, (V, \trianglelefteq), R) = \{(A, B) \in \mathcal{B}(U, V, R) \mid \text{for each } y_1, y_2 : y_1 \in B, y_1 \trianglelefteq y_2 \text{ implies } y_2 \in B\}.$$

First of all, we deal with the properties of $\mathcal{B}(U, (V, \trianglelefteq), R)$ in lattice theory, i.e, the results beyond Theorem 3 and below.

Lemma 1. *Let* $(X_j, Y_j) \in \mathcal{B}(U, (V, \trianglelefteq), R), (j \in \mathfrak{Y})$*. Then* $\bigwedge_{j \in \mathfrak{Y}} (X_j, Y_j) \in \mathcal{B}(U, (V, \trianglelefteq), R)$*.*

Proof. From Section 2, we show

$$\bigwedge_{j \in \mathfrak{Y}} (X_j, Y_j) = ((\bigcap_{j \in \mathfrak{Y}} X_j)^{\Box\Diamond}, \bigcap_{j \in \mathfrak{Y}} Y_j) \in \mathcal{B}(U, V, R).$$

Let $y_1, y_2 \in V, y_1 \trianglelefteq y_2$ and $y_1 \in \bigcap_{j \in \mathfrak{Y}} Y_j$. Since $y_1 \in \bigcap_{j \in \mathfrak{Y}} Y_j \subseteq Y_j, y_1 \trianglelefteq y_2$ and $(X_j, Y_j) \in \mathcal{B}(U, (V, \trianglelefteq), R)$ together follows $y_2 \in Y_j, (j \in \mathfrak{Y})$, we attain $y_2 \in \bigcap_{j \in \mathfrak{Y}} Y_j$. Thus, we obtain $\bigwedge_{j \in \mathfrak{Y}} (X_j, Y_j) \in \mathcal{B}(U, (V, \trianglelefteq), R)$.

Theorem 2. *An object oriented concept* $(X, Y) \in \mathcal{B}(U, V, R)$ *is compatible with* $\trianglelefteq$ *if and only if* (X, Y) *is an intersection of some elements in* $\mathcal{B}(U, (V, \trianglelefteq), R)$*, i.e. there is* $\mathfrak{Y} \subseteq \mathcal{B}(U, (V, \trianglelefteq), R)$ *such that* $(X, Y) = \wedge \mathfrak{Y}$*.*

Proof. (⇒) Let $(X_j, Y_j) \in \mathcal{B}(U,(V,\trianglelefteq),R)$ be all the elements satisfying $(X,Y) \leq (X_j, Y_j)$ in $\mathcal{B}(U,V,R), (j \in \mathfrak{A})$. By the definition of $\mathcal{B}(U,V,R)$, $(X,Y) \leq (X_j,Y_j)$ means $X \subseteq X_j$ and $Y \subseteq Y_j, (j \in \mathfrak{A})$. $(X,Y) \in \mathcal{B}(U,(V,\trianglelefteq),R)$ implies that there is $j_0 \in \mathfrak{A}$ satisfying $(X_{j_0}, Y_{j_0}) = (X,Y)$. Let $\mathfrak{Y} = \{(X_j,Y_j) : j \in \mathfrak{A}\}$. Considering Lemma 1, it is easy to obtain $(X,Y) = \wedge\mathfrak{Y}$.

(⇐) Since $(X,Y) = \wedge\mathfrak{Y}$ holds, where $\mathfrak{Y} = \{(X_j,Y_j) : j \in \mathfrak{A}\} \subseteq \mathcal{B}(U,(V,\trianglelefteq),R)$. By the definition of $\mathcal{B}(U,V,R)$ and $(X,Y) \in \mathcal{B}(U,V,R)$, we receive

$$(X,Y) = \wedge\mathfrak{Y} = \bigwedge_{j\in\mathfrak{A}} (X_j,Y_j).$$

By Lemma 1 and $(X_j,Y_j) \in \mathcal{B}(U,(V,\trianglelefteq),R)$, we obtain $\bigwedge_{j\in\mathfrak{A}} (X_j,Y_j) \in \mathcal{B}(U,(V,\trianglelefteq),R)$. Therefore, we know $(X,Y) \in \mathcal{B}(U,(V,\trianglelefteq),R)$.

The restriction of the subconcept-superconcept hierarchy $\leq$ which is defined on $\mathcal{B}(U,V,R)$ makes $\mathcal{B}(U,(V,\trianglelefteq),R)$ itself a partially ordered set $(\mathcal{B}(U,(V,\trianglelefteq),R),\leq)$. The following theorem shows that $\mathcal{B}(U,(V,\trianglelefteq),R)$ is itself a complete lattice which is a reasonable substructure of the whole object oriented concept lattice $\mathcal{B}(U,V,R)$.

Theorem 3. *$\mathcal{B}(U,(V,\trianglelefteq),R)$ equipped with $\leq$ is a complete lattice in which arbitrary infima coincide with infima in $\mathcal{B}(U,V,R)$, i.e. it is a complete $\wedge$-sublattice of $\mathcal{B}(U,V,R)$.*

Proof. $\mathcal{B}(U,(V,\trianglelefteq),R) \subseteq \mathcal{B}(U,V,R)$ is obvious by definitions.

Let $(X_1,Y_1),(X_2,Y_2) \in \mathcal{B}(U,(V,\trianglelefteq),R)$. Theorem 1 means

$$(X_1,Y_1) \wedge (X_2,Y_2) \in \mathcal{B}(U,(V,\trianglelefteq),R).$$

This hints the infima in $\mathcal{B}(U,(V,\trianglelefteq),R)$ coincide with infima in $\mathcal{B}(U,V,R)$.

Next we prove the suprema in $\mathcal{B}(U,(V,\trianglelefteq),R)$ is existed with the order $\leq$ in $\mathcal{B}(U,V,R)$.

By Section 2, $(U,^{\Box\Diamond})$ contains the universe U and is the family of the extension of all object oriented concepts. The definition of $^{\Box}$ leads to

$$U^{\Box} = \{y \in V \mid Ry \subseteq U\} = V.$$

Thus, the greatest element in $\mathcal{B}(U,V,R)$ is (U,V). It is evident $(U,V) \in \mathcal{B}(U,(V,\trianglelefteq),R)$.

Let $\mathfrak{I} = \{(A_i,B_i) \in \mathcal{B}(U,(V,\trianglelefteq),R) \mid (X_j,Y_j) \leq (A_i,B_i), (j = 1,2; i \in \mathcal{I})\}$. Evidently, we get $(U,V) \in \mathfrak{I}$, and so, $\mathfrak{I} \neq \emptyset$. Theorem 1 pledges $\bigwedge_{i\in\mathcal{I}} (A_i,B_i) \in \mathcal{B}(U,(V,\trianglelefteq),R)$. Therefore, $(X_1,Y_1),(X_2,Y_2) \leq \bigwedge_{i\in\mathcal{I}} (A_i,B_i)$ holds and $\bigwedge_{i\in\mathcal{I}} (A_i,B_i)$ is the suprema of (X_1,Y_1) and (X_2,Y_2) in $\mathcal{B}(U,(V,\trianglelefteq),R)$. This follows that $\mathcal{B}(U,(V,\trianglelefteq),R)$ is a complete lattice equipped with $\leq$.

Remark 1. *Let* $(A_1, B_1), (A_2, B_2) \in \mathcal{B}(U, (V, \trianglelefteq), R)$. *The suprema of* (A_1, B_1) *and* (A_2, B_2) *in* $\mathcal{B}(U, (V, \trianglelefteq), R)$ *is denoted by* $(A_1, B_1) \vee_{\trianglelefteq} (A_2, B_2)$. *Theorem 2 and* $\mathcal{B}(U, (V, \trianglelefteq), R) \subseteq \mathcal{B}(U, V, R)$ *together implies*

$$(A_1, B_1) \vee (A_2, B_2) \leq (A_1, B_1) \vee_{\trianglelefteq} (A_2, B_2).$$

We assert that it does not always have

$$(A_1, B_1) \vee (A_2, B_2) = (A_1, B_1) \vee_{\trianglelefteq} (A_2, B_2).$$

Since $(A_1, B_1) \vee (A_2, B_2) = (A_1 \cup A_2, (B_1 \cup B_2)^{\diamond\Box})$

$$= (A_1 \cup A_2, (B_1^{\diamond} \cup B_2^{\diamond})^{\Box})$$

$$= (A_1 \cup A_2, B_1 \cup B_2 \cup \{y \in V \mid Ry \subseteq B_1 \cup B_2 \text{ but } Ry \nsubseteq B_1, \text{ and } Ry \nsubseteq B_2\}).$$

Let $y_1, y_2 \in V, Ry_1 = B_1 \cup B_2$ *and* $y_1 \trianglelefteq y_2$, *where* $\trianglelefteq$ *is defined as* $a \trianglelefteq b \Leftrightarrow Ra \subset Rb$. *Under this case, it will not have* $y_2 \in (B_1^{\diamond} \cup B_2^{\diamond})^{\Box}$. *In other words, for some context and some hierarchy order on attributes, it could not pledge the following to be true:*

$$(A_1, B_1) \vee (A_2, B_2) = (A_1, B_1) \vee_{\trianglelefteq} (A_2, B_2)$$

Therefore, in general, $\mathcal{B}(U, (V, \trianglelefteq), R)$ *is not a* $\vee$*-sublattice of* $\mathcal{B}(U, V, R)$.

The next theorem shows a natural result saying that the more restrictions, the less object oriented concepts satisfying the restrictions.

Theorem 4. *If* $\trianglelefteq_1 \subseteq \trianglelefteq_2$, *then* $\mathcal{B}(U, (V, \trianglelefteq_2), R) \subseteq \mathcal{B}(U, (V, \trianglelefteq_1), R)$.

Proof. Let $(X, Y) \in \mathcal{B}(U, (V, \trianglelefteq_2), R)$, $y_1, y_2 \in V, y_1 \trianglelefteq_1 y_2$ and $y_1 \in Y$. In view of $\trianglelefteq_1 \subseteq \trianglelefteq_2$, we obtain $y_1 \trianglelefteq_2 y_2$. Owing to $y_1 \in Y$ and $(X, Y) \in \mathcal{B}(U, (V, \trianglelefteq_2), R)$, we earn $y_2 \in Y$. Therefore, $(X, Y) \in \mathcal{B}(U, (V, \trianglelefteq_1), R)$ holds.

Given a formal context (U, V, R) and binary relation $\trianglelefteq$ on V, a natural question arises for what binary relations Q on V, we have $\mathcal{B}(U, (V, \trianglelefteq), R) = \mathcal{B}(U, (V, Q), R)$, that is, what Q are restrictive to the same extent as $\trianglelefteq$. We will answer the question with respect to the operations of transitive closure. Namely, given a binary relation $\trianglelefteq$ on attributes specifying their relative importance, the transitive closure of $\trianglelefteq$ represents an intuitively sound extension of $\trianglelefteq$. For a binary relation R, the transitive closure will be denoted by R^+. By definition, R^+ is the least transitive closure relation containing R. We need the following lemma.

Lemma 5. *For an HA-context* $(U, (V, \trianglelefteq), R)$, *we have* $\mathcal{B}(U, (V, \trianglelefteq), R) = \mathcal{B}(U, (V, \trianglelefteq^+), R)$.

Proof. By definition, it follows $\trianglelefteq \subseteq \trianglelefteq^+$. Thus, Theorem 3 hints

$$\mathcal{B}(U, (V, \trianglelefteq^+), R) \subseteq \mathcal{B}(U, (V, \trianglelefteq), R).$$

Conversely, let $(X, Y) \in \mathcal{B}(U, (V, \trianglelefteq), R), y_1 \in Y$ and $y_1 \trianglelefteq^+ y_2$. As $y_1 \trianglelefteq^+ y_2$, from the well-known description of transitive closure, we believe that there are $z_1, \ldots, z_n \in Y$ such that

$y_1 \trianglelefteq z_1, z_1 \trianglelefteq z_2, \ldots, z_{n-1} \trianglelefteq z_n, z_n \trianglelefteq y_2.$
Taking into account that $(X,Y) \in \mathcal{B}(U,(V,\trianglelefteq),R), y_1 \in Y$ and $y_1 \trianglelefteq z_1$, we present $z_1 \in Y$. Taken $z_1 \in Y$ and $z_1 \trianglelefteq z_2$ together, we produce $z_2 \in Y$. Repeating this argument, we finally get $y_2 \in Y$. This shows $(X,Y) \in \mathcal{B}(U,(V,\trianglelefteq^+),R)$.

For a binary relation Q on V, we denote by Q^2 the composition $Q \circ Q$, i.e. $(y_1,y_2) \in Q^2$ if and only if there is some $y \in V$ such that $(y_1,y) \in Q$ and $(y,y_2) \in Q$. In addition, if x and y do not satisfy Q, it is denoted by $x\neg Qy$. Now we have the following theorem.

Theorem 6. *For an HA-context $(U,(V,\trianglelefteq),R)$, we have*
$$\mathcal{B}(U,(V,\trianglelefteq - \trianglelefteq^2),R) = \mathcal{B}(U,(V,\trianglelefteq),R) = \mathcal{B}(U,(V,\trianglelefteq^+),R).$$
Furthermore, for each binary relation Q on V satisfying $\trianglelefteq - \trianglelefteq^2 \subseteq Q \subseteq \trianglelefteq^+$, we have $\mathcal{B}(U,(V,Q),R) = \mathcal{B}(U,(V,\trianglelefteq),R)$.

Proof. First to prove $(\trianglelefteq - \trianglelefteq^2)^+ = \trianglelefteq^+$.
$(\trianglelefteq - \trianglelefteq^2)^+ \subseteq \trianglelefteq^+$ is evident. Next we demonstrate $\trianglelefteq^+ \subseteq (\trianglelefteq - \trianglelefteq^2)^+$.
For any $x_1, x_2 \in U, x_1 \trianglelefteq^+ x_2$ means that there exist $z_1, z_2, \ldots, z_n \in U$ satisfying
$$x_1 \trianglelefteq z_1, \quad z_1 \trianglelefteq z_2, \quad \ldots, \quad z_n \trianglelefteq x_2.$$
For $x_1 \trianglelefteq z_1$, we give the following analysis.
If $x_1 \trianglelefteq z_1$ and $x_1 \neg \trianglelefteq^2 z_1$.
It means the hold of $x_1(\trianglelefteq - \trianglelefteq^2)z_1$, and so $x_1(\trianglelefteq - \trianglelefteq^2)^+ z_1$.
If $x_1 \trianglelefteq z_1$ and $x_1 \trianglelefteq^2 z_1$.
Then, there is $z \in U$ satisfying $z \neq x_1, z \neq z_1, x_1 \trianglelefteq z$ and $z \trianglelefteq z_1$.
Suppose $x_1 \trianglelefteq z$ and $x_1 \neg \trianglelefteq^2 z$.
This hints the true of $x_1(\trianglelefteq - \trianglelefteq^2)z$.
Suppose $x_1 \trianglelefteq z$ and $x_1 \trianglelefteq^2 z$.
This follows $x_1 \trianglelefteq a$ and $a \trianglelefteq z$ for some $a \in U$. Repeating the above argument for $x_1 \trianglelefteq a$, considering with $|U| < \infty$ together, we may obtain a series $p_1, p_2, \ldots, p_l \in U$ satisfying
$$x_1 \trianglelefteq p_1, \quad x_1 \neg \trianglelefteq^2 p_1, \quad p_1 \trianglelefteq p_2, \quad p_1 \neg \trianglelefteq^2 p_2, \quad \ldots, \quad p_l \trianglelefteq a \text{ and } p_l \neg \trianglelefteq^2 a.$$
This hints
$$x_1(\trianglelefteq - \trianglelefteq^2)p_1, \quad p_1(\trianglelefteq - \trianglelefteq^2)p_2, \quad \ldots, \quad p_l(\trianglelefteq - \trianglelefteq^2)a.$$
Similarly discussion to $a \trianglelefteq z$, it will bring about $p_{l+1}, \ldots, p_t \in U$ satisfying
$$a(\trianglelefteq - \trianglelefteq^2)p_{l+1}, \quad p_{l+1}(\trianglelefteq - \trianglelefteq^2)p_{l+2}, \quad \ldots, \quad p_t(\trianglelefteq - \trianglelefteq^2)z.$$
In other words, if $x_1 \trianglelefteq z$ and $x_1 \trianglelefteq^2 z$, we will get $p_1, \ldots, p_t \in U$ satisfying
$$x_1(\trianglelefteq - \trianglelefteq^2)p_1, \quad p_1(\trianglelefteq - \trianglelefteq^2)p_2, \quad \ldots, \text{ and } p_t(\trianglelefteq - \trianglelefteq^2)z.$$
Analogously to $z_1, z_2, \ldots, z_n$, finally, we obtain the correct of the following formulas
$$x_1(\trianglelefteq - \trianglelefteq^2)y_1, \ldots, y_s(\trianglelefteq - \trianglelefteq^2)x_2$$
for some $y_1, y_2, \ldots, y_s \in U$. Simply to say, $x_1(\trianglelefteq - \trianglelefteq^2)^+ x_2$ is right.

Therefore, we refer to

$$\trianglelefteq^{+} \subseteq (\trianglelefteq - \trianglelefteq^{2})^{+}.$$

Afterwards, by Lemma 2, we believe

$$\mathcal{B}(U, (V, (\trianglelefteq - \trianglelefteq^{2})^{+})), R) = \mathcal{B}(U, (V, (\trianglelefteq - \trianglelefteq^{2})), R).$$

Summing up, we attain

$$\mathcal{B}(U, (V, (\trianglelefteq - \trianglelefteq^{2})), R) = \mathcal{B}(U, (V, (\trianglelefteq - \trianglelefteq^{2})^{+}), R) = \mathcal{B}(U, (V, \trianglelefteq^{+}), R).$$

For the second part, Theorem 3, Lemma 2 and the above taken together will pledge its true.

Theorem 4 shows natural bounds (in terms of transitive reduction and closure) on relation Q which are equally restrictive as $\trianglelefteq$.

4. Discussion

We assume that the reader is familiar with Hasse diagrams which will be used for visualization of object oriented concept lattices and attribute hierarchies. The following example shows the effect of a binary relation $\trianglelefteq$ on V.

Example 1. Consider the formal context from Table I and a relation $\trianglelefteq$ given by $y_1 \trianglelefteq y_2$, that is, for any $(A, B) \in \mathcal{B}(U, V, R)$, there is

$$y_1 \in B \Rightarrow y_2 \in B.$$

Then, the diagram of $\mathcal{B}(U, V, R)$ and $\mathcal{B}(U, (V, \trianglelefteq), R)$ is Figure 1 and Figure 2 respectively.

Table I A formal context

	y_1	y_2	y_3
x_1	1	0	0
x_2	0	1	0
x_3	0	0	1

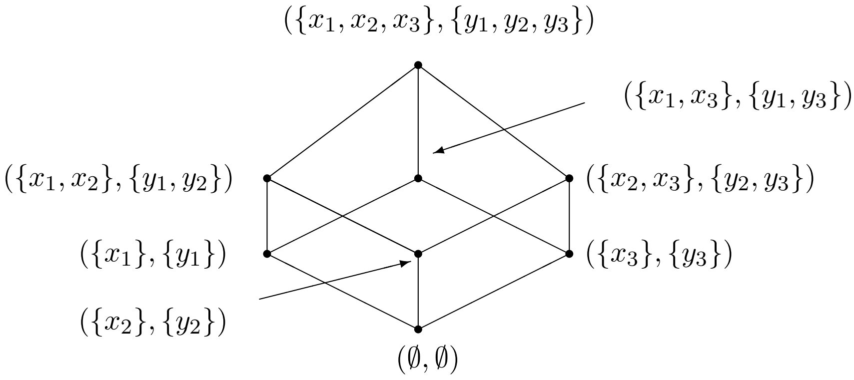

Figure 1 Object oriented concept lattice $\mathcal{B}(U, V, R)$ from Table I

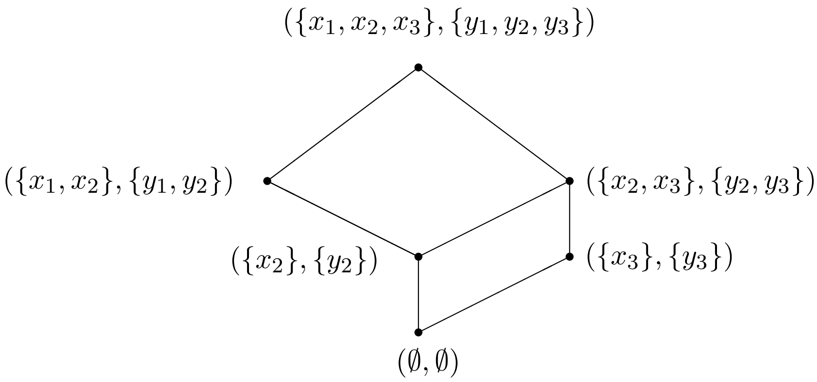

Figure 2 Constrained object oriented concept lattice $\mathcal{B}(U, (V, \unlhd), R)$ from Table I

Let (U, V, R) be a formal context and $(U, (V, \unlhd), R)$ be an HA-context relative to (U, V, R). Let $\mathfrak{B}(U, V, R), \mathfrak{B}(U, (V, \unlhd), R)$ be the correspondent formal concept lattice and the family of all constrained formal concepts respectively. Let $\mathcal{B}(U, V, R), \mathcal{B}(U, (V, \unlhd), R)$ be the correspondent object oriented concept lattice and the family of all constrained object oriented concepts respectively. For two lattices L_1 and L_2, $L_1 \cong L_2$ and $L_1 \ncong L_2$ stands for the isomorphism of L_1 and L_2 and the non-isomorphism of L_1 and L_2 respectively.

Comparing the properties of $\mathfrak{B}(U, (V, \unlhd), R)$ presented by [5] with the results for $\mathcal{B}(U, (V, \unlhd), R)$ in this paper, we could read out some differences and similarities between them as follows.

(1) Both of $\mathfrak{B}(U, (V, \unlhd), R)$ and $\mathcal{B}(U, (V, \unlhd), R)$ are complete lattices.

(2) For what binary relations Q on V, we have $\mathfrak{B}(U, (V, \unlhd), R) = \mathfrak{B}(U, (V, Q), R)$, i.e. what Q are restrictive to the same extent as $\unlhd$. The Theorem 8 provided by [5]

answers the question with respect to the operation of transitive closure for $\trianglelefteq^+$.

For what binary relations Q on V, we have $\mathcal{B}(U,(V,\trianglelefteq),R) = \mathcal{B}(U,(V,Q),R)$, i.e. what Q are restrictive to the same extent as $\trianglelefteq$. Theorem 4 answers the question with respect to the operation of transitive closure for $\trianglelefteq^+$.

We find that the Theorem 8 provided by [5] is similar to Theorem 4.

(3) The Example 5 and Example 8 provided by [10] show $\mathfrak{B}(U,V,R) \ncong \mathcal{B}(U,V,R)$ generally. From [5] and this paper, we may infer to $\mathfrak{B}(U,(V,id_V),R) = \mathfrak{B}(U,V,R)$ and $\mathcal{B}(U,(V,id_V),R) = \mathcal{B}(U,V,R)$ respectively. Hence, it follows that for $\trianglelefteq$ on attributes, it does not always pledge $\mathfrak{B}(U,(V,\trianglelefteq),R) \cong \mathcal{B}(U,(V,\trianglelefteq),R)$.

(4) The Theorem 10 provided by [5] states that $\mathfrak{B}(U,(V,\trianglelefteq),R)$ is a $\vee$-sublattice of $\mathfrak{B}(U,V,R)$, but not a $\wedge$-sublattice of $\mathfrak{B}(U,V,R)$. Theorem 2 and Remark 1 together indicates that $\mathcal{B}(U,(V,\trianglelefteq),R)$ is a $\wedge$-sublattice of $\mathcal{B}(U,V,R)$, but not a $\vee$-sublattice of $\mathcal{B}(U,V,R)$.

The authors point [5] that since the present state of art of formal concept analysis does not offer a way to cope with attribute hierarchies, the reduction techniques have to be applied even if the user is able to formulate a natural attribute hierarchy. In fact, the research for object oriented concept lattice is the same as the above description for formal concept lattices. From this view, the present approach offers a direct way to reduce the number of object oriented concepts by keeping only object oriented concepts compatible with an attribute hierarchy. To discussing object oriented concepts deeply, as the future research for formal concept lattices provided by [5], our future research for object oriented concept lattices constrained by hierarchically ordered attributes will be directed to the investigation of other natural forms of an additional information accompanying (U,V,R) and the corresponding constraining rules. The next step is also to consider constraining rules by which an object oriented concept (A,B) satisfies a constraint if it is true that whenever the attribute y belongs to B, then at least one of the attributes $y_1,\ldots,y_n$ belong to B. Clearly, for $n = 1$, this becomes just the restrictions considered in this paper.

Acknowledgements. This research is granted by NSF of China (61073121, 11101115) and NSF of Hebei Province (A2013201119, F2012402037, A2014201033).

References

[1] I. Düntsch, G. Gediga, *Modal-style operators in qualitative data analysis*, In: Proceedings of the 2002 IEEE International Conference on Data Mining, Maebashi, (2002), 155-162.

[2] M. Wolski, *Galois connections and data analysis*, Fund. Infor. 60, (1-4)(2003), 401-415.

[3] Y. Yao, *Concept lattices in rough set theory*, In: Proceedings of Annual Meeting of the North American Fuzzy Information Processing Society, Banff, (2004), 796-801.

[4] B. Ganter, R. Wille, *Formal Concept Analysis: Mathematical Foundations*, Springer-Verlag, New York, (1999).

[5] R. Bělohlávek, V. Sklenář, J. Zacpal, *Formal concept analysis with hierarchically ordered attributes*, Inter. J. of Gene. Syst. 33, 4(2004), 383-394.

[6] R. Bělohlávek, V. Sklenář, J. Zacpal, *Concept lattices constrained by equivalence relations*, In: Proceedings of the International Workshop on Concept Lattices and Their Applications, Ostrava, (2004), 58-66.

[7] R. Bělohlávek, V. Sklenář, J. Zacpal, *Concept lattices constrained by systems of partitions*, In: Proceeding Znalosti 2005, 4th Annual Congress, Stara Lesna, (2005), 5-8.

[8] G. Grätzer, *General Lattice Theory, 2nd. ed.*, Birkhäuser Verlag, Basel, (1998).

[9] A. Davey, A. Priestley, *Introduction to Lattices and Order, 2nd. ed.*, Cambridge University Press, Cambridge, (2003).

[10] Y. Chen, Y. Yao, *A multiview approach for intelligent data analysis based on data operators*, Infor. Sci. 178(2008), 1-20.

Hua Mao
Department of Mathematics,
Hebei University,
Baoding 071002,
China
email: *yushengmao@263.net*