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Finite and Infinite Collections of Multiplication Modules

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Abstract. All rings are commutative with identity and all modules are unitary. In this note we give some properties of a finite collection of submodules such that the sum of any two distinct members is multiplication, generalizing those which characterize arithmetical rings. Using these properties we are able to give a concise proof of Patrick Smith's theorem stating conditions ensuring that the sum and intersection of a finite collection of multiplication submodules is a multiplication module. We give necessary and sufficient conditions for the intersection of a collection (not necessarily finite) of multiplication modules to be a multiplication module, generalizing Smith's result. We also give sufficient conditions on the sum and intersection of a collection (not necessarily finite) for them to be multiplication. We apply D. D. Anderson's new characterization of multiplication modules to investigate the residual of multiplication modules.

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0. Introduction

Let R be a ring and M an R-module. For submodules K and L of M, the residual of K by L, denoted by $[K : L]$, is the set of all x in R such that $xL \subseteq K$. An R-module M is called a multiplication module if for each submodule N of M there exists an ideal I of

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R such that $N = IM$ [3]. It is clear that every cyclic module is multiplication, and that a multiplication module over a local ring is cyclic [3]. Let N be a submodule of a multiplication module M. There exists an ideal I of R such that $N = IM$. Note that $I \subseteq [N : M]$ and $N = IM \subseteq [N : M]M \subseteq N$ so that $N = [N : M]M$. It follows that M is a multiplication R-module if and only if $N = [N : M]M$ for all submodules N of M. An ideal A of R which is a multiplication module is called a multiplication ideal.

In Section 2 (Theorem 2.1) we establish several properties of a finite collection of submodules N_i of an R-module M which satisfy the condition that $N_i + N_j$ is multiplication for all $i < j$. The interest in these properties lies in the fact that they generalize the standard characterization of arithmetical rings, see [6], [9] and [10]. Using these properties, we offer a short proof of Patrick Smith's result giving conditions for the sum and intersection of a finite collection of multiplication modules to be a multplication module [18, Theorem 8]. Some examples will be given to highlight these properties by showing that they fail without the assumption of finiteness and that their converses are not true in general.

In Section 3 we give (see Theorem 3.2) necessary conditions for the intersection of a collection (not necessarily finite) of multiplication modules to be a multiplication module. We also give (see Theorems 3.6 and 4.2) conditions on the sum and intersection of an arbitrary collection of modules sufficient for them to be multliplication modules, generalizing Smith's theorem.

In Section 4 we apply D.D. Anderson's new characterization of multiplication modules, [2, Theorem 2.1], to obtain yet another characterization. In Theorem 4.3 we apply it to residuals of multiplication modules.

For the basic concepts we refer the reader to [5], [7], [11], [12] and [17].

1. Preliminaries

Let R be a ring and $N_i (1 \leq 1 \leq n)$ a finite collection of submodules of an R-module M. Throughout this note we use the following notation:

$$S = \sum_{l=1}^{n} N_l, \quad N = \bigcap_{l=1}^{n} N_l, \quad \hat{S}_i = \sum_{j \neq i} N_j \quad \text{and} \quad \hat{N}_i = \bigcap_{j \neq i} N_j.$$

Lemma 1.1. *Let R be a ring and $N_i (1 \leq i \leq n)$ a finite collection of submodules of an R-module M such that*

$$[N_i : N_j] + [N_j : N_i] = R \quad \textit{for all} \quad i < j.$$

Then

$$\text{(i)} \quad \sum_{i=1}^{n} [\hat{S}_i : S] = R, \qquad \text{(ii)} \quad \sum_{i=1}^{n} [N : \hat{N}_i] = R.$$

Proof. (i) From the assumption we obtain $\sum_{i=1}^{n} \sum_{j \neq i} [N_j : N_i] = R$. Obviously

$$\sum_{j \neq i} [N_j : N_i] \subseteq [\hat{S}_i : N_i] \quad \text{for all} \quad 1 \leq i \leq n.$$

Hence $\sum_{i=1}^{n} [\hat{S}_i : N_i] = R$. But $[\hat{S}_i : N_i] = [\hat{S}_i : S]$, and hence (i) is proved. The second part is similar. $\square$

Compare the following result with [18, Proposition 4] and [7, Theorem 25.2].

Corollary 1.2. *Let R be a ring and $N_i (1 \leq i \leq n)$ a finite collection of submodules of an R-module M such that $[N_i : N_j] + [N_j : N_i] = R$ for all $i < j$. Then*

(i) $[S : K] = \sum_{i=1}^{n} [N_i : K]$ *for every submodule K of M,*

(ii) $\sum_{i=1}^{n} [N_i : S] = R$,

(iii) $[K : N] = \sum_{i=1}^{n} [K : N_i]$ *for every submodule K of M,*

(iv) $\sum_{i=1}^{n} [N : N_i] = R$,

(v) $K \bigcap S = \sum_{i=1}^{n} (K \bigcap N_i)$ *for every submodule K of M,*

(vi) $K + N = \bigcap_{i=1}^{n} (K + N_i)$ *for every submodule K of M,*

(vii) $IN = \bigcap_{l=1}^{n} IN_l$ *for every ideal I of R.*

Proof. We prove (i) by induction on n. The result is true for $n = 2$ [18, Proposition 4]. Assume $n > 2$ and the result is true for $n - 1$, i.e. for every submodule K of M,

$$[\hat{S}_i : K] = \sum_{j \neq i} [N_j : K], \quad \text{for all} \quad 1 \leq i \leq n.$$

Suppose K is a submodule of M. Clearly $\sum_{l=1}^{n} [N_l : K] \subseteq [S : K]$. By Lemma 1.1, there exists $x_i \in [\hat{S}_i : S]$ such that $1 = \sum_{i=1}^{n} x_i$. Let $z \in [S : K]$. Then $z = \sum_{i=1}^{n} z x_i$ and $z x_i K \subseteq x_i S \subseteq \hat{S}_i$ for all $1 \leq i \leq n$. It follows that

$$z \in \sum_{i=1}^{n} [\hat{S}_i : K] = \sum_{i=1}^{n} [N_i : K]$$

so that $[S : K] \subseteq \sum_{i=1}^{n} [N_i : K]$, and (i) is proved. For (ii) take $K = S$ in (i). (iii) is similar to (i). For (iv) take $K = N$ in (iii).

To prove (v), let K be any submodule of M. Clearly $\sum_{l=1}^{n} K \bigcap N_l \subseteq K \bigcap S$. Using part (ii), there exist $x_i \in [N_i : S]$ such that $1 = \sum_{i=1}^{n} x_i$. Let $u \in K \bigcap S$. Then $u = \sum_{i=1}^{n} u x_i$ and $u x_i \in K \bigcap N_i$ for all $1 \leq i \leq n$. Thus $u \in \sum_{i=1}^{n} K \bigcap N_i$ and hence $K \bigcap S \subseteq \sum_{i=1}^{n} K \bigcap N_i$.

To prove (vi), assume that K is a submodule of M. Then clearly $K+N \subseteq \bigcap_{l=1}^{n}(K+N_l)$. By part (iv) there exist $x_i \in [N : N_i]$ such that $1=\sum_{i=1}^{n} x_i$. Let $v \in \bigcap_{j=1}^{n}(K+N_j)$. Then $v = k_j + n_j$ for some $k_j \in K$, $n_j \in N_j$ and all $1 \le j \le n$. It follows that $x_j v = x_j k_j + x_j n_j \in K + N$, and hence $v = \sum_{j=1}^{n} x_j v \in K+N$ so that $\bigcap_{l=1}^{n}(K+N_l) \subseteq K+N$ and (vi) follows.

Finally, let I be an ideal of R. Clearly $IN \subseteq \bigcap_{l=1}^{n} IN_l$. Again by (vi), $1 = \sum_{i=1}^{n} x_i$ for some $x_i \in [N : N_i]$. Let $y \in \bigcap_{r=1}^{n} IN_r$. Then $y \in IN_r$ for all $1 \le r \le n$. Hence $y = \sum_{k=1}^{m} u_{rk} n_{rk}$ where $u_{rk} \in I$ and $n_{rk} \in N_r$ for all $1 \le r \le n$ and all $1 \le k \le m$. It follows that

$$y = \sum_{r=1}^{n} y x_r = \sum_{k=1}^{m}\sum_{r=1}^{n} u_{rk}(n_{rk} x_r).$$

But $n_{rk} x_r \in N$ and hence $u_{rk}(n_{rk}x_r) \in IN$ for all $1 \le r \le n$ and all $1 \le k \le m$. This implies that $y \in IN$ and thus completes the proof of (vii). □

We observe that if R is a ring and $N_i (1 \le i \le n)$ is a finite collection of finitely generated submodules of an R-module M such that $N_i + N_j$ is multiplication for all $i < j$, then the conclusions of Corollary 1.2 are satisfied, see [16, Lemma 3.3] and [18, Corollary 3 of Theorem 1].

An R-module M is called a weak-cancellation module if, for all ideals I and J of R, if $IM \subseteq JM$ then $I \subseteq J + \mathrm{ann}(M)$. It is clear that every cyclic module is weak-cancellation, from which it follows that finitely generated multiplication modules are weak-cancellation, see [4, Theorem 3.1] and [18, Corollary of Theorem 9]. However, the following holds for multiplication modules in general.

Lemma 1.3. *Let R be a ring and M a multiplication R-module. Let I and J be ideals of R. Then $IM \subseteq JM$ if and only if $I \subseteq J + \mathrm{ann}(m)$ for all $m \in M$.*

Proof. Suppose first that I and J are ideals such that $IM \subseteq JM$. By [18, Theorem 9] either $I \subseteq J + \mathrm{ann}(M)$ and the result follows immediately, or $M = [(J+\mathrm{ann}(M)) : I]M$. Let $m \in M$. Then $Rm = AM$ for some ideal A of R. Now

$$\begin{aligned} Rm &= AM = A[(J+\mathrm{ann}(M)) : I]M \\ &= [(J+\mathrm{ann}(M)) : I]AM = [(J+\mathrm{ann}(M)) : I]m, \end{aligned}$$

and hence $R = [(J+\mathrm{ann}(M)) : I] + \mathrm{ann}(m)$. Finally

$$\begin{aligned} I &= RI = [(J+\mathrm{ann}(M)) : I]I + \mathrm{ann}(m)I \\ &\subseteq J + \mathrm{ann}(M) + \mathrm{ann}(m)I \subseteq J + \mathrm{ann}(m), \end{aligned}$$

as required. The converse is trivial. □

Corollary 1.4. *Let R be a ring and $N_\lambda(\lambda \in \Lambda)$ a collection of submodules of an R-module M, and let $S = \sum_{\lambda\in\Lambda} N_\lambda$ be a multiplication module. Then*

(i) $\sum_{\lambda\in\Lambda}[N_{\lambda P} : S_P] = R_P$ *for every maximal ideal P of R.*

(ii) $\sum_{\lambda\in\Lambda}[N_\lambda : S] + \mathrm{ann}(a) = R$ *for every $a \in S$.*

In particular, if K and L are submodules of an R-module M such that $K+L$ is multiplication, then

(i) $[K_P : L_P] + [L_P : K_P] = R_P$ *for every maximal ideal P of R.*

(ii) $[K : L] + [L : K] + ann(a) = R$ *for every $a \in K + L$.*

Proof. (i) As S is a multiplication module, we have $N_\lambda = [N_\lambda : S]S$ for all $\lambda \in \Lambda$. Hence $S = \sum_{\lambda\in\Lambda}[N_\lambda : S]S$. Assume P is a maximal ideal of R. Then S_P is a weak-cancellation R_P-module. It follows that

$$R_P = \sum_{\lambda\in\Lambda}[N_\lambda : S]_P + \mathrm{ann}(S_P) \subseteq \sum_{\lambda\in\Lambda}[N_{\lambda P} : S_P] + \mathrm{ann}(S_P).$$

But $\mathrm{ann}(S_P) \subseteq [N_{\lambda P} : S_P]$ for all $\lambda \in \Lambda$. Thus (i) is proved.
Part (ii) follows immediately by the preceding Lemma, since $S = \sum_{\lambda\in\Lambda}[N_\lambda : S]S$. □

We remark that Patrick Smith [18, Theorem 2] proved (ii) under the additional assumption that the submodules N_λ of M are multiplication.

2. Finite collections

The following theorem shows several properties of a finite collection of submodules $N_i(1 \le i \le n)$ such that $N_i + N_j$ is multiplication for all $i < j$. These properties generalize the characteristic properties of Prüfer domains (see for example [7, Theorem 25.2], [12, Theorem 6.6]) and of arithmetical rings (see [9, Lemma 2] and [10, Theorem 3]). Later we use these properties to give a concise proof of Smith's theorem [18, Theorem 8].

Theorem 2.1. *Let R be a ring and $N_i(1 \le i \le n)$ a finite collection of submodules of an R-module M such that $N_i + N_j$ is multiplication for all $i < j$. Then*

(i) $\sum_{i=1}^{n}[\hat{S}_i : S] + \mathrm{ann}(a) = R$ *for all $a \in S$,*

(ii) $\sum_{i=1}^{n}[N : \hat{N}_i] + \mathrm{ann}(a) = R$ *for all $a \in S$,*

(iii) $[S : K] = \sum_{i=1}^{n}[N_i : K] \pmod{\mathrm{ann}(a)}$ *for all submodules K of M and all $a \in S$,*

(iv) $\sum_{i=1}^{n}[N_i : S] + \mathrm{ann}(a) = R$ *for all $a \in S$,*

(v) $[K : N] = \sum_{i=1}^{n}[K : N_i] \pmod{\mathrm{ann}(a)}$ *for all submodules K of M and all $a \in S$,*

(vi) $\sum_{i=1}^{n}[N : N_i] + \mathrm{ann}(a) = R$ *for all $a \in S$,*

(vii) $K \bigcap S = \sum_{i=1}^{n} (K \bigcap N_i)$ *for every submodule* K *of* M,

(viii) $K + N = \bigcap_{i=1}^{n} (K + N_i)$ *for every submodule* K *of* M,

(ix) $IN = \bigcap_{l=1}^{n} IN_l$ *for every ideal* I *of* R.

Proof. As $N_i + N_j$ is multiplication for all $i < j$, we infer from Corollary 1.4 that

$$[N_i : N_j] + [N_j : N_i] + \operatorname{ann}(a) = R \quad \text{for all} \quad a \in N_i + N_j.$$

It follows that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} [N_i : N_j] + [N_j : N_i] + \operatorname{ann}(a) = R,$$

and hence by Lemma 1.1 we get that

$$\sum_{i=1}^{n} [\hat{S}_i : S] + \operatorname{ann}(a) = R \quad \text{for all} \quad a \in N_k + N_l, k < l.$$

Let $m \in S$. Then $m = \sum_{k<l} a_{kl}$ where $a_{kl} \in N_k + N_l$. Hence

$$R = \sum_{i=1}^{n} [\hat{S}_i : S] + \bigcap_{k<l} \operatorname{ann}(a_{kl}) = \sum_{i=1}^{n} [\hat{S}_i : S] + \operatorname{ann}(\sum_{k<l} R a_{kl}) \subseteq \sum_{i=1}^{n} [\hat{S}_i : S] + \operatorname{ann}(m),$$

so that

$$R = \sum_{i=1}^{n} [\hat{S}_i : S] + \operatorname{ann}(m) \quad \text{for all} \quad m \in S.$$

(ii) is similar. For (iii), let K be a submodule of M. By induction it suffices to assume $n = 2$. By (i),

$$[N_1 : N_1 + N_2] + [N_2 : N_1 + N_2] + \operatorname{ann}(a) = R \quad \text{for all } a \in N_1 + N_2.$$

Let $a \in N_1 + N_2$. Clearly

$$[N_1 : K] + [N_2 : K] + \operatorname{ann}(a) \subseteq [N_1 + N_2 : K] + \operatorname{ann}(a).$$

Now let $w \in [N_1 + N_2 : K] + \operatorname{ann}(a)$. Then $w = w_1 + w_2$ where $w_1 \in [N_1 + N_2 : K]$ and $w_2 \in \operatorname{ann}(a)$. Also, there exist $x_1, x_2, z \in R$ such that $x_1 \in [N_1 : N_1 + N_2]$, $x_2 \in [N_2 : N_1 + N_2]$, $z \in \operatorname{ann}(a)$ and $1 = x_1 + x_2 + z$. But $w = w_1(x_1 + x_2 + z) + w_2$ and $w_1 x_1 K \subseteq x_1(N_1 + N_2) \subseteq N_1$, and hence $w_1 x_1 \in [N_1 : K]$. Similarly, $w_1 x_2 \in [N_2 : K]$. Also, $w_1 z + w_2 \in \operatorname{ann}(a)$, and this shows that

$$[N_1 + N_2 : K] + \operatorname{ann}(a) \subseteq [N_1 : K] + [N_2 : K] + \operatorname{ann}(a).$$

For (iv), take $K = S$ in (iii). (v) is similar to (iii), and for (vi), take $K = N$ in (v).
The last three parts of the theorem are true locally as seen by using Corollary 1.2 (parts (v), (vi), and (vii)) and Corollary 1.4, and hence they are true globally. □

Lüneburg [14, Theorem 3] proved that for ideals I, J of a domain R, if $I + J$ is invertible, then $(I + J)(I \bigcap J) = IJ$. The following corollary extends this result.

Corollary 2.2. *Let R be a ring and I, J ideals of R such that $I + J$ is a multiplication ideal of R. Then*

$$IJ = (I + J)(I \bigcap J).$$

Proof. As $I + J$ is multiplication, we infer from Theorem 2.1 that

$$(I + J)(I \bigcap J) = (I + J)I \bigcap (I + J)J \supseteq JI \bigcap IJ = IJ.$$

The other inclusion is always satisfied. □

We apply Theorem 2.1 to give a concise proof of a theorem of P. Smith.

Theorem 2.3. [18, Theorem 8] *Let R be a ring and $N_i (1 \leq i \leq n)$ a collection of submodules of an R-module M such that $N_i + N_j$ is multiplication for all $i < j$. Then:*

(i) *S is multiplication.*

(ii) *If each N_i is multiplication, then N is multiplication.*

Proof. (i) Let $k \in \{1, \ldots, n\}$. It follows from Theorem 2.1(i) that

$$\hat{S}_k = (\sum_{i=1}^{n} [\hat{S}_i : S])\hat{S}_k.$$

Then

$$\hat{S}_k = [\hat{S}_k : S]\hat{S}_k + (\sum_{i \neq k} [\hat{S}_i : S])\hat{S}_k = [\hat{S}_k : S]\hat{S}_k + (\sum_{i \neq k} [\hat{S}_i : \hat{S}_k])\hat{S}_k.$$

By induction suppose that $\hat{S}_j$ is multiplication for all $j \in \{1, \ldots, n\}$. Then $[\hat{S}_i : \hat{S}_k]\hat{S}_k = [\hat{S}_k : \hat{S}_i]\hat{S}_i = [\hat{S}_k : S]\hat{S}_i$ for all $i \neq k$, and hence

$$\hat{S}_k = [\hat{S}_k : S]\hat{S}_k + \sum_{i \neq k} [\hat{S}_k : S]\hat{S}_i = [\hat{S}_k : S]S.$$

Let K be any submodule of M. Then by Theorem 2.1 (vii) we have that

$$K \bigcap S = \sum_{i=1}^{n} K \bigcap N_i \subseteq \sum_{i=1}^{n} K \bigcap \hat{S}_i = \sum_{i=1}^{n} [K : \hat{S}_i]\hat{S}_i = \sum_{i=1}^{n} [K : \hat{S}_i][\hat{S}_i : S]S \subseteq [K : S]S \subseteq K \bigcap S,$$

so that $K \bigcap S = [K : S]S$. This shows that S is a multiplication module.
For the second part, let K be any submodule of M. By Theorem 2.1 (viii),

$$K \bigcap N = \bigcap_{i=1}^{n} K \bigcap N_i = \bigcap_{i=1}^{n} [K : N_i]N_i \subseteq \bigcap_{i=1}^{n} [K : N]N_i = [K : N]N \subseteq K \bigcap N,$$

so that $K \bigcap N = [K : N]N$, and N is a multiplication module. □

We prove two corollaries. The first is a generalization of [16, Corollary 3.4] and [18, Propositon 12], and the second shows that Theorem 2.1(vii) is true for an arbitrary collection of modules.

Corollary 2.4. *Let R be a ring and $N_i (1 \leq i \leq n)$ a finite collection of finitely generated multiplication submodules of an R-module M that can be generated by m_i elements respectively, and let $N = \bigcap_{i=1}^{n} N_i$. If $N_i + N_j$ is multiplication for all $i < j$, then N is a finitely generated multiplication module that can be generated by $\sum_{i=1}^{n} m_i$ elements.*

Proof. As $N_i + N_j$ is multiplication, it follows from the remark made after Corollary 1.2 that $\sum_{i=1}^{n} [N : N_i] = R$. Then there exist elements $x_i \in [N : N_i]$ such that $\sum_{i=1}^{n} x_i = 1$. It follows that

$$N = \sum_{i=1}^{n} x_i N \subseteq \sum_{i=1}^{n} x_i N_i \subseteq N,$$

so that $N = \sum_{i=1}^{n} x_i N_i$ is a submodule of M generated by $\sum_{i=1}^{n} m_i$ elements. That N is multiplication follows by Theorem 2.3(ii). Alternatively, we may observe that $x_i N \subseteq x_i N_i$ for all i, and hence $x_i \in [x_i N_i : N]$. It follows that $\sum_{i=1}^{n} [x_i N_i : N] = R$. But $N = \sum_{j=1}^{n} x_j N_j$, and $x_i N_i$ is multiplication [4, Corollary 1.4]. Thus by [18, Corollary 1 of Theorem 1], N is multiplication. □

Corollary 2.5. *Let R be a ring and $N_\lambda (\lambda \in \Lambda)$ a collection of submodules of an R-module M such that $N_\lambda + N_\mu$ is multiplication for all $\lambda \neq \mu$. Let $S = \sum_{\lambda \in \Lambda} N_\lambda$. Then $K \bigcap S = \sum_{\lambda \in \Lambda} K \bigcap N_\lambda$ for every submodule K of M. In particular, if $N_\lambda (\lambda \in \Lambda)$ is a collection of multiplication modules such that*

$$[N_\lambda : N_\mu] + [N_\mu : N_\lambda] = R \quad \text{for all} \quad \lambda \neq \mu,$$

then the result holds.

Proof. Let K be a submodule of M. Clearly $\sum_{\lambda \in \Lambda} (K \bigcap N_\lambda) \subseteq K \bigcap S$. For each $x \in K \bigcap S$ there exists a finite subset Λ_x of Λ such that $x \in \sum_{\lambda \in \Lambda_x} N_\lambda$. It follows that $K \bigcap S \subseteq \sum_{x \in K \bigcap S} (K \bigcap \sum_{\lambda \in \Lambda_x} N_\lambda)$. As $N_\lambda + N_\mu$ is multiplication for all $\lambda, \mu \in \Lambda_x$ $(\lambda \neq \mu)$, we infer from Theorem 2.1(vii) that $K \bigcap \sum_{\lambda \in \Lambda_x} N_\lambda = \sum_{\lambda \in \Lambda_x} (K \bigcap N_\lambda)$. Therefore $K \bigcap S \subseteq \sum_{\lambda \in \Lambda} K \bigcap N_\lambda$, and the proof of the first assertion is complete. The second assertion follows now by [18, Corollary 3 of Theorem 2]. □

Examples 1–3 below show that the conclusions of Theorem 2.1 (other than (vii)) are not valid without the finiteness assumption, and Example 4 shows, among other things, that their converses are not true.

Example 1. Let $R = \mathbb{C}[[x]]$. R is a Noetherian local domain and the unique maximal ideal of R is Rx. Let $N_k = Rx^k$. Then N_k is a multiplication ideal of R for each $k \geq 1$ and so too is $N_k + N_n$ for any positive integers $k < n$, because $N_k + N_n = N_k$. On the other hand $N = \bigcap_{k \geq 1} N_k = 0$ and hence $\sum_{l \geq 1} [N : N_l] = 0 \neq R$.

Example 2. Let $R = [\mathbb{Z} \quad \mathbb{Q}]$, the commutative ring of all matrices of the form $\begin{pmatrix} n & r \\ 0 & n \end{pmatrix}$ with $n \in \mathbb{Z}$, $r \in \mathbb{Q}$.

(i) For $s \in \mathbb{Q}$, let $N_s = R\begin{pmatrix} 0 & s \\ 0 & 0 \end{pmatrix} = [0 \quad \mathbb{Z}s]$. N_s is a multiplication ideal of R (being principal). If $s, t \in \mathbb{Q}$, say $s = \frac{u}{v}$, $t = \frac{x}{y}$ with $\gcd(u, v) = \gcd(x, y) = 1$, then let $r = \gcd(s, t) = \frac{1}{vy}\gcd(u, x)\gcd(v, y)$. (See [13] for properties of gcd in this sense.) Then for any $s, t \in \mathbb{Q}$, $N_s + N_t$ is multiplication because $N_s + N_t = N_r$, $r = \gcd(s, t)$. But $S = \sum_{s\in\mathbb{Q}} N_s = [0 \quad \mathbb{Q}]$ which is neither finitely generated nor multiplication, and $\sum_{s\in\mathbb{Q}}[N_s : S] \neq R$ since $[N_s : S] = [0 \quad \mathbb{Q}]$ for all $s \in \mathbb{Q}$.

(ii) For $i \geq 1$, let $N_i = R\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} = [\mathbb{Z}i \quad \mathbb{Q}]$. Then N_i is multiplication and so too is $N_i + N_j$ for all $i, j \geq 1$ because $N_i + N_j = N_d$ where $d = \gcd(i, j)$. But $N = \bigcap_{i\geq 1} N_i = [0 \quad \mathbb{Q}]$ is not multiplication, and $\sum_{j\geq 1}[N : N_j] = 0 \neq R$.

(iii) Let p be an odd prime integer and $N_p = [\mathbb{Z}p \quad \mathbb{Q}]$. Then $\bigcap_{p\neq 2} N_p = [0 \quad \mathbb{Q}]$, and hence $N_2 + \bigcap_{p\neq 2} N_p = N_2$. But $\bigcap_{p\neq 2}(N_2 + N_p) = [\mathbb{Z} \quad \mathbb{Q}] = R$. Also it is easy to verify that $[N_2 : \bigcap_{p\neq 2} N_p] = R$ but $\sum_{p\neq 2}[N_2 : N_p] = N_2$.

Example 3. [8, Example 31] Let R be a Prüfer domain which is not Noetherian. There is a maximal ideal P of R which is not finitely generated. P is not multiplication, and hence $R \neq \theta(p) = \sum_{p\in P}[pR : P]$, [1, Theorem 1] and [2, Theorem 2.1]. However, $aR + bR$ is multiplication for all $a, b, \in R$.

Example 4. Let Q be the ring of all sequences of elements of $\mathbb{Z}_2$, and put $e_n = (0, 0, \ldots, 1, 0, \ldots)$. Let $R = Q[x, y]$. Then $I = \sum_{n\geq 1} e_n R$ is a multiplication ideal of R since it is generated by idempotents, [1] and [4]. Let $I_n = (e_1, \ldots, e_n, e_{n+1}x, e_{n+1}y)R$. We show that I_n is not a multiplication ideal by showing that I_n is not locally principal, [1]. Let $M_{n+1} = (1 - e_{n+1}, x, y)R$. M_{n+1} is a maximal ideal of R (in fact $R/M_{n+1} \approx \mathbb{Z}_2$). For $f = \sum_{i,j} f_{ij}x^i y^j \in R$, define $\varphi(f) = \sum (f_{ij})_{n+1}x^i y^j$, where $(f_{ij})_k$ is the k^{th} term of the sequence f_{ij}. Then $\varphi : R \to \mathbb{Z}_2[x, y]$ is a homomorphism, and ker $\varphi = (1 - e_{n+1})R \subset M_{n+1}$. Moreover, $\varphi(M_{n+1}) = (x, y)\mathbb{Z}_2[x, y]$. So φ extends to a homomorphism $\varphi : R_{M_{n+1}} \to \mathbb{Z}_2[x, y]_{(x,y)}$, and $\varphi(I_n) = (x, y)\mathbb{Z}_2[x, y]_{(x,y)}$ is not principal, hence neither is $I_n R_{M_{n+1}}$. Now, $I_1 \subset I_2 \subset \cdots$, and $I = \bigcup_{n\geq 1} I_n$. One may show directly that, as Corollary 1.4(ii) concludes, $\sum_{n\geq 1}[I_n : I] + \text{ann}(a) = R$ for all $a \in I$, a conclusion which does not follow from Smith's theorem [18, Theorem 2]. It also shows that the converse of Corollary 1.4 is not true, because

$$[I_i : I_j] + [I_j : I_i] + \text{ann}(a) = R$$

for all $i \neq j$ and all $a \in I_i + I_j$, but $I_i + I_j = I_k$ where $k = \max\{i, j\}$, which is not multiplication. This example further shows that the converses of Theorem 2.1 and Corollary 2.2 are not true. For this purpose, let $S = \sum_{i=1}^{n} I_i$ and $N = \bigcap_{i=1}^{n} I_i$. Then $S = I_n$ and $N = I_1$ are not multiplication. Also, for all i, j, $I_i + I_j$ is not multiplication.

3. Infinite collections

Naoum and Hasan [16] gave a sufficient condition for the intersection of two multiplication modules to be multiplication. That result was generalized by Smith [18] to a finite collection of multiplication modules. Using different methods, we extend the results to intersections of arbitrary collections of modules. First, we need a lemma.

Lemma 3.1. *Let R be a ring and $N_\lambda (\lambda \in \Lambda)$ a collection of submodules of an R-module M and let $N = \bigcap_{\lambda \in \Lambda} N_\lambda$. If $\sum_{\lambda \in \Lambda} [N : N_\lambda] = R$, then $IN = \bigcap_{\lambda \in \Lambda} IN_\lambda$ for every ideal I of R.*

Proof. Let I be an ideal of R. Clearly $IN \subseteq \bigcap_{\lambda \in \Lambda} IN_\lambda$. If the condition is satisfied, there exist a finite subset Λ' of Λ and elements $x_\lambda \in [N : N_\lambda] (\lambda \in \Lambda')$ such that $\sum_{\lambda \in \Lambda'} x_\lambda = 1$. Let $w \in \bigcap_{\lambda \in \Lambda} IN_\lambda$. Then $w \in IN_\lambda$ for all $\lambda \in \Lambda'$. For each $\lambda \in \Lambda'$, there exists a finite set Λ_λ such that $w = \sum_{\mu \in \Lambda_\lambda} u_\mu y_{\lambda\mu}$ where $u_\mu \in I$ and $y_{\lambda\mu} \in N_\lambda$. It follows that $w = \sum_{\lambda \in \Lambda'} \sum_{\mu \in \Lambda_\lambda} u_\mu (x_\lambda y_{\lambda\mu})$. But $x_\lambda y_{\lambda\mu} \in N$ for all $\lambda \in \Lambda'$ and all $\mu \in \Lambda_\lambda$. Thus $w \in IN$, and the result follows. □

Theorem 3.2. *Let R be a ring and $N_\lambda (\lambda \in \Lambda)$ a collection of multiplication submodules of an R-module M and let $N = \bigcap_{\lambda \in \Lambda} N_\lambda$. Then*

$$\sum_{\lambda \in \Lambda} [N : N_\lambda] + \operatorname{ann}(a) = R \quad \textit{for all} \quad a \in N$$

if and only if these conditions are satisfied:

(i) $IN = \bigcap_{\lambda \in \Lambda} IN_\lambda$ *for every ideal I of R.*

(ii) *N is a multiplication module.*

Proof. Suppose first that

$$\sum_{\lambda \in \Lambda} [N : N_\lambda] + \operatorname{ann}(a) = R \quad \text{for all} \quad a \in N.$$

We can prove (i) even without the assumption that the N_λ are multiplication. It suffices to prove it locally. Thus we may assume that R is a local ring. If $N = 0$, there is nothing to prove. Otherwise $N \neq 0$, and hence there exists $0 \neq a \in N$ so that $\mathrm{ann}(a) \neq R$. It follows that

$$\sum_{\lambda\in\Lambda}[N : N_\lambda] = R,$$

and the result follows from Lemma 3.1. To prove (ii), let K be any submodule of M. Then

$$K\bigcap N = \bigcap_{\lambda\in\Lambda}(K\bigcap N_\lambda) = \bigcap_{\lambda\in\Lambda}[K : N_\lambda]N_\lambda \subseteq \bigcap_{\lambda\in\Lambda}[K : N]N_\lambda = [K : N]N \subseteq K\bigcap N,$$

so that $K\bigcap N = [K : N]N$, and N is multiplication. Conversely, assume that (i) and (ii) are satisfied. As N_λ is multiplication for all $\lambda \in \Lambda$, we infer that

$$N = [N : N_\lambda]N_\lambda \subseteq (\sum_{\mu\in\Lambda}[N : N_\mu])N_\lambda \quad \text{for all} \quad \lambda \in \Lambda.$$

Hence

$$N \subseteq \bigcap_{\lambda\in\Lambda}(\sum_{\mu\in\Lambda}[N : N_\mu])N_\lambda = (\sum_{\mu\in\Lambda}[N : N_\mu])N.$$

By Lemma 1.3, the result follows. □

In the following example, condition (i) but not (ii) of Theorem 3.2 holds.

Example 5. Let $R = Q[[x]]$, $I = Rx$. (R, I) is a discrete valuation ring. Let $D = \mathbb{Z} + I$. Then I is a Prüfer domain [7, p.319]. Let $N_i = Dp_i$ where $p_1 < p_2 < \cdots$ is the sequence of positive primes of $\mathbb{Z}$. Let $N = \bigcap_{i\geq 1} Dp_i$. Then N is not a finitely generated ideal of R and hence it is not multiplication. One can easily verify that for all i, $[N : N_i] = N$ and hence $\sum_{i\geq 1}[N : N_i] \neq D$.

We mention three corollaries to Theorem 3.2. The first is an immediate consequence while the second and third give sufficient conditions for the radical of a module to be multiplication.

Corollary 3.3. *Let R be a ring and $N_\lambda(\lambda \in \Lambda)$ a collection of multiplication submodules of an R-module M and let $N = \bigcap_{\lambda\in\Lambda} N_\lambda$. If $\sum_{\lambda\in\Lambda}[N : N_\lambda] = R$, then N is a multiplication module.*

Let R be a ring and M an R-module. A submodule P of M is called a prime submodule if whenever $rm \in P$, for some $m \in M$, $r \in R$, then $m \in P$ or $r \in [P : M]$. The M-radical of a submodule N of M is defined as the intersection of all prime submodules of M containing N, (see [15]).

Corollary 3.4. *Let R be a ring and N a submodule of an R-module M such that*

$$\sum[\mathrm{rad}\, N : P] = R,$$

where the sum is over all prime submodules P of M containing N.

(i) *If every P is multiplication, then so too is* rad N.

(ii) *If every P is finitely generated, then so too is* rad N.

(iii) *If every P is faithful, then so too is* rad N.

Proof. (i) Corollary 3.3.

(ii) There exist a finite set of prime submodules $\{P_1, P_2, \dots, P_n\}$ of M containing N and elements $x_i \in [\text{rad } N : P_i]$ such that $\sum_{i=1}^{n} x_i = 1$. It follows that rad $N = \sum_{i=1}^{n} x_i P_i$, and hence is finitely generated.

(iii) As in (ii), $1 = \sum_{i=1}^{n} x_i$ with $x_i \in [\text{rad } N : P_i]$. Let $w \in \text{ann}(\text{rad}N)$. Then $wx_iP_i = 0$, and $wx_i \in \text{ann}(P_i) = 0$. But $w = \sum wx_i = 0$. So rad$N$ is faithful. □

The Jacobson radical of a module M over a ring is defined [11] to be the intersection of all maximal submodules of M.

Corollary 3.5. *Let R be a semi-local ring with maximal ideals $P_1, \dots, P_n$. If each P_i is multiplication (i.e. principal), then $J(R)$, the Jacobson radical of R is multiplication (i.e. principal). If M is a multiplication R-module (i.e. cyclic) with $P_iM \neq M$, then $J(M)$, the Jacobson radical of M is also multiplication (i.e. cyclic).*

Proof. The first assertion is clear by Theorem 2.3(ii). On the other hand, by [4, Theorem 2.5], P_iM is a maximal submodule of M and by [4, Corollary 1.4], P_iM is multiplication. In fact P_iM are the only maximal submodules of M. For, if Q is any maximal submodule different from P_iM, then again by [4, Theorem 2.5] there exists a maximal ideal $P \in \{P_1, P_2, \dots, P_n\}$ such that $Q = PM$. Hence $P_iM \neq PM$ and hence $P_i \neq P$, a contradiction. It follows that $J(M) = \bigcap_{i=1}^{n} P_iM$. Since $P_iM + P_jM = M$ for all $i \neq j$, the result is clear by Theorem 2.3(ii). □

Let R be a ring such that every maximal ideal P of R is multiplication and $\sum_{P \text{ maximal}} [J(R):P] = R$. Then by Corollary 3.3, $J(R)$ is multiplication. Suppose that M is a multiplication module such that for all maximal ideals P of R, $PM \neq M$. Then $J(M) = \bigcap_{P \text{ maximal}} PM$, and PM is a multiplication submodule of M. Also it is easy to verify that $\sum_{P \text{ maximal}} [J(M) : PM] = R$, and hence again by Corollary 3.3, $J(M)$ is multiplication.

Example 6. Contrary to what happens in the finite case, in Example 2(ii) and Example 5 an intersection of multiplication modules is not multplication. On the other hand, let $R = k[x, y]$, k an infinite field. Then R has an infinite number of maximal ideals M_λ, $\lambda \in \Lambda$. The Jacobson radical $J = \bigcap_{\lambda \in \Lambda} M_\lambda = 0$ is a multiplication ideal. Also $M_\lambda + M_\mu = R$, a multiplication ideal for all $\lambda \neq \mu$, but the M_λ are not all multiplication ideals. Thus an intersection of modules may be multiplication even if the components are not. In each of these examples, however, the sum of any two distinct modules is multiplication.

The next theorem establishes necessary and sufficient conditions for the intersection of a collection (not necessarily finite) of multiplication submodules to be a multiplication module. It is a generalization of [18, Theorem 8 (ii)].

Theorem 3.6. *Let R be a ring and $N_\lambda(\lambda \in \Lambda)$ a collection of submodules of an R-module M. Let $N = \bigcap_{\lambda\in\Lambda} N_\lambda$ and $S = \sum_{\lambda\in\Lambda} N_\lambda$. Suppose that $N_\lambda + N_\mu$ is a multiplication module for all $\lambda \neq \mu$. Let $A = \sum_{\lambda\in\Lambda} [N : N_\lambda]$, and suppose that $A + \mathrm{ann}(n) = R$ for all $n \in S$. Then N is multiplication if and only if N_λ is multiplication for all $\lambda \in \Lambda$.*

Proof. Suppose first that N is multiplication. Let $\lambda \in \Lambda$ and K a submodule of M such that $K \subseteq N_\lambda$. Clearly $[K : N_\lambda]N_\lambda \subseteq K$. Let $x \in K$ and set

$$H = \{r \in R \quad | \quad rx \in [K : N_\lambda]N_\lambda\}.$$

If $H \neq R$, then there exists a maximal ideal P of R such that $H \subseteq P$. We discuss two cases.
Case 1: $A \subseteq P$. As $A + \mathrm{ann}(n) = R$ for all $n \in S$, we have $N_\lambda + N_\mu = A(N_\lambda + N_\mu)$ for all $\mu \neq \lambda$. Then

$$N_\lambda + N_\mu = A(N_\lambda + N_\mu) \subseteq P(N_\lambda + N_\mu) \subseteq N_\lambda + N_\mu$$

so that $N_\lambda + N_\mu = P(N_\lambda + N_\mu)$. Also $N_\lambda + N_\mu$ is multiplication for all $\mu \neq \lambda$. Thus $Rx = I(N_\lambda + N_\mu)$ for some ideal I of R and hence

$$Rx = I(N_\lambda + N_\mu) = IP(N_\lambda + N_\mu) = Px.$$

There exists $p \in P$ such that $(1 - p)x = 0$, and hence $1 - p \in H \subseteq P$, a contradiction.
Case 2: $A \nsubseteq P$. Then

$$[N : N_\lambda] + \sum_{\mu\neq\lambda} [N : N_\mu] \nsubseteq P.$$

If $[N : N_\lambda] \nsubseteq P$, there exists $q \in P$ such that $(1 - q)N_\lambda \subseteq N$, and hence $(1 - q)K \subseteq N$. Since N is multiplication, there exists an ideal J of R such that $(1 - q)K = JN$. Now

$$(1 - q)JN_\lambda = J(1 - q)N_\lambda \subseteq JN = (1 - q)K \subseteq K$$

so that $(1 - q)J \subseteq [K : N_\lambda]$. It follows that

$$(1 - q)^2x \in (1 - q)^2K = (1 - q)JN \subseteq (1 - q)JN_\lambda \subseteq [K : N_\lambda]N_\lambda.$$

Hence $(1-q)^2 \in H \subseteq P$, a contradiction. Finally, if $\sum_{\mu\neq\lambda} [N : N_\mu] \nsubseteq P$, then there exists $\mu \neq \lambda$, such that $[N : N_\mu] \nsubseteq P$, and hence there exists $q' \in P$ such that $(1 - q')N_\mu \subseteq N \subseteq N_\lambda$. Then $(1 - q')(N_\lambda + N_\mu) \subseteq N_\lambda$. Next

$$K = [K : (N_\lambda + N_\mu)](N_\lambda + N_\mu) \subseteq [K : N_\lambda](N_\lambda + N_\mu),$$

and hence

$$(1-q')x \in (1-q')K \subseteq [K:N_\lambda](1-q')(N_\lambda+N_\mu) \subseteq [K:N_\lambda]N_\lambda.$$

But this gives $(1-q') \in H \subseteq P$, a contradiction. Thus $H=R$ and $x \in [K:N_\lambda]N_\lambda$ so that $K=[K:N_\lambda]N_\lambda$ and this proves that N_λ is a multiplication module. The converse follows by Theorem 3.2. □

4. Applications of Anderson's new characterization of multiplication modules

D. D. Anderson [2, Theorem 2.1] has proved that a submodule A of an R-module M is multiplication if and only if for each maximal ideal P of R with $A_P \neq 0_P$, A_P is cyclic and $[N:A]_P=[N_P:A_P]$ for each submodule N of M. We use this new characterization to obtain two further characterizations of multiplication modules which we then apply to investigate the residual of multiplication modules. We further illustrate uses of the characterization by providing alternative proofs of some of our results in Sections 2 and 3.

Proposition 4.1. *Let R be a ring and A a submodule of an R-module M. The following conditions are equivalent:*

(i) *A is a multiplication module.*

(ii) *For each maximal ideal P of R with $A_P \neq 0_P$, there exists a multiplication submodule B containing A and an element $p \in R$ such that $(1-p)B \subseteq A$.*

(iii) *For each maximal ideal P of R with $A_P \neq 0_P$, there exists a multiplication submodule B of A and an element $p \in P$ such that $(1-p)A \subseteq B$.*

Proof. (i) ⇒ (ii) and (i) ⇒ (iii) are obvious by [2, Theorem 2.1 Part 5] and by taking $A=B$, $B=Ra$ for some $a \in A$ respectively.

(ii) ⇒ (i): Let P be a maximal ideal of R such that $A_P \neq 0_P$. There exists a multiplication submodule B of M containing A and an element $p \in P$ such that $(1-p)B \subseteq A$. It follows that $A_P=B_P$ and for each submodule N of M,

$$[N_P:A_P]=[N_P:B_P]=[N:B]_P \subseteq [N:A]_P \subseteq [N_P:A_P],$$

and by [2, Theorem 2.1], A is multiplication.

(iii) ⇒ (i): Let P be a maximal ideal of R such that $A_P \neq 0_P$. There exists a multiplication submodule B of A and an element $p \in P$ such that $(1-p)A \subseteq B$. Hence $A_P=B_P$ and for each submodule N of M,

$$\begin{aligned}[N:B]_P &\subseteq [N:(1-p)A]_P \\ &= [[N:A]:(1-p)R]_P = [[N:A]_P:(1-p)R_P] = [N:A]_P \subseteq [N:B]_P,\end{aligned}$$

and therefore $[N:A]_P=[N:B]_P=[N_P:B_P]=[N_P:A_P]$, and again by [2, Theorem 2.1], A is multiplication. □

In Example 4 at the end of Section 2, we showed that a sum of modules may be multiplication even if no summand is multiplication. It is known [4, Theorem 2.2] that if a multiplication module S is a direct sum of submodules, then all of the summands are multiplication. We show that the same conclusion follows if the assumption of directness is weakened by assuming only that the intersection of distinct summands is multiplication.

Theorem 4.2. *Let R be a ring and $N_\lambda (\lambda \in \Lambda)$ a collection of submodules of an R-module M and let $S = \sum_{\lambda\in\Lambda} N_\lambda$. If S is multiplication and $N_\lambda \cap N_\mu$ is multiplication for all $\lambda \neq \mu$, then all N_λ are multiplication.*

Proof. Let P be a maximal ideal of R. Let $\lambda \in \Lambda$, and suppose that $N_{\lambda P} \neq 0_P$. Then $S_P \neq 0_P$, and hence for some $a \in S$, $\operatorname{ann}(a) \subseteq P$. By Corollary 1.4, $\sum_{\mu\in\Lambda} [N_\mu : S] \nsubseteq P$, and hence there exist $\mu \in \Lambda$ and $p \in P$ such that $(1-p)S \subseteq N_\mu$. Let K be a submodule of M. If $\lambda = \mu$, then

$$[K_P : N_{\lambda P}] = [K_P : S_P] = [K : S]_P \subseteq [K : N_\lambda]_P \subseteq [K_P : N_{\lambda P}],$$

and by [2, Theorem 2.1], N_λ is multiplication. Otherwise, $\lambda \neq \mu$, and hence $(1-p)N_\lambda \subseteq N_\lambda \cap N_\mu$. It follows that $(N_\lambda \cap N_\mu)_P = N_{\lambda P}$, and hence

$$\begin{aligned}[K_P : N_{\lambda P}] &= [K_P : (N_\lambda \cap N_\mu)_P] \\ &= [K : (N_\lambda \cap N_\mu)]_P \subseteq [K : (1-p)N_\lambda]_P = [K : N_\lambda]_P \subseteq [K_P : N_{\lambda P}],\end{aligned}$$

and again by [2, Theorem 2.1], N_λ is multiplication. □

We remark that this same method applying Anderson's new characterization of multiplication modules may be used for example to give alternative proofs for our results 2.3, 3.2 and 3.6.

Naoum and Hasan [16, Theorem 2.5] proved that if R is an arithmetical ring and A and B are finitely generated ideals in R such that $\operatorname{ann}(B)$ is finitely generated, then $[A : B]$ is finitely generated and hence a multiplication ideal. Patrick Smith [18, Theorem 10] showed that if M is a finitely generated faithful multiplication module, then N is a multiplication submodule of M if and only if $[N : M]$ is multiplication. Compare Smith's theorem with the following in which Anderson's characterization of multiplication modules is applied to give an alternative proof. An R-module M is torsion if $\operatorname{ann}(m) \neq 0$ for all $m \in M$, otherwise it is called non-torsion [17]. Clearly every non-torsion module is faithful.

Theorem 4.3. *Let R be a ring and B a non-torsion multiplication submodule of an R-module M. Then*

(i) *$I = [IB : B]$ for every ideal I of R,*
(ii) *a submodule A of B is multiplication if and only if $[A : B]$ is a multiplication ideal of R,*
(iii) *IB is a multiplication module if and only if I is a multiplication ideal of R,*
(iv) *a submodule A of B is non-torsion if and only if $[A : B]$ is non-torsion.*

Proof. (i) Let P be a maximal ideal of R. Since B is non-torsion, $B_P \neq 0_P$. Since B is multiplication, it follows that $IB = [IB : B]B$, and from [2, Theorem 2.1] that $\operatorname{ann}(B_P) = \operatorname{ann}(B)_P = 0_P$. Hence, $I_P B_P = [IB : B]_P B_P$. Since B_P is cyclic, we conclude from [18, Corollary to Theorem 9] that $I_P + \operatorname{ann}(B_P) = [IB : B]_P + \operatorname{ann}(B_P)$. But $\operatorname{ann}(B_P) = 0_P$, so $I_P = [IB : B]_P$. Since the equality holds locally, it holds globally.

(ii) Assume A is a multiplication submodule of B, and suppose P is a maximal ideal of R with $[A : B]_P \neq 0_P$. Since B is non-torsion, we infer that $B_P \neq 0_P$ and $[A : B]_P = [A_P : B_P]$. Hence, $A_P \neq 0_P$. It is easy to show that $[A_P : B_P]$ is multiplication, and hence $[A : B]_P$ is principal. Let I be an ideal of R. Since B is non-torsion and multiplication, it is easy to show that $[IB : A] = [I : [A : B]]$. Since A is multiplication,

$$[I_P : [A : B]_P] = [I_P : [A_P : B_P]] = [I_P B_P : A_P] = [IB : A]_P = [I : [A : B]]_P.$$

It follows from [2, Theorem 2.1] that $[A : B]$ is multiplication. The converse follows by [4, Corollary 1.4].

(iii) follows from (i) and (ii), (iv) is routine. □

Finally, we observe that it follows easily from Lemma 1.3 that every non-torsion multiplication module over any ring R is a cancellation module and hence is finitely generated. On the other hand if R is an integral domain, then any faithful multiplication R-module M is non-torsion. In fact, since M is multiplication, $M = M\theta(M)$, so that $\theta(M) \neq 0$. For some $m \in M$, $[Rm : M] \neq 0$. Hence $rM \subseteq Rm$ for some $0 \neq r \in R$, so that $\operatorname{ann}(m) \subseteq \operatorname{ann}(rM) = 0$.

References

[1] Anderson, D. D.: *Some remarks on multiplication ideals.* Math. Japonica **4** (1980), 463–469.

[2] Anderson, D. D.: *Some remarks on multiplication ideals II.* Comm. in Algebra **28** (2000), 2577–2583.

[3] Barnard, A.: *Multiplication modules.* J. Algebra **71** (1981), 174–178.

[4] El-Bast, Z. A.; Smith, P. F.: *Multiplication modules.* Comm. in Algebra **16** (1988), 755–779.

[5] Fontana, M.; Huckaba, J.; Papick, I.: *Prüfer Domains.* Marcel Dekker 1997.

[6] Fuchs, L.: *Über die Ideale arithmetischer Ringe.* Comment. Math. Helv. **23** (1949), 334–341.

[7] Gilmer, R.: *Multiplicative Ideal Theory.* Kingston, Queen's 1992.

[8] Hutchins, H. C.: *Examples of Commutative Rings.* New Jersey 1981.

[9] Jensen, C. U.: *A remark on arithmetical rings.* Proc. Amer. Math. Soc. **15** (1964), 951–954.

[10] Jensen, C. U.: *Arithmetical rings.* Acta Math. Acad. Sci. Hungar. **17** (1966), 115–123.

[11] Kaplansky, I.: *Commutative Rings.* Boston 1970.

[12] Larsen, M. D.; McCarthy, P. J.: *Multiplicative Theory of Ideals.* New York 1971.

[13] Lüneburg, H.: *Was ist das kleinste gemeinsame Vielfache zweier rationaler Zahlen?* Math. Phys. Sember. **23** (1976), 6–13.

[14] Lüneburg, H.: *Intorno ad una questione aritmetica in un dominio di Prüfer.* Ricerche di Matematica **38** (1989), 249–259.

[15] McCasland, R. L.; Moore, M. E.: *On radicals of submodules of finitely generated modules.* Canad. Math. Bull. **29** (1986), 37–39.

[16] Naoum, A. G.; Hasan, M. A. K.: *The residual of finitely generated multiplication modules.* Arch. der Math. **46** (1986), 225–230.

[17] Ribenboim, P.: *Algebraic Numbers.* Wiley 1972.

[18] Smith, P. F.: *Some remarks on multiplication modules.* Arch. der Math. **50** (1988), 223–235.

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