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Quadratic Embeddings

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Abstract. The quadratic Veronese embedding ρ maps the point set $\mathcal{P}$ of $\mathrm{PG}(n,F)$ into the point set of $\mathrm{PG}(\binom{n+2}{2}-1,F)$ (F a commutative field) and has the following well-known property: If $\mathcal{M}\subset\mathcal{P}$, then the intersection of all quadrics containing $\mathcal{M}$ is the inverse image of the linear closure of $\mathcal{M}^\rho$. In other words, ρ transforms the closure from quadratic into linear. In this paper we use this property to define "quadratic embeddings". We shall prove that if ν is a quadratic embedding of $\mathrm{PG}(n,F)$ into $\mathrm{PG}(n',F')$ (F a commutative field), then $\rho^{-1}\nu$ is dimension-preserving. Moreover, up to some exceptional cases, there is an injective homomorphism of F into F'. An additional regularity property for quadratic embeddings allows us to give a geometric characterization of the quadratic Veronese embedding.

1. Introduction

The aim of this paper is to examine geometric properties of a *quadratic embedding*, i.e. a mapping between projective spaces sharing some properties of the classical quadratic Veronese embedding. We follow an approach that has been used in discussing embeddings of Grassmann spaces (cf. [6] and [16]) and product spaces (cf. [17]). See also [8, chapter 25] for combinatorial characterizations of Veronese varieties over finite fields.

Let F be a commutative field and $(\mathcal{P},\mathcal{L}) := \mathrm{PG}(n,F)$. Write

$$\Phi := \{\mathcal{S}\subset\mathcal{P} \mid \mathcal{S} \text{ is a quadric of } \mathrm{PG}(n,F)\}\cup\{\mathcal{P}\}.$$

If $\mathcal{M}\subset\mathcal{P}$, then the *quadratic closure* of $\mathcal{M}$ is

$$\overline{\overline{\mathcal{M}}} := \bigcap_{\mathcal{M}\subset\mathcal{S},\,\mathcal{S}\in\Phi} \mathcal{S}.$$

We call $\mathcal{M}$ a *closed set* if $\mathcal{M} = \overline{\overline{\mathcal{M}}}$. The linear closure of a set $\mathcal{M}$ of points will be denoted by $\overline{\mathcal{M}}$. Each hyperplane of $\mathrm{PG}(n,F)$ is a quadric, namely a repeated hyperplane. Hence $\mathcal{M} \subset \overline{\overline{\mathcal{M}}} \subset \overline{\mathcal{M}}$.

Definition 1. *Let* $(\mathcal{P},\mathcal{L}) := \mathrm{PG}(n,F)$ *and* $(\mathcal{P}',\mathcal{L}') := \mathrm{PG}(n',F')$*, where the field* F *is commutative. A mapping* $\nu : \mathcal{P} \to \mathcal{P}'$ *is a* quadratic embedding *if*

$$\overline{\overline{\mathcal{M}}} = (\overline{\mathcal{M}^\nu})^{\nu^{-1}} \text{ for all } \mathcal{M} \subset \mathcal{P}, \tag{1}$$

and

$$\overline{\operatorname{im}\nu} = \mathcal{P}'. \tag{2}$$

We give some examples of quadratic embeddings:

Example 1. The classical *quadratic Veronese embedding* ρ is defined in the case $F' = F$, $n' = \binom{n+2}{2} - 1$, by

$$F(x_0,\dots,x_n) \overset{\rho}{\longmapsto} F(y_{ij})_{0\le i\le j\le n}, \text{ with } y_{ij} := x_i x_j.$$

There are many equivalent definitions. Cf., e.g., [5], [7], [9].

Example 2. Let $n' = \binom{n+2}{2} - 1$. If $\alpha : F \to F'$ is an injective homomorphism, then α induces a canonical embedding ϵ of $\mathrm{PG}(n',F)$ into $\mathrm{PG}(n',F')$. The mapping $\rho\epsilon$ turns out to be a quadratic embedding. Since there are examples of fields admitting an injective, but not surjective homomorphism $\alpha : F \to F$, there exist quadratic embeddings different from the classical one, even if we demand that F and F' are isomorphic.

Example 3. If $n = 1$ then $\mathcal{M} \subset \mathcal{P}$ is closed if, and only if, $|\mathcal{M}| \le 2$ or $\mathcal{M} = \mathcal{P}$. Thus a mapping $\nu : \mathcal{P} \to \mathcal{P}'$ is a quadratic embedding if, and only if, $n' = 2$, ν is injective and $\operatorname{im}\nu$ is an arc.

Example 4. If $\mathrm{PG}(n,F) = \mathrm{PG}(2,2)$ and $\mathcal{F}$ is a frame in $\mathrm{PG}(5,F')$ then any injection $\nu : \mathcal{P} \to \mathcal{P}'$ such that $\operatorname{im}\nu = \mathcal{F}$ is a quadratic embedding. This is immediate from the fact that each subset $\mathcal{M}$ of $\mathcal{P}$ is closed, unless $|\mathcal{M}| = 6$.

2. Properties of quadratic embeddings

In this section ν is a quadratic embedding of $\mathrm{PG}(n,F) = (\mathcal{P},\mathcal{L})$ $(n \ge 1)$ into $\mathrm{PG}(n',F') = (\mathcal{P}',\mathcal{L}')$.

Proposition 2.1. *Let* $\mathcal{K}_1$ *and* $\mathcal{K}_2$ *be two distinct closed sets in* $\mathrm{PG}(n,F)$*. Then* $\overline{\mathcal{K}_1^\nu} \ne \overline{\mathcal{K}_2^\nu}$*. Consequently, the mapping* ν *is injective and satisfies*

$$\overline{\overline{\mathcal{M}}}^\nu = \overline{\mathcal{M}^\nu} \cap \operatorname{im}\nu \text{ for all } \mathcal{M} \subset \mathcal{P}. \tag{3}$$

If $\mathcal{U}' \subset \mathcal{P}'$ *is a subspace, then* $\mathcal{U}'^{\nu^{-1}} \subset \mathcal{P}$ *is a closed set.*

Proof. By the definition of a quadratic embedding,

$$(\overline{\mathcal{K}_1^\nu})^{\nu^{-1}} = \overline{\overline{\mathcal{K}_1}} = \mathcal{K}_1 \neq \mathcal{K}_2 = \overline{\overline{\mathcal{K}_2}} = (\overline{\mathcal{K}_2^\nu})^{\nu^{-1}},$$

so that $\overline{\mathcal{K}_1^\nu} \neq \overline{\mathcal{K}_2^\nu}$. Moreover, ν is injective, since any subset of $\mathcal{P}$ with a single element is closed. Hence (3) is true. Finally, let $\mathcal{M} := \mathcal{U}'^{\nu^{-1}}$. Then

$$\overline{\overline{\mathcal{M}}} = (\overline{\mathcal{M}^\nu})^{\nu^{-1}} \subset \mathcal{U}'^{\nu^{-1}} = \mathcal{M}. \qquad \square$$

Theorem 1. *If ν is a quadratic embedding of* $\mathrm{PG}(n,F)$ *into* $\mathrm{PG}(n',F')$, *then* $n' = \binom{n+2}{2} - 1$.

Proof. Define $\delta(t) := \binom{t+2}{2}$, $t \in \mathbb{N}$. Let $\{\mathbf{e}_0, \ldots, \mathbf{e}_n\}$ be a basis of F^{n+1} and

$$\mathcal{X} := \{F(\mathbf{e}_i + \mathbf{e}_j) | 0 \leq i < j \leq n\} \cup \{F\mathbf{e}_i | i = 0, \ldots, n\}.$$

Since $|\mathcal{X}| = \delta(n)$ and $\overline{\overline{\mathcal{X}}} = \mathcal{P}$, we have $\operatorname{im}\nu \subset \overline{\mathcal{X}^\nu}$, hence

$$n' \leq \delta(n) - 1. \tag{4}$$

We now prove that in (4) the equality holds. We give a definition, by recursion on $d = 0, \ldots, n$, of distinct closed sets in $\mathrm{PG}(n,F)$, say $\mathcal{K}_{\delta(d-1)}$, $\mathcal{K}_{\delta(d-1)+1}$, $\ldots$, $\mathcal{K}_{\delta(d)-1}$, such that

$$\mathcal{K}_{\delta(d-1)} \subset \mathcal{K}_{\delta(d-1)+1} \subset \ldots \subset \mathcal{K}_{\delta(d)-1},$$

with ${}^d\mathcal{U} := \mathcal{K}_{\delta(d)-1}$ being a d-subspace of $\mathrm{PG}(n,F)$. For $d = 0$, choose a point Q and let $\mathcal{K}_0 = {}^0\mathcal{U} := \{Q\}$. Now let $d > 0$ and $\mathcal{K}_{\delta(d-1)-1} = {}^{d-1}\mathcal{U}$. Take a d-subspace ${}^d\mathcal{U}$ containing ${}^{d-1}\mathcal{U}$ and a basis $\mathcal{B} = \{P_0, \ldots, P_d\}$ of ${}^d\mathcal{U}$ such that ${}^{d-1}\mathcal{U} \cap \mathcal{B} = \emptyset$. Since the union of two subspaces of $\mathrm{PG}(n,F)$ is a closed set, we can define

$$\mathcal{K}_{\delta(d-1)+i} := {}^{d-1}\mathcal{U} \cup \overline{\{P_0, \ldots, P_i\}}, \qquad i \in \{0, \ldots, d\}.$$

By Proposition 2.1,

$$\emptyset \subset \overline{\mathcal{K}_0^\nu} \subset \overline{\mathcal{K}_1^\nu} \subset \ldots \subset \mathcal{K}_{\delta(n)-1}^\nu$$

is a chain of distinct subspaces of $\mathrm{PG}(n',F')$. $\square$

Proposition 2.2. *If $\mathcal{M} \subset \mathcal{P}$, then* $\dim(\overline{\mathcal{M}^\nu})$ *is equal to the largest $i \in \mathbb{N}$, such that there exists a chain*

$$\emptyset \subset \mathcal{K}_0 \subset \mathcal{K}_1 \subset \ldots \subset \mathcal{K}_i = \overline{\overline{\mathcal{M}}}, \tag{5}$$

consisting of $i+2$ distinct closed subsets of $\overline{\overline{\mathcal{M}}}$. Consequently, $\dim(\overline{\mathcal{M}^\nu}) = \dim(\overline{\mathcal{M}^\rho})$, *where ρ denotes the quadratic Veronese embedding.*

Proof. By Proposition 2.1, the subspaces $\overline{\mathcal{K}_0^\nu}$, $\overline{\mathcal{K}_1^\nu}$, $\ldots$, $\overline{\mathcal{K}_i^\nu}$ are distinct and

$$\overline{\mathcal{K}_i^\nu} = \overline{\overline{\mathcal{M}^\nu} \cap \operatorname{im}\nu} = \overline{\mathcal{M}^\nu},$$

whence $\dim(\overline{\mathcal{M}^\nu}) \geq i$.

Now assume $\dim(\overline{\mathcal{M}^\nu}) > i$. Then there exists an integer j, $0 \le j < i$, such that

$$\dim(\overline{\mathcal{K}_{j+1}^\nu}) \ne \dim(\overline{\mathcal{K}_j^\nu}) + 1.$$

Let $P \in \mathcal{K}_{j+1}^\nu \setminus \overline{\mathcal{K}_j^\nu}$. Then $\mathcal{K} := (\overline{\{P\} \cup \mathcal{K}_j^\nu})^{\nu^{-1}}$ is a closed set (cf. Proposition 2.1), and $\mathcal{K}_j \subset \mathcal{K} \subset \mathcal{K}_{j+1}$, $\mathcal{K} \ne \mathcal{K}_j$. The maximality of the chain (5) implies $\mathcal{K} = \mathcal{K}_{j+1}$. Therefore

$$\dim(\overline{\mathcal{K}_{j+1}^\nu}) = \dim(\overline{\{P\} \cup \mathcal{K}_j^\nu}) = \dim(\overline{\mathcal{K}_j^\nu}) + 1,$$

a contradiction. □

Proposition 2.3. *If $\mathcal{T}$ is a hyperplane of* $\mathrm{PG}(n, F)$ *and $\mathcal{M} \subset \mathcal{P} \setminus \mathcal{T}$, then*

$$\dim(\overline{(\mathcal{T} \cup \mathcal{M})^\nu}) = \binom{n+1}{2} + \dim(\overline{\mathcal{M}}). \tag{6}$$

Proof. By Theorem 1, $\dim(\overline{\mathcal{T}^\nu}) = \binom{n+1}{2} - 1$. If $\mathcal{B} = \{P_0, \ldots, P_t\} \subset \mathcal{M}$ is a basis of $\overline{\mathcal{M}}$, then the closed sets

$$\mathcal{K}_i := \mathcal{T} \cup \overline{\{P_0, \ldots, P_i\}}, \qquad i \in \{0, \ldots, t\},$$

form a saturated chain

$$\mathcal{T} \subset \mathcal{K}_0 \subset \ldots \subset \mathcal{K}_t.$$

Now the assertion is a consequence of Theorem 1 and Proposition 2.2. □

Proposition 2.4. *Let $\mathcal{T}$ be a hyperplane of* $\mathrm{PG}(n, F)$. *If $\overline{\mathcal{T}^\nu}$ and $\mathcal{E}'$ are complementary subspaces of* $\mathrm{PG}(n', F')$, *then the mapping*

$$\iota : \mathcal{P} \setminus \mathcal{T} \longrightarrow \mathcal{E}' : A \longmapsto \overline{(\mathcal{T} \cup \{A\})^\nu} \cap \mathcal{E}' \tag{7}$$

has the following property:

$$\dim(\overline{\mathcal{M}}) = \dim(\overline{\mathcal{M}^\iota}) \text{ for all } \mathcal{M} \subset \mathcal{P} \setminus \mathcal{T}. \tag{8}$$

Consequently, ι is preserving both collinearity and non-collinearity of points. So, the mapping ι is a (linear) embedding of the affine space $\mathcal{P} \setminus \mathcal{T}$ into the projective space $\mathcal{E}'$.

Proof. By (6), $\dim \overline{\mathcal{T}^\nu} = \binom{n+1}{2} - 1$, whence $\dim \mathcal{E}' = n$. Applying (6) again yields

$$\begin{aligned}\dim(\overline{\mathcal{M}^\iota}) &= \dim(\overline{\mathcal{T}^\nu \cup \mathcal{M}^\nu} \cap \mathcal{E}') = \dim(\overline{\mathcal{T}^\nu \cup \mathcal{M}^\nu}) - (\dim(\overline{\mathcal{T}^\nu}) + 1)\\ &= \dim(\overline{\mathcal{M}}).\end{aligned}$$

□

Proposition 2.5. *Let $|F| > 2$ and $n \ge 2$. If $|F| \ne 3$ or $n \ne 2$, then the embedding (7) can be extended to exactly one embedding $\beta : \mathcal{P} \to \mathcal{E}'$.*

Proof. The case $n = 2$ is dealt with in [12]. The case $|F| > 3$ is covered by [2, Theorem 3.5]. Thus only $|F| = 3$ and $n > 2$ remains open. For F' being finite, the assertion follows from a result in [10, chapitre 2.3] (cf. also [11, théorème 1]), and by slight modifications, this carries over to an infinite F'.

On the other hand we sketch a direct proof for $n > 2$: Let $P \in \mathcal{T}$. If $g, h \in \mathcal{L}$ and $P \in g \cap h$, then the lines $\overline{(g \setminus \{P\})^\iota}$ and $\overline{(h \setminus \{P\})^\iota}$ are coplanar by Proposition 2.4. Since there exist three non coplanar lines through P, all lines of the kind $\overline{(g \setminus \{P\})^\iota}$, with $P \in g$, share one point $P' \in \mathcal{E}'$. Then we define $P^\beta := P'$. By repeatedly using Proposition 2.4, we have that β is an embedding. The restriction of β to a plane $\mathcal{A}$ of $\mathcal{P}$, not contained in $\mathcal{T}$, is an extension of $\iota|(\mathcal{A} \setminus \mathcal{T})$, and thus we obtain the uniqueness of β. □

As a corollary, we have:

Theorem 2. *Let $|F| > 2$ and $n \geq 2$. If $|F| \neq 3$ or $n \neq 2$, then the existence of a quadratic embedding of* $\mathrm{PG}(n, F)$ *into* $\mathrm{PG}(n', F')$ *implies that the field F is isomorphic to a subfield of F'.* □

Whenever for some fixed hyperplane $\mathcal{T} \subset \mathcal{P}$ and an adequately chosen subspace $\mathcal{E}' \subset \mathcal{P}'$ the mapping (7) is uniquely extendable to an embedding $\beta : \mathcal{P} \to \mathcal{E}'$, then $\mathcal{T}$ gives rise to an embedding

$$\nu_{\mathcal{T}} : \mathcal{P} \longrightarrow \mathcal{P}'/\overline{\mathcal{T}^\nu} : X \longmapsto \{X^\beta\} \vee \overline{\mathcal{T}^\nu}; \tag{9}$$

here $\mathcal{P}'/\overline{\mathcal{T}^\nu}$ denotes the point set of the quotient space $\mathrm{PG}(n', F')$ modulo $\overline{\mathcal{T}^\nu}$. Moreover, we can associate with $\mathcal{T}$ the following hyperplane of $\mathrm{PG}(n', F')$:

$$\overline{\mathcal{T}^\nu \cup \mathcal{T}^\beta} =: \mathcal{H}'_{\mathcal{T}}.$$

Both definitions do not depend on the choice of $\mathcal{E}'$. Since $\mathcal{H}'_{\mathcal{T}} \cap \mathcal{E}' \cap \operatorname{im} \iota = \emptyset$, we have

$$(\mathcal{H}'_{\mathcal{T}})^{\nu^{-1}} = \mathcal{T}. \tag{10}$$

Proposition 2.6. *Let Σ be the collection of all hyperplanes of* $\mathrm{PG}(n, F)$. *If $\mathcal{H}'_{\mathcal{T}}$ is well defined for all $\mathcal{T} \in \Sigma$, then*

$$\widehat{\nu} : \Sigma \longrightarrow \Sigma' : \mathcal{T} \longmapsto \mathcal{H}'_{\mathcal{T}} \tag{11}$$

is an injective mapping. □

The previous results give sufficient conditions for the existence of the mapping $\widehat{\nu}$.

3. Regular quadratic embeddings

In the following we shall assume that ν is a quadratic embedding of $\mathrm{PG}(n, F)$, $n \geq 1$, into $\mathrm{PG}(n', F')$, $n' = \binom{n+2}{2} - 1$.

Definition 2. *A quadratic embedding ν is called* (P, ℓ)-regular *if there exists an incident point-line pair (P, ℓ) of* $\mathrm{PG}(n, F)$ *such that the plane arc ℓ^ν has a unique unisecant line which is running through P^ν and contained in the plane $\overline{\ell^\nu}$. If ν is (P, ℓ)-regular for all incident pairs (P, ℓ), then ν is said to be a* regular *quadratic embedding.*

Proposition 3.1. *Suppose that $n \geq 2$ and that ν is (P, ℓ)-regular. Then ν is regular.*

Proof. By $n \geq 2$, there exists a hyperplane $\mathcal{T}$ of $\mathrm{PG}(n, F)$ such that $P \in \mathcal{T}$, $\ell \not\subset \mathcal{T}$. Define an embedding $\iota : \mathcal{P} \setminus \mathcal{T} \to \mathcal{E}'$ according to (7). The (P, ℓ)-regularity of ν implies that

$$|\overline{(\ell \setminus \{P\})^\iota} \setminus (\ell \setminus \{P\})^\iota| = 1. \tag{12}$$

If the settings of Proposition 2.5 are true, then ι extends to an embedding $\beta : \mathcal{P} \to \mathcal{E}'$ with $\ell^\beta = \overline{(\ell \setminus \{P\})^\iota}$ by (12). Hence β is a collineation.

Otherwise $|F| =: p \in \{2, 3\}$ so that $|F'| = p$ by (12). If $X \in \mathcal{P} \setminus \mathcal{T}$, then there is a certain number of lines through X and on each such line there are p points of $\mathcal{P} \setminus \mathcal{T}$. In $\mathcal{E}'$ the same number of lines is running through X^ι and, by Proposition 2.4, there are p points of $\operatorname{im} \iota$ on each such line. This in turn means that on each line in $\mathcal{E}'$ through a point of $\mathcal{E}' \setminus \operatorname{im} \iota$ there is either no point of $\operatorname{im} \iota$ or no other point of $\mathcal{E}' \setminus \operatorname{im} \iota$, whence $\mathcal{E}' \setminus \operatorname{im} \iota$ is a subspace. More precisely, $\mathcal{T}' := \mathcal{E}' \setminus \operatorname{im} \iota$ is a hyperplane of $\mathcal{E}'$. Two distinct lines of the affine space $\mathcal{P} \setminus \mathcal{T}$ are parallel if, and only if, they are disjoint and coplanar. By Proposition 2.4 these properties carry over to the ι-images of these lines, whence ι is an affinity of $\mathcal{P} \setminus \mathcal{T}$ onto $\mathcal{E}' \setminus \mathcal{T}'$. (This is trivial when $p = 3$.) Thus ι is also extendable to a collineation $\beta : \mathcal{P} \to \mathcal{E}'$ if Proposition 2.5 cannot be applied.

Next choose any point $X_1 \in \mathcal{T}$ and any line $\ell_1 \not\subset \mathcal{T}$, $X_1 \in \ell_1$. Then $(\{X_1^\beta\} \vee \overline{\mathcal{T}^\nu}) \cap \overline{\ell_1^\nu}$ is the only unisecant of ℓ_1^ν at X_1^ν within the plane $\overline{\ell_1^\nu}$, since $|\overline{(\ell_1 \setminus \{X_1\})^\iota} \setminus (\ell_1 \setminus \{X_1\})^\iota| = 1$. Hence ν is (X_1, ℓ_1)-regular. Repeatedly using this last idea yields that ν is regular. □

As an immediate consequence of the proof of Proposition 3.1 we have

Proposition 3.2. *Let ν be a regular quadratic embedding and $n \geq 2$. Choose any hyperplane $\mathcal{T} \subset \mathcal{P}$. Then the embedding $\iota : \mathcal{P} \setminus \mathcal{T} \to \mathcal{E}'$, defined according to (7), is extendable to a unique collineation $\beta : \mathcal{P} \to \mathcal{E}'$. Consequently, F and F' are isomorphic fields, and $\nu_\mathcal{T} : \mathcal{P} \to \mathcal{P}'/\overline{\mathcal{T}^\nu}$ (cf. (9)) is a collineation.* □

If $n = 1$, then the (P, ℓ)-regularity of ν implies

$$|F| = |\ell \setminus \{P\}| = |\ell^\nu \setminus \{P^\nu\}| = |F'|.$$

This does not imply, however, that ν is regular. If $n = 1$ and ν is regular, then $\operatorname{im} \nu$ obviously is an oval but not necessarily a conic. Cf. [3, 4, 14] for topological conditions that force an oval to be a conic.

These results can be improved if we assume that $|F| =: q$ is finite. Then $n = 1$ and ν being (P, ℓ)-regular yield $|F| = |F'| = q$ so that $\operatorname{im} \nu$ is a $(q+1)$-arc in $\mathrm{PG}(2, F') \cong \mathrm{PG}(2, q)$. Hence $\operatorname{im} \nu$ is an oval, which in turn shows that ν is regular. Moreover, by Segre's theorem, $\operatorname{im} \nu$ is a (regular) conic if q is odd; the last result is also true when $q \in \{2, 4\}$.

The case $n = 1$ is excluded from our further discussions. Hence we may assume without loss of generality that $F = F'$, by virtue of Proposition 3.2.

The following result will be used in order to characterize the ν-images of lines.

Lemma 1. *Let P_0 and P_2 be two distinct points of* $\mathrm{PG}(2, F)$ *and let σ be a collineation of* $\mathrm{PG}(2, F)$ *taking P_0 to P_2, but not fixing the line $\overline{P_0 P_2}$. Then*

$$\mathcal{C} := \{X | \{X\} = x \cap x^\sigma, x \text{ is a line through } P_0\}$$

is containing three distinct collinear points if, and only if, σ is a non-projective collineation.

Proof. If σ is projective, then $\mathcal{C}$ is a regular conic, whence it does not contain three distinct collinear points.

If σ is not projective, then let $\alpha \in \mathrm{Aut}(F)$ be the companion automorphism of σ. Set

$$\{P_1\} := (\overline{P_0P_2})^{\sigma^{-1}} \cap (\overline{P_0P_2})^{\sigma}.$$

Choose some line e running through P_0 but not containing P_1 or P_2 and define $\{E\} := e \cap e^{\sigma}$. Then (P_0, P_1, P_2, E) is an ordered quadrangle; we may assume that this is the standard frame of reference. A straightforward calculation yields

$$\mathcal{C} = \{F(u_0u_0^{\alpha}, u_0u_1^{\alpha}, u_1u_1^{\alpha}) | (0,0) \neq (u_0, u_1) \in F^2\}.$$

By $\alpha \neq \mathrm{id}_F$, there exists an element $c \in F$ with $c \neq c^{\alpha}$. Define $v \in F$ via $c = v^{\alpha\alpha}c^{\alpha}$. Thus $v \neq 0, 1$ and $F(1,1,1)$, $F(1, v^{\alpha}, vv^{\alpha})$ are distinct points of $\mathcal{C}$. With

$$w := \frac{1 + vv^{\alpha}c}{1 + v^{\alpha}c}$$

we obtain

$$\frac{1}{1+c}(1,1,1) + \frac{c}{1+c}(1, v^{\alpha}, vv^{\alpha}) = (1, w^{\alpha}, ww^{\alpha}),$$

whence $\mathcal{C}$ is containing three distinct collinear points. □

Proposition 3.3. *If g is a line of* $\mathrm{PG}(n,F)$*, $n \geq 2$, and ν is a regular quadratic embedding, then g^{ν} is a regular conic.*

Proof. Choose hyperplanes $\mathcal{T}, \check{\mathcal{U}} \subset \mathcal{P}$ such that $g \cap \mathcal{T} \cap \check{\mathcal{U}} = \emptyset$. Set $\{T\} := g \cap \mathcal{T}$ and define a collineation $\nu_{\mathcal{T}}$ according to (9). Write

$$\mathcal{L}'_T := \{x' \in \mathcal{L}' | T^{\nu} \in x' \subset \overline{g^{\nu}}\}$$

and

$$\pi_T : \mathcal{L}'_T \longrightarrow g^{\nu_{\mathcal{T}}} : x' \longmapsto x' \vee \overline{\mathcal{T}^{\nu}}.$$

This π_T is a projectivity from a pencil of lines onto a pencil of subspaces. Replacing $\mathcal{T}$ by $\check{\mathcal{U}}$ gives a point U and a projectivity π_U. Since $\nu_{\mathcal{T}}^{-1}\nu_{\check{\mathcal{U}}}$ is a collineation of quotient spaces,

$$\pi_T \nu_{\mathcal{T}}^{-1} \nu_{\check{\mathcal{U}}} \pi_U^{-1} : \mathcal{L}'_T \longrightarrow \mathcal{L}'_U$$

is extendable to a collineation, say σ, of the plane $\overline{g^{\nu}}$ onto itself. We have

$$\begin{aligned} \overline{g^{\nu}} \cap \overline{\mathcal{T}^{\nu}} &\overset{\sigma}{\longmapsto} \overline{T^{\nu}U^{\nu}} \neq \overline{g^{\nu}} \cap \overline{\mathcal{T}^{\nu}}, \\ \overline{T^{\nu}U^{\nu}} &\overset{\sigma}{\longmapsto} \overline{g^{\nu}} \cap \overline{\check{\mathcal{U}}^{\nu}}, \\ \overline{T^{\nu}X^{\nu}} &\overset{\sigma}{\longmapsto} \overline{U^{\nu}X^{\nu}} \; (X \in g \setminus \{T, U\}) \end{aligned}$$

and

$$g^{\nu} = \{X' | \{X'\} = x' \cap x'^{\sigma}, x' \in \mathcal{L}'_T\}.$$

Since any two distinct points of g form a closed set, no three points of g^{ν} are collinear. We read off from Lemma 1 that σ is projective, whence g^{ν} is a regular conic. □

We remark that $\nu_{\mathcal{T}}^{-1}\nu_{\check{\mathcal{U}}}$ is a projective collineation of quotient spaces.

Proposition 3.4. *Let $(P_0, P_1, \ldots, P_n, E)$ be an ordered frame of* $\mathrm{PG}(n, F)$*, $n \geq 2$, and let ν be a regular quadratic embedding. Write $Q'_{ii} := P_i^\nu$, $E' := E^\nu$, and Q'_{ij} for the common point of the tangent lines of the conic $(\overline{P_i P_j})^\nu$ at P_i^ν and P_j^ν, $i, j \in \{0, 1, \ldots, n\}$, $i \neq j$. Then*

$$\{Q'_{ij} | 0 \leq i \leq j \leq n\} \cup \{E'\} \tag{13}$$

is a frame of $\mathrm{PG}(n', F)$.

Proof. For any $i, j \in \{0, 1, \ldots, n\}$, $i < j$, take a $P_{ij} \in \overline{P_i P_j} \setminus \{P_i, P_j\}$. Then $\{Q'_{00}, Q'_{11}, \ldots, Q'_{nn}\} \cup \{P_{ij}^\nu | 0 \leq i < j \leq n\}$ is a basis of $\mathrm{PG}(n', F)$ by Theorem 1. Since Q'_{ii}, Q'_{jj}, P_{ij}^ν, Q'_{ij} is a plane quadrangle, the exchange lemma yields that

$$\mathcal{B}' := \{Q'_{ij} | 0 \leq i \leq j \leq n\}$$

is a basis of $\mathrm{PG}(n', F)$.

Define hyperplanes

$$\mathcal{T}_i := \overline{\{P_k | k \in \{0, 1, \ldots, n\} \setminus \{i\}\}} \subset \mathcal{P},$$

and $\mathcal{X}'_i := \mathcal{T}_i^{\widehat{\nu}}$ (cf. (11)) for $i \in \{0, 1, \ldots, n\}$. Obviously $Q'_{jk} \in \mathcal{X}'_i$ for all $j, k \in \{0, 1, \ldots, n\} \setminus \{i\}$. Moreover, if $j \in \{0, 1, \ldots, n\} \setminus \{i\}$, then

$$\overline{\{Q'_{ii}, Q'_{ij}, Q'_{jj}\}} \cap \mathcal{X}'_i$$

is the tangent line of the conic $(\overline{P_i P_j})^\nu$ at $P_j^\nu = Q'_{jj}$, so that $Q'_{ij} \in \mathcal{X}'_i$. We infer that

$$\mathcal{X}'_i = \overline{\mathcal{B}' \setminus \{Q'_{ii}\}}.$$

Now $E \notin \mathcal{T}_i$ implies $E' \notin \mathcal{X}'_i$. Finally, $\mathcal{T}_i \cup \mathcal{T}_j$ is a closed set not containing E. Hence

$$E' \notin \overline{(\mathcal{T}_i \cup \mathcal{T}_j)^\nu} = \overline{\mathcal{B}' \setminus \{Q'_{ij}\}}.$$

This completes the proof. $\square$

Theorem 3. *If ν is a regular quadratic embedding of* $\mathrm{PG}(n, F)$ *into* $\mathrm{PG}(n', F)$*, $n \geq 2$, $n' = \binom{n+2}{2} - 1$ and ρ denotes the quadratic Veronese embedding, then there exists a collineation κ of* $\mathrm{PG}(n', F)$ *such that $\nu = \rho\kappa$.*

Proof. We adopt the notation of Proposition 3.4. The coordinates with respect to $(P_0, P_1, \ldots, P_n, E)$ of a point $X \in \mathcal{P}$ are written as $F(x_0, x_1, \ldots, x_n)$, and the coordinates of X^ν with respect to $(Q'_{00}, Q'_{01}, \ldots, Q'_{nn}, E')$ (cf. (13)) are denoted by $F(y_{00}, y_{01}, \ldots, y_{nn})$. In order to simplify notation we put $y_{ij} := y_{ji}$ for $i > j$.

Choose an index $i \in \{0, 1, \ldots, n\}$ and set

$$\mathcal{E}'_i := \overline{\{Q'_{i0}, Q'_{i1}, \ldots, Q'_{in}\}}.$$

Hence $\mathcal{E}'_i$ is a complement of $\overline{\mathcal{T}_i^\nu}$ (cf. the proof of Proposition 3.4) and, by Proposition 3.2, we obtain a collineation $\beta_i : \mathcal{P} \to \mathcal{E}'_i$ with $P_j \mapsto Q'_{ij}$ ($j \in \{0, 1, \ldots, n\}$) and $E \mapsto E'_i$, where