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A CHARACTERIZATION OF CONSTANT RATIO CURVES IN EUCLIDEAN 3-SPACE $\mathbb{E}^3$

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Abstract. A twisted curve in Euclidean 3-space $\mathbb{E}^3$ can be considered as a curve whose position vector can be written as linear combination of its Frenet vectors. In the present study we study the twisted curves of constant ratio in $\mathbb{E}^3$ and characterize such curves in terms of their curvature functions. Further, we obtain some results of T-constant and N-constant type twisted curves in $\mathbb{E}^3$. Finally, we give some examples of equiangular spirals which are constant ratio curves.

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1. Introduction

A curve $x : I \subset \mathbb{R} \to \mathbb{E}^3$ in Euclidean 3-space is called a *twisted curve* if it has nonzero Frenet curvatures $\kappa_1(s)$ and $\kappa_2(s)$. From the elementary differential geometry it is well known that a curve $x(s)$ in $\mathbb{E}^3$ lies on a sphere if its position vector (denoted also by x) lies on its normal plane at each point. If the position vector x lies on its rectifying plane then $x(s)$ is called *rectifying curve* [3]. Rectifying curves characterized by the simple equation

$$x(s) = \lambda(s)T(s) + \mu(s)N_2(s), \tag{1}$$

where $\lambda(s)$ and $\mu(s)$ are smooth functions and $T(s)$ and $N_2(s)$ are tangent and binormal vector fields of x respectively [3]. In the same paper B. Y. Chen gave a simple characterization of rectifying curves. In particular it is shown in [6] that there exists a simple relation between rectifying curves and centrodes, which play an important roles in mechanics kinematics as well as in differential geometry in defining the curves of constant procession. It is also provide that a twisted curve is congruent to a non constant linear function of s [4]. Further, in the Minkowski 3-space $\mathbb{E}^3_1$, the rectifying curves are investigated in ([7, 12, 13, 14]). In [14] a characterization of the spacelike, the timelike and the null rectifying curves in the Minkowski 3-space in terms of centrodes is given. For the characterization of rectifying curves in three dimensional compact Lie groups or in dual spaces see [19] or [1] respectively.

For a regular curve $x(s)$, the position vector x can be decompose into its tangential and normal components at each point:

$$x = x^T + x^N. \tag{2}$$

A curve $x(s)$ with $\kappa_1(s) > 0$ is said to be of *constant ratio* if the ratio $\left\|x^T\right\| : \left\|x^N\right\|$ is constant on $x(I)$ where $\left\|x^T\right\|$ and $\left\|x^N\right\|$ denote the length of x^T and x^N, respectively [2]. Clearly a curve x in $\mathbb{E}^3$ is of constant ratio if and only if $x^T = 0$ or $\left\|x^T\right\| : \|x\|$ is constant [3]. The distance function $\rho = \|x\|$ satisfies $\|grad\rho\| = c$ for some constant c if and only if we have $\left\|x^T\right\| = c\,\|x\|$. In particular, if $\|grad\rho\| = c$ then $c \in [0, 1]$.

A curve in $\mathbb{E}^n$ is called T-*constant* (resp. N-*constant*) if the tangential component x^T (resp. the normal component x^N) of its position vector x is of constant length [2]. It is known that a twisted curve in $\mathbb{E}^3$ is congruent to a N-constant curve if and only if the ratio $\frac{\kappa_2}{\kappa_1}$ is a non-constant linear function of an arc-length function s, i.e., $\frac{\kappa_2}{\kappa_1}(s) = c_1 s + c_2$ for some constants c_1 and c_2 with $c_1 \neq 0$ [2].

In the present study, we give a generalization of the rectifying curves in Euclidean 3-space $\mathbb{E}^3$. We consider a twisted curve in Euclidean 3-space $\mathbb{E}^3$ whose position vector satisfies the parametric equation

$$x(s) = m_0(s)T(s) + m_1(s)N_1(s) + m_2(s)N_2(s), \tag{3}$$

for some differentiable functions, $m_i(s)$, $0 \leq i \leq 2$. If $m_1(s) = 0$ then $x(s)$ becomes a rectifying curve. We characterize the twisted curves in terms of their curvature functions $m_i(s)$ and give the necessary and sufficient conditions for the twisted curves to become T -constant or N-constant. We give necessary and sufficient conditions for twisted curves in $\mathbb{E}^3$ to become W-curves. We also show that every N-constant twisted curve with nonzero constant $\left\|x^N\right\|$ is a rectifying curve of $\mathbb{E}^3$. Finally, we give some examples of equiangular spirals which are constant ratio curves. We give a characterization of a T-constant curve of second kind in $\mathbb{E}^3$ to become a conchospiral.

2. Basic Notations

Let $x : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed curve in Euclidean 3-space $\mathbb{E}^3$. Let us denote $T(s) = x'(s)$ and call $T(s)$ as a unit tangent vector of x at s. We denote the curvature of x by $\kappa_1(s) = \|x''(s)\|$. If $\kappa_1(s) \neq 0$, then the unit principal normal vector $N_1(s)$ of the curve x at s is given by $x^{''}(s) = \kappa_1(s)N_1(s)$. The unit vector $N_2(s) = T(s) \times N_1(s)$ is called the unit binormal vector of x at s. Then we have the Serret-Frenet formulae:

$$\begin{aligned} T'(s) &= \kappa_1(s)N_1(s), \\ N_1'(s) &= -\kappa_1(s)T(s) + \kappa_2(s)N_2(s), \\ N_2'(s) &= -\kappa_2(s)N_1(s), \end{aligned} \tag{4}$$

where $\kappa_2(s)$ is the torsion of the curve x at s (see, [9] and [17]).

If the Frenet curvature $\kappa_1(s)$ and torsion $\kappa_2(s)$ of x are constant functions then x is called a screw line or a helix [8]. Since these curves are the traces of 1-parameter family of the groups of Euclidean transformations then F. Klein and S. Lie called them *W-curves* [15]. It is known that a twisted curve x in $\mathbb{E}^3$ is called a *general helix* if the ratio $\kappa_2(s)/\kappa_1(s)$ is a nonzero constant on the given curve [16].

For a space curve $x : I \subset \mathbb{R} \to \mathbb{E}^3$, the planes at each point of $x(s)$ the spanned by $\{T, N_1\}$, $\{T, N_2\}$ and $\{N_1, N_2\}$ are known as the *osculating plane*, the *rectifying plane* and *normal plane* respectively. If the position vector x lies on its rectifying plane then $x(s)$ is called *rectifying curve*. Similarly, the curve for which the position vector x always lies in its osculating plane is called *osculating curve*. Finally, x is called *normal curve* if its position vector x lies in its normal plane.

From elementary differential geometry it is well known that a curve in $\mathbb{E}^3$ lies in a plane if its position vector lies in its osculating plane at each point, and lies on a sphere if its position vector lies in its normal plane at each point [3].

3. Constant Ratio Curves in $\mathbb{E}^3$

In the present section we characterize the twisted curves in $\mathbb{E}^3$ in terms of their curvatures. Let $x : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed twisted curve with curvatures $\kappa_1(s) > 0$ and $\kappa_2(s)$. By definition of the position vector of the curve (also defined by x) satisfies the vectorial equation (3), for some differential functions $m_i(s)$, $0 \leq i \leq 2$. By taking the derivative of (3) with respect to arclength parameter s and using the Serret-Frenet equations (4), we obtain

$$\begin{aligned} x'(s) = {} & (m_0'(s) - \kappa_1(s)m_1(s))T(s) \\ & +(m_1'(s) + \kappa_1(s)m_0(s) - \kappa_2(s)m_2(s))N_1(s) \\ & +(m_2'(s) + \kappa_2(s)m_1(s))N_2(s). \end{aligned} \tag{5}$$

It follows that

$$\begin{aligned} m_0' - \kappa_1 m_1 &= 1, \\ m_1' + \kappa_1 m_0 - \kappa_2 m_2 &= 0, \\ m_2' + \kappa_2 m_1 &= 0. \end{aligned} \tag{6}$$

The following result explicitly determines all twisted W-curves in $\mathbb{E}^3$.

Proposition 1. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^3$ be a twisted curve with $\kappa_1 > 0$ and let s be its arclength function. If x is a W-curve of $\mathbb{E}^3$ then the position vector x is given by the curvature functions*

$$\begin{aligned} m_0(s) &= -\kappa_1 \left(\frac{c_3 \sin as - c_2 \cos as}{a} - bs \right) + s + c_0, \\ m_1(s) &= c_2 \sin as + c_3 \cos as - b, \\ m_2(s) &= \kappa_2 \left(\frac{c_3 \sin as - c_2 \cos as}{a} - bs \right) + c_1, \end{aligned} \tag{7}$$

where c_i, $(0 \leq i \leq 3)$ are integral constants and $a = \sqrt{\kappa_1^2 + \kappa_2^2}$, $b = \frac{\kappa_1}{a^2}$ are real constants.

Proof. Let x be a twisted W-curve in $\mathbb{E}^3$, then by the use of the equations (6) we get

$$\begin{aligned} m_0' &= \kappa_1 m_1 + 1, \\ m_1'' &= -(\kappa_1^2 + \kappa_2^2) m_1 - \kappa_1, \\ m_2' &= -\kappa_2 m_1. \end{aligned} \tag{8}$$

Further, one can show that the system of equations (8) has a non-trivial solution (7). Thus, the proposition is proved.

Definition 1. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^n$ be a unit speed regular curve in $\mathbb{E}^n$. Then the position vector x can be decompose into its tangential and normal components at each point:*

$$x = x^T + x^N.$$

if the ratio $\left\| x^T \right\| : \left\| x^N \right\|$ is constant on $x(I)$ then x is said to be of constant-ratio, or equivalently $\left\| x^T \right\| : \|x\| = c =$constant [2].

For a unit speed regular curve x in $\mathbb{E}^n$, the gradient of the distance function $\rho = \|x(s)\|$ is given by

$$grad\rho = \frac{d\rho}{ds} x'(s) = \frac{< x(s), x'(s) >}{\|x(s)\|} T \tag{9}$$

where T is the tangent vector field of x.

The following results characterize constant-ratio curves.

Theorem 1. *[5] Let $x : I \subset \mathbb{R} \to \mathbb{E}^n$ be a unit speed regular curve in $\mathbb{E}^n$. Then x is of constant-ratio with $\left\|x^T\right\| : \|x\| = c$ if and only if $\|grad\rho\| = c$ which is constant.*

In particular, for a curve of constant-ratio we have $\|grad\rho\| = c \leq 1$.

Example 1. *For any real numbers a,c with $0 \leq a \leq c < 1$, the curve*

$$x(s) = \left(\sqrt{c^2 - a^2} s \sin\left(\frac{\sqrt{1-c^2}}{\sqrt{c^2-a^2}} \ln s\right), \sqrt{c^2 - a^2} s \cos\left(\frac{\sqrt{1-c^2}}{\sqrt{c^2-a^2}} \ln s\right), as\right)$$

in $\mathbb{E}^3$ is a unit speed curve satisfying $\|grad\rho\| = c$ (see, [5]).

Theorem 2. *[5] Let $x : I \subset \mathbb{R} \to \mathbb{E}^n$ be a unit speed regular curve in $\mathbb{E}^n$. Then $\|grad\rho\| = c$ holds for a constant c if and only if one of the following three cases occurs:*

(i) $x(I)$ is contained in a hypersphere centered at the origin.

(ii) $x(I)$ is an open portion of a line through the origin.

(iii) $x(s) = csy(s)$, $c \in (0, 1)$, where $y = y(u)$ is a unit curve on the unit sphere of $\mathbb{E}^n$ centered at the origin and $u = \frac{\sqrt{1-c^2}}{c} \ln s$.

As a consequence of Theorem 2, one can get the following result.

Corollary 3. *[5] Let $x : I \subset \mathbb{R} \to \mathbb{E}^n$ be a unit speed regular curve in $\mathbb{E}^n$. Then up to a translation of the arc length function s, we have*

i) $\|grad\rho\| = 0 \Longleftrightarrow x(I)$ is contained in a hypersphere centered at the origin.

ii) $\|grad\rho\| = 1 \Longleftrightarrow x(I)$ is an open portion of a line through the origin.

iii) $\|grad\rho\| = c \Longleftrightarrow \rho = \|x(s)\| = cs$, for $c \in (0, 1)$.

iv) If $n = 2$ and $\|grad\rho\| = c$ for $c \in (0, 1)$, then the curvature of x satisfies

$$\kappa^2 = \frac{1 - c^2}{c^2(s^2 + b)},$$

for some real constant b.

For twisted curves in $\mathbb{E}^3$ we obtain the following results.

Proposition 2. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed twisted curve in $\mathbb{E}^3$. If x is of constant-ratio then the position vector of the curve has the parametrization of the form*

$$x(s) = \left(c^2 s + cb\right) T(s) + \left(\frac{c^2 - 1}{\kappa_1}\right) N_1(s) + \left(\frac{\kappa_1 c\left(c^2 + b\right)}{\kappa_2} - \frac{\left(c^2 - 1\right)\kappa_1'}{\kappa_2 \kappa_1^2}\right) N_2(s),$$

for some differentiable functions, $b \in \mathbb{R}, c \in [0, 1)$.

Proof. Let x be a regular curve of constant-ratio. Then, from the previous result the distance function ρ of x satisfies the equality $\rho = \|x(s)\| = cs + b$ for some differentiable functions, $b, c \in [0, 1)$. Further, using (9) we get

$$\|grad\rho\| = \frac{< x(s), x'(s) >}{\|x(s)\|} = c.$$

Since, x is a twisted curve of $\mathbb{E}^3$, then it satisfies the equality (3). So, we get $m_0 = c^2 s + cb$. Hence, substituting this value into the equations in (6) one can get

$$\begin{aligned} m_1(s) &= \tfrac{c^2-1}{\kappa_1}, \\ m_2(s) &= \tfrac{\kappa_1\left(c^2 s + cb\right)}{\kappa_2} - \tfrac{\left(c^2-1\right)\kappa_1'}{\kappa_2\kappa_1^2}. \end{aligned}$$

Substituting these values into (3), we obtain the desired result.

3.1. T-constant Twisted Curves in $\mathbb{E}^3$

Definition 2. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^n$ be a unit speed curve in $\mathbb{E}^n$. If $\left\|x^T\right\|$ is constant then x is called a T-constant curve. For a T-constant curve x, either $\left\|x^T\right\| = 0$ or $\left\|x^T\right\| = \lambda$ for some non-zero smooth function λ [3]. Further, a T-constant curve x is called first kind if $\left\|x^T\right\| = 0$, otherwise second kind.*

As a consequence of (6), we get the following result.

Theorem 4. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed twisted curve in $\mathbb{E}^3$ with the curvatures $\kappa_1 > 0$ and $\kappa_2 \neq 0$. Then x is a T-constant curve of first kind, if and only if*

$$\frac{\kappa_2}{\kappa_1} - \left(\frac{\kappa_1'}{\kappa_1^2\kappa_2}\right)' = 0. \tag{10}$$

Proof. Let x be a T-constant twisted curve of first kind. Then, from the first and third equalities in (6) we get $m_2 = \frac{m_1'}{\kappa_2}$ and $m_2' + m_1\kappa_2 = 0$. Further, substituting the differentiation of the first equation and $m_1 = -\frac{1}{\kappa_1}$ into the first equation we get the result.

Remark 1. *Any twisted curve satisfying the equality (10) is a spherical curve lying on a sphere $S^2(r)$ of $\mathbb{E}^3$. So every T-constant twisted curves of first kind are spherical (see, [18]).*

By the use of (6) with (10) one can construct the following examples.

Example 2. *The twisted curve given with the parametrization*

$$x(s) = -\cos\left(\int \kappa_2 ds\right) N_1(s) + \sin\left(\int \kappa_2 ds\right) N_2(s), \tag{11}$$

is a T-constant twisted curve of first kind .

Example 3. *The twisted curve given with the curvatures $\kappa_1 = s$ and $\kappa_2 = \frac{1}{(\ln s+a)s^2}$ is a T-constant twisted curve of first kind.*

As a consequence of (6), we get the following result.

Theorem 5. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^3$ be a twisted curve in $\mathbb{E}^3$. Then x is a T-constant curve of second kind if and only if*

$$\left(\frac{\kappa_1' + m_0\kappa_1^3}{\kappa_1^2\kappa_2}\right)' - \frac{\kappa_2}{\kappa_1} = 0, \tag{12}$$

holds, for some constant function m_0.

Proof. Suppose that x is a T-constant curve of second kind. Then, by the use of (6) we get

$$0 = m_2' + m_1\kappa_2,\ m_2 = \frac{m_1' + \kappa_1 m_0}{\kappa_2}. \tag{13}$$

Further, substituting the differentiation of second equation and using $m_1 = -\frac{1}{\kappa_1}$ with first equation, we get the result.

Corollary 6. *Let $x \in \mathbb{E}^3$ be a twisted curve in $\mathbb{E}^3$. If x is a T-constant of second kind with non-zero constant first curvature κ_1 then*

$$\kappa_2(s) = \mp\frac{\sqrt{a}}{\sqrt{2s + c_1 a}}, \tag{14}$$

holds, for some constant functions c_1 and $a = \kappa_1^2 m_0$.

Proof. Suppose, first curvature κ_1 is a constant function then by the use of (12), we get,

$$\left(\frac{1}{\kappa_2}\right)' m_0\kappa_1 - \frac{\kappa_2}{\kappa_1} = 0,$$

which has a non-trivial solution (14).

For T-constant curves of second kind we give the following results;

Theorem 7. *Let $x \in \mathbb{E}^3$ be a T-constant twisted curve of second kind. Then the distance function $\rho = \|x\|$ satisfies*

$$\rho = \pm\sqrt{c_1 s + c_2}. \tag{15}$$

for some constants $c_1 = m_0$ and c_2.

Proof. Let $x \in \mathbb{E}^3$ be a T-constant twisted curve of second kind then by definition the curvature function $m_0(s)$ of x is constant. So differentiating the squared distance function $\rho^2 = \langle x(s), x(s)\rangle$ and using (12) we get $\rho\rho' = m_0$. It is an easy calculation to show that, this differential equation has a nontrivial solution (15).

Theorem 8. *Let $x \in \mathbb{E}^3$ be a T-constant twisted curve of second kind. Then x is a general helix of $\mathbb{E}^3$ if and only if*

$$\kappa_1(s) = \mp\frac{1}{\sqrt{\lambda s^2 + 2c_2 s + c_1}}, \tag{16}$$

holds, where $\lambda = \frac{\kappa_2}{\kappa_1}$ is a non-zero constant and $c_2 = m_0 - \lambda c_1$, $c_1 \in \mathbb{R}$.

Proof. Assume that x is a T-constant twisted curve of second kind. Then by the use of (12), we obtain

$$\kappa_1' + \kappa_1^3\left(m_0 - \lambda^2 s - \lambda c_1\right) = 0. \tag{17}$$

Consequently, this differential equation has a non-trivial solution(16), where c_1, $c_2 = m_0 - \lambda c_1$ are integral constants and $\lambda = \frac{\kappa_2}{\kappa_1}$ a non-zero constant. This completes proof of the corollary.

3.2. N-constant Twisted Curves in $\mathbb{E}^3$

Definition 3. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed curve in $\mathbb{E}^3$. If $\left\|x^N\right\|$ is constant then x is called a N-constant curve.For a N-constant curve x, either $\left\|x^N\right\| = 0$ or $\left\|x^N\right\| = \mu$ for some non-zero smooth function μ [3]. Further, a N-constant curve x is called first kind if $\left\|x^N\right\| = 0$, otherwise second kind.*

So, for a N-constant twisted curve x

$$\left\|x^N(s)\right\|^2 = m_1^2(s) + m_2^2(s) \tag{18}$$

becomes a constant function.

As a consequence of (3) and (6) with (18) we get the following result.

Lemma 9. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed curve in $\mathbb{E}^3$. Then x is a N-constant twisted curve if and only if*

$$\begin{aligned} m_0' &= 1 + \kappa_1 m_1, \\ m_1' &= \kappa_2 m_2 - \kappa_1 m_0, \\ m_2' &= -\kappa_2 m_1, \\ 0 &= m_1 m_1' + m_2 m_2', \end{aligned} \tag{19}$$

hold, where $m_0(s), m_1(s)$ and $m_2(s)$ are differentiable functions.

For the N-constant twisted curves of first kind we give the following result.

Proposition 3. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed curve in $\mathbb{E}^3$. Then x is a N-constant twisted curve of first kind if and only if $x(I)$ is an open portion of a line through the origin.*

Proof. Suppose that x is N-constant twisted curve of $\mathbb{E}^3$, then the equality (18) holds. Further, if x is of first kind then from (18) $m_1 = m_2 = 0$ which implies that $\kappa_1 = \kappa_2 = 0$. So x becomes a part of a straight line.

Definition 4. *A space curve $x : I \subset \mathbb{R} \to \mathbb{E}^3$ whose position vector always lies in its rectifying plane is called a rectifying curve. So, for a rectifying curve $x : I \subset \mathbb{R} \to \mathbb{E}^3$, the position vector $x(s)$ satisfies the simple equation*

$$x(s) = \lambda(s)T(s) + \mu(s)N_2(s),$$

for some differentiable functions $\lambda(s)$ and $\mu(s)$ [3].

The following result of B.Y. Chen provides some simple characterizations of rectifying curves.

Theorem 10. *[3] Let $x : I \subset \mathbb{R} \to \mathbb{E}^3$ be a rectifying curve in $\mathbb{E}^3$ with $\kappa_1 > 0$ and let s be its arclength function. Then:*

i) The distance function $\rho = \|x\|$ satisfies $\rho^2 = s^2 + c_l s + c_2$ for some constants c_1 and c_2.

ii) The tangential component of the position vector of the curve is given by $\langle x, T\rangle = s + b$ for some constant b.

iii) The normal component x^N of the position vector of the curve has constant length and the distance function ρ is nonconstant.

iv) The torsion κ_2 is nonzero, and the binormal component of the position vector is constant, i.e., $\langle x, N_2\rangle$ is constant.

Conversely, if $x : I \subset \mathbb{R} \to \mathbb{E}^3$ is a curve with $\kappa_1 > 0$ and if one of $(i), (ii), (iii)$, or (iv) holds, then x is a rectifying curve.

The following result of B.Y. Chen provides some simple characterizations of rectifying curves in terms of the ratio $\frac{\kappa_2}{\kappa_1}$.

Theorem 11. *[3] Let $x\in \mathbb{E}^3$ be a curve in $\mathbb{E}^3$ with $\kappa_1 > 0$. Then x is congruent to a rectifying curve if and only if the ratio of the curvatures of the curve is a nonconstant linear function in arclength functions, i.e., $\frac{\kappa_2}{\kappa_1}(s) = c_1 s + c_2$ for some constants c_1 and c_2, with $c_1 \neq 0$.*

By the use of Lemma 9, we obtain the following result.

Theorem 12. *Let $x(s) \in \mathbb{E}^3$ be a twisted curve in $\mathbb{E}^3$ with $\kappa_1 > 0$ and s be its arclength function. If x is a N-constant curve of second kind, then the position vector x of the curve has the parametrization*

$$x(s) = (s+\lambda)\, T(s) + \mu N_2(s), \quad \lambda, \mu \in \mathbb{R}. \tag{20}$$

Proof. Let x be a N-constant twisted curve of second kind then the equation (19) holds. So we get $m_1 \left(m_1' - \kappa_2 m_2 \right) = 0$. Hence, there are two possible cases; $m_1' - \kappa_2 m_2 = 0$ or $m_1 = 0$. For the first case one can get $\kappa_1 = 0$, $\kappa_2 = 0$ which implies that x is N-constant curve of first kind. Hence, one can get $m_1 = 0$ and $m_2 = 0$, which contradicts the fact that $\left\| x^N(s) \right\| = \sqrt{m_1^2(s) + m_2^2(s)}$ is nonzero constant. So this case does not occur. For the second case we get $m_0 = s+\lambda$, $m_1 = 0$ and $m_2 = \mu$ for some constant functions λ and μ. This completes the proof of the theorem.

Corollary 13. *Let $x \in \mathbb{E}^3$ be a N-constant twisted curve of second kind in $\mathbb{E}^3$ with $\kappa_1 > 0$. Then the ratio of the curvatures of the curve is a nonconstant linear function of arclength functions.*

Proof. Let x be a N-constant twisted curve of then the equation $m_1' - \kappa_2 m_2 = 0$ holds. So, substituting the values $m_0 = s+\lambda$, $m_1 = 0$ and $m_2 = \mu$ into the previous one, we get $\frac{\kappa_2}{\kappa_1}(s) = \frac{s+\lambda}{\mu}$ for some real constants λ and μ.

4. The Equiangular Spirals

Definition 5. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^n$ be a regular curve in $\mathbb{E}^n$. If the angle between the position vector field and the tangent vector field of the curve x is constant (i.e., the angle between x and T is constant) then it is called equiangular ([11]).*

The equiangular curves in $\mathbb{E}^n$ are characterized by the following result.

Proposition 4. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^n$ be an equiangular regular curve in $\mathbb{E}^n$, given with arclength parameter. Then x is of constant-angle curve in $\mathbb{E}^n$.*

Proof. Let $x(I) \subset \mathbb{E}^n$ be an equiangular curve in $\mathbb{E}^n$. Then, by definition

$$\cos\alpha = \frac{< x(s), T(s) >}{\rho} = \|grad\rho\|,$$

is a constant function. So, x becomes a constant-angle curve of $\mathbb{E}^n$.

Remark 2. *In the plane $\mathbb{E}^2$ the equiangular spirals have constant angle $\alpha \in \left[0, \frac{\pi}{2}\right]$ between x and T. It is a well-known result that the equiangular spirals of $\mathbb{E}^2$ are characterized by the property that their radius of curvature $R = 1/\kappa$ is a first degree function of their arclength $s : R = as + b$ for some real constants a and b, (including the straight lines and the circles as the particular cases of 0-spirals and $(\pi/2)$-spirals, respectively) (see, [11]).*

For the twisted equiangular spirals we get the following result.

Proposition 5. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^2$ be a unit speed curve in $\mathbb{E}^2$. If x is an equiangular spiral then the position vector of x has the parametrization*

$$\begin{aligned} m_0(s) &= c_1 \cos\varphi(s) + c_2 \sin\varphi(s) + \tfrac{a(as+b)}{a^2+1}, \\ m_1(s) &= c_1 \sin\varphi(s) - c_2 \cos\varphi(s) + \tfrac{as+b}{a^2+1}. \end{aligned} \tag{21}$$

where $\varphi(s) = \frac{1}{a}\ln\left(s + \frac{b}{a}\right)$ is a differentiable function.

Proof. Let $x : I \subset \mathbb{R} \to \mathbb{E}^2$ be a unit speed curve in $\mathbb{E}^2$. Then from (6), $m_0' - \kappa m_1 = 1$ and $m_1' + \kappa m_0 = 0$, hold. Further, assume that x is an equiangular spiral in $\mathbb{E}^2$. Then, substituting $\kappa = \frac{1}{as+b}$ into the equations above we obtain a system of differential equations which has a non-trivial solution (21).

Definition 6. *A concho-spiral in $\mathbb{E}^3$ is characterized by the property that its first and second radii of curvature, $R_1 = 1/\kappa_1, R_2 = 1/\kappa_2$ (i.e., the inverse of its first and second Serret-Frenet curvatures κ_1 and κ_2) are both first degree functions of their arclenghts (see, [10]):*

$$\begin{aligned} R_1 &= 1/\kappa_1 = a_1 s + b_1, \\ R_2 &= 1/\kappa_2 = a_1 s + b_1. \end{aligned} \tag{22}$$

For the twisted equiangular spirals we get the following results.

Proposition 6. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed T-constant curve of second kind in $\mathbb{E}^3$. If x is a concho-spiral in $\mathbb{E}^3$, then the position vector (3) of x has the* *paramerization*

$$
\begin{aligned}
m_1(s) &= -(as+b), \\
m_2(s) &= \left(\tfrac{as+b\ln(as+b)}{c} - \tfrac{ad\ln(as+b)}{c^2}\right) + c_1, \\
m_0(s) &= \tfrac{as+b}{cs+d}\left\{\left(\tfrac{as+b\ln(as+b)}{c} - \tfrac{ad\ln(as+b)}{c^2}\right) + c_1\right\} + a(as+b),
\end{aligned} \tag{23}
$$

where a, b, c and d are real constants.

Proof. Suppose that x is a concho-spiral in $\mathbb{E}^3$ given with the curvatures $\kappa_1 = \frac{1}{as+b}, \kappa_2 = \frac{1}{cs+d}$. If x is T-constant curve of second kind. Then, by the use of (6) we get

$$
\begin{aligned}
m_1(s) &= -as+b, \\
m_2(s) &= \int \left(\frac{as+b}{cs+d}\right) ds, \\
m_0(s) &= \frac{\kappa_2 m_2 - a}{\kappa_1}.
\end{aligned}
$$

Further, integrating the second equation and using the third one we get the result.

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