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About a Determinant of Rectangular $2 \times n$ Matrix and its Geometric Interpretation

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Abstract. A determinant of rectangular $2 \times n$ matrix is considered. Some of its properties in connection with geometric interpretation are stated in this paper.

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1. Introduction

In [2] the following definition of a determinant of rectangular matrix is given: The determinant of a $m \times n$ matrix A with columns $A_1, \dots, A_n$ and $m \leq n$, is the sum

$$\sum_{1 \leq j_1 < j_2 < \cdots < j_m \leq n} (-1)^{r+s} |A_{j_1}, \dots, A_{j_m}|, \tag{1.1}$$

where $r = 1 + \cdots + m$, $s = j_1 + \cdots + j_m$.

This determinant is a skew-symmetric multilinear functional with respect to the rows and therefore has many well-known standard properties, for example, the general Laplace's expansion along rows.

Here are some examples.

Example 1. Let $[a_1, a_2, a_3]$ be a 1×3 matrix. Then by (1.1) we have

$$|a_1, a_2, a_3| = (-1)^{1+1} a_1 + (-1)^{1+2} a_2 + (-1)^{1+3} a_3 = a_1 - a_2 + a_3.$$

Example 2. Let $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$ be a 2×3 matrix. Then

$$\begin{aligned}
\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} &= (-1)^{(1+2)+(1+2)} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} + (-1)^{(1+2)+(1+3)} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \\
&+ (-1)^{(1+2)+(2+3)} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}.
\end{aligned}$$

Using Laplace's expansion along first row we have

$$\begin{aligned}
\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} &= a_1(-1)^{1+1}|b_2, b_3| + a_2(-1)^{1+2}|b_1, b_3| + a_3(-1)^{1+3}|b_1, b_2| \\
&= a_1 b_2 - a_2 b_1 - (a_1 b_3 - a_3 b_1) + a_2 b_3 - a_3 b_2.
\end{aligned}$$

Of course, $|b_i, b_j| = b_i - b_j$ by definition given by (1.1).

In this paper we shall consider in more detail the special case of the determinant given by (1.1) for $m = 2$:

$$\begin{vmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{vmatrix} = \sum_{1 \le i < j \le n} (-1)^{1+2+(i+j)} \begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix} \tag{1.2}$$

In this case, as will be seen, the determinant may serve as an elegant tool for some problems concerning polygons in a plane. For convenience in the following expression we shall call it *generalized determinant* or g-*determinant.*

2. Some properties of the g-determinant and their geometric interpretation

Our aim in this section is to prove in a simple way some important properties of g-determinant given by (1.2) in connection with its geometric interpretation.

First about notation which will be used.

Let $A_1 \cdots A_n$ be a polygon in the plane R^2 and let $A_i(x_i, y_i), i = 1, \dots, n$. Then g-determinant

$$\begin{vmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{vmatrix}$$

will also be written in each of the following two ways:

$$\det(A_1, \dots, A_n),\ |A_1, \dots, A_n|.$$

Also let us remark that every g-determinant

$$\begin{vmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{vmatrix}$$

will be often briefly written as $|A_1, \dots, A_n|$, where $A_1, \dots, A_n$ are columns of the corresponding matrix.

Now we shall consider the following g-determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \end{vmatrix}$$

or shorter written $|A_1, A_2, A_3, A_4, A_5|$. According to (1.2) for $n = 5$ we can write

$$\begin{aligned} |A_1, A_2, A_3, A_4, A_5| = |A_1, A_2| - |A_1, A_3| + |A_1, A_4| - |A_1, A_5| + \\ |A_2, A_3| - |A_2, A_4| + |A_2, A_5| + \\ +|A_3, A_4| - |A_3, A_5| + \\ + |A_4, A_5| \end{aligned}$$

or

$$\begin{aligned} |A_1, A_2, A_3, A_4, A_5| = |A_1, A_2 - A_3 + A_4 - A_5| + \\ |A_2, A_3 - A_4 + A_5| + \\ |A_3, A_4 - A_5| + \\ |A_4, A_5|. \end{aligned} \tag{2.1}$$

Let us remark that, for example, it holds

$$|A_1, A_2 - A_3 + A_4 - A_5| = |A_1, A_2| - |A_1, A_3| + |A_1, A_4| - |A_1, A_5|.$$

Now we can state the following theorem.

Theorem 1. *Let $|A_1, \dots, A_n|$ be a $2 \times n$ matrix with $n \geq 2$. Then*

$$\begin{aligned} |A_1, A_2, \dots, A_n| = |A_1, A_2 - A_3 + A_4 - \cdots + (-1)^n A_n| + \\ |A_2, A_3 - A_4 + \cdots + (-1)^{n-1} A_n| + \\ \cdots\cdots\cdots \\ |A_{n-1}, A_n|. \end{aligned} \tag{2.2}$$

Proof. Follows directly from the definition given by (1.2). For example, if $n = 5$, then holds (2.1). □

Here let us remark that using this theorem can be easily seen that for g-determinant of $2 \times n$ matrix holds Laplace's expansion along row. So, using equality (2.1) it is easy to see that

$$\begin{aligned} \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \end{vmatrix} &= \begin{vmatrix} a_1, & a_2 - a_3 + a_4 - a_5 \\ b_1, & b_2 - b_3 + b_4 - b_5 \end{vmatrix} + \begin{vmatrix} a_2, & a_3 - a_4 + a_5 \\ b_2, & b_3 - b_4 + b_5 \end{vmatrix} \\ &\quad + \begin{vmatrix} a_3, & a_4 - a_5 \\ b_3, & b_4 - b_5 \end{vmatrix} + \begin{vmatrix} a_4, & a_5 \\ b_4, & b_5 \end{vmatrix} \\ &= a_1(b_2 - b_3 + b_4 - b_5) - a_2(b_1 - b_3 + b_4 - b_5) \\ &\quad + a_3(b_1 - b_2 + b_4 - b_5) - a_4(b_1 - b_2 + b_3 - b_5) \\ &\quad + a_5(b_1 - b_2 + b_3 - b_4). \end{aligned}$$

Theorem 2. *It holds*

$$|A_1, A_2, \ldots, A_{n-1}, A_n| = |A_1, A_2, \ldots, A_{n-1}| + (-1)^n |A_1 - A_2 + \cdots + (-1)^n A_{n-1}, A_n|. \quad (2.3)$$

Proof. It is easy to see that

$$|A_1, A_2, A_3| = |A_1, A_2| - |A_1 - A_2, A_3|,$$

$$|A_1, A_2, A_3, A_4| = |A_1, A_2, A_3| + |A_1 - A_2 + A_3, A_4|,$$

$$|A_1, A_2, A_3, A_4, A_5| = |A_1, A_2, A_3, A_4| - |A_1 - A_2 + A_3 - A_4, A_5|$$

and so on. □

Remark 1. For convenience in the following we shall suppose that a considered polygon is positively oriented, that is, numeration of its vertices is such that corresponding determinant is not negative.

Theorem 3. *Let $A_1 \cdots A_n$ be a polygon in R^2. Then*

$$2 \text{ area of } A_1 \cdots A_n = |A_1 + A_2, A_2 + A_3, \ldots, A_{n-1} + A_n, A_n + A_1|.$$

Proof. It is well known that

$$2 \text{ area of } A_1 \cdots A_n = |A_1, A_2| + |A_2, A_3| + \cdots + |A_{n-1}, A_n| + |A_n, A_1|.$$

Thus, we have to prove that

$$\begin{aligned} &|A_1 + A_2, A_2 + A_3, \ldots, A_{n-1} + A_n, A_n + A_1| = \\ &\qquad |A_1, A_2| + |A_2, A_3| + \cdots + |A_{n-1}, A_n| + |A_n, A_1|. \end{aligned} \quad (2.4)$$

The proof will use the method of mathematical induction.

First we have that Theorem 3 holds for $n = 3$, that is

$$|A_1 + A_2, A_2 + A_3, A_3 + A_1| = |A_1, A_2| + |A_2, A_3| + |A_3, A_1|.$$

Supposing that (2.4) holds for a given $n \geq 3$ and using Theorem 2 (where now are not columns $A_1, A_2, \ldots$ but $A_1 + A_2, A_2 + A_3, \ldots$) we can write

$$\begin{aligned} &|A_1 + A_2, A_2 + A_3, \ldots, A_{n-1} + A_n, A_n + A_1| \\ &= |A_1 + A_2, A_2 + A_3, \ldots, A_{n-1} + A_n| + (-1)^n |A_1 + (-1)^n A_n, A_n + A_1| \\ &= |A_1 + A_2, A_2 + A_3, \ldots, A_{n-1} + A_n| + (-1)^n |A_1, A_n| + |A_n, A_1|, \end{aligned} \quad (2.5)$$

$$
\begin{aligned}
&|A_1 + A_2, A_2 + A_3, \ldots , A_{n-1} + A_n, A_n + A_{n+1}, A_{n+1} + A_1| \\
=&|A_1 + A_2, A_2 + A_3, \ldots , A_{n-1} + A_n| + |A_n + A_{n+1}, A_{n+1} + A_1| \\
&+ (-1)^n|A_1 + (-1)^n A_n, A_n + A_{n+1} - A_{n+1} - A_1| \\
=&|A_1 + A_2, A_2 + A_3, \ldots , A_{n-1} + A_n| + |A_n, A_{n+1}| + |A_n, A_1| + |A_n, A_1| \\
&+ (-1)^n|A_1, A_n| - |A_n, A_1| \\
=&|A_1 + A_2, A_2 + A_3, \ldots , A_{n-1} + A_n| + (-1)^n|A_1, A_n| + |A_n, A_1| \\
&+ |A_n, A_{n+1}| + |A_{n+1}, A_1| - |A_n, A_1|,
\end{aligned}
$$

from which according to (2.5) and (2.4) it follows that

$$
\begin{aligned}
&|A_1 + A_2, A_2+A_3, \ldots , A_n + A_{n+1}, A_{n+1} + A_1| = \\
&\qquad |A_1, A_2| + |A_2, A_3| + \cdots + |A_n, A_{n+1}| + |A_{n+1}, A_1|. \qquad \square
\end{aligned}
$$

Here is one more way of proving of Therorem 3 using method of mathematical induction. It may be interesting that induction from n to $n + 1$ may be as follows:

$$
\begin{aligned}
&|A_1, A_2| + |A_2, A_3| + \cdots + |A_{n-1}, A_n| + |A_n, A_1| + |A_1, A_n| + |A_n, A_{n+1}| + |A_{n+1}, A_1| = \\
&|A_1 + A_2, A_2 + A_3, \ldots , A_{n-1} + A_n, A_n + A_1| + |A_n + A_1, A_n + A_{n+1}, A_{n+1} + A_1| = \\
&|A_1 + A_2, A_2 + A_3, \ldots , A_{n-1} + A_n, A_n + A_{n+1}, A_{n+1} + A_1|
\end{aligned} \tag{2.6}
$$

since

$$
\begin{aligned}
&|A_1 + A_2, A_2 + A_3, \ldots , A_{p-2} + A_{p-1}, A_{p-1} + A_p| = \\
&\qquad = |A_1 + A_2, A_2 + A_3, \ldots , A_{p-2} + A_{p-1}| + |(-1)^{p-1}A_1 + A_{p-1}, A_{p-1} + A_p|
\end{aligned} \tag{2.7}
$$

and reducing both sides of (2.6) to

$$= |A_1 + A_2, A_2 + A_3, \ldots , A_{n-1} + A_n|.$$

The relation (2.7) follows directly from the given definition of g-determinant and may be interesting in itself.

In connection with Theorem 3 we shall also point out the following relations. First, using relation (2.1), we can write

$$
\begin{aligned}
|A_1 + A_2, A_2 + A_3, A_3 + A_4, A_5 + A_1| &= |A_1 + A_2, A_2 - A_1| \\
&+ |A_2 + A_3, A_3 + A_1| \\
&+ |A_3 + A_4, A_4 - A_1| \\
&+ |A_4 + A_5, A_5 + A_1| \\
= |A_1, A_2| + |A_2, A_3| + |A_3, A_4| &+ |A_4, A_5| + |A_5, A_1|.
\end{aligned}
$$

In the same way can be seen that

$$
\begin{aligned}
|A_1 + A_2, A_2 + A_3, \ldots, A_{n-1} + A_n, A_n + A_1| &= |A_1 + A_2, A_2 + jA_1| \\
&\quad + |A_2 + A_3, A_3 - jA_1| \\
&\quad + |A_3 + A_4, A_4 + jA_1| \\
&\quad \cdots\cdots\cdots\cdots \\
&\quad + |A_{n-2} + A_{n-1}, A_{n-1} - A_1| \\
&\quad + |A_{n-1} + A_n, A_n + A_1| \\
&= |A_1, A_2| + |A_2, A_3| + \cdots + |A_{n-1}, A_n| + |A_n, A_1|.
\end{aligned}
$$

where $j = 1$ if n is even and -1 if n is odd, that is, $j = (-1)^n$.

Now we shall state some of the corollaries of Theorem 3.

Corollary 3.1. *Let $B_1, \ldots, B_n$ be given by*

$$B_1 = \frac{A_1 + A_2}{2}, B_2 = \frac{A_2 + A_3}{2}, \ldots, B_n = \frac{A_n + A_1}{2}.$$

Then

$$4|B_1, B_2, \ldots, B_n| = |A_1, A_2| + |A_2, A_3| + \cdots + |A_n, A_1|.$$

Proof. The g-determinant given by (1.2) has the property that for every two numbers a and b it holds

$$\begin{vmatrix} ax_1 & \cdots & ax_n \\ by_1 & \cdots & by_n \end{vmatrix} = ab \begin{vmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{vmatrix}.$$

□

Corollary 3.2. *Let $A_1 \cdots A_n$ be a polygon in the Gauss plane and let $z_1, \ldots, z_n$ be complex numbers corresponding respectively to the vertices $A_1, \ldots, A_n$. Then*

$$2 \text{ area of } A_1 \cdots A_n = \frac{i}{2} \begin{vmatrix} z_1 + z_2 & z_2 + z_3 & \cdots & z_n + z_1 \\ \bar{z}_1 + \bar{z}_2 & \bar{z}_2 + \bar{z}_3 & \cdots & \bar{z}_n + \bar{z}_1 \end{vmatrix}$$

Proof. We shall use the property of g-determinant that its value is unchanged if a multiple of one row is added to the other row. Thus, we can write

$$
\begin{aligned}
&\begin{vmatrix} x_1 + x_2 + i(y_1 + y_2) & \cdots & x_n + x_1 + i(y_n + y_1) \\ x_1 + x_2 - i(y_1 + y_2) & \cdots & x_n + x_1 - i(y_n + y_1) \end{vmatrix} \\
&= 2 \begin{vmatrix} x_1 + x_2 + i(y_1 + y_2) & \cdots & x_n + x_1 + i(y_n + y_1) \\ 0 - i(y_1 + y_2) & \cdots & 0 - i(y_n + y_1) \end{vmatrix} \\
&= 2 \begin{vmatrix} x_1 + x_2 & \cdots & x_n + x_1 \\ -i(y_1 + y_2) & \cdots & -i(y_n + y_1) \end{vmatrix} \\
&= -2i \begin{vmatrix} x_1 + x_2 & \cdots & x_n + x_1 \\ y_1 + y_2 & \cdots & y_n + y_1 \end{vmatrix}.
\end{aligned}
$$

□

Definition 1. *Let $A_1 \cdots A_n$ and $B_1 \cdots B_n$ be polygons in R^2 such that*

$$B_j = \frac{A_j + A_{j+1} + \cdots + A_{j+k-1}}{k}, \ j = 1, \ldots, n. \tag{2.8}$$

Then the polygon $B_1 \cdots B_n$ is said to be k-inscribed to the polygon $A_1 \cdots A_n$ and the polygon $A_1 \cdots A_n$ is said to be k-outscribed to the polygon $B_1 \cdots B_n$.

For example, let $A_1A_2A_3$ be a triangle in the plane R^2 and let the triangle $A_1^{(2)}A_2^{(2)}A_3^{(2)}$ be defined by

$$A_1^{(2)} = A_1 - A_2 + A_3, \ A_2^{(2)} = A_2 - A_3 + A_1, \ A_3^{(2)} = A_3 - A_1 + A_2. \tag{2.9}$$

Then the triangle $A_1^{(2)}A_2^{(2)}A_3^{(2)}$ is 2-outscribed to the triangle $A_1A_2A_3$ since

$$A_1^{(2)} + A_2^{(2)} = 2A_1, \quad A_2^{(2)} + A_3^{(2)} = 2A_2, \quad A_3^{(2)} + A_1^{(2)} = 2A_3.$$

Generally, let $A_1 \cdots A_n$ be a polygon in R^2, where n is odd. Then the polygon $A_1^{(2)} \cdots A_n^{(2)}$ defined by

$$A_i^{(2)} = \sum_{j=0}^{n-1} (-1)^j A_{i+j}, \ i = 1, \ldots, n \tag{2.10}$$

is 2-outscribed to the polygon $A_1 \cdots A_n$. (Of course, indices $i + j$ are calculated modulo n.)

It is easy to see that the system

$$Z_1 + Z_2 = 2A_1, \ Z_2 + Z_3 = 2A_2, \ldots, \ Z_n + Z_1 = 2A_n$$

if n is odd, has the unique solution $Z_i = A_i^{(2)}, \ i = 1, \ldots, n$, where $A_i^{(2)}$ is given by (2.10). Thus, the polygon $A_1^{(2)} \cdots A_n^{(2)}$ given by (2.10) is the unique one which is 2-outscribed to the polygon $A_1 \cdots A_n$.

It may be interesting that points $A_i^{(2)}, \ i = 1, \ldots, n$, can be easily constructed. For example, if $n = 5$ then point $A_1^{(2)}$ can be constructed as shown in Figure 2.1. The quadrilaterals $A_1A_2A_3S$ and $SA_4A_5A_1^{(2)}$ are parallelograms. Let us remark that from

$$S = A_1 - A_2 + A_3, \quad S = A_4 - A_5 + A_1^{(2)}$$

follows $A_1^{(2)} = A_1 - A_2 + A_3 - A_4 + A_5$.

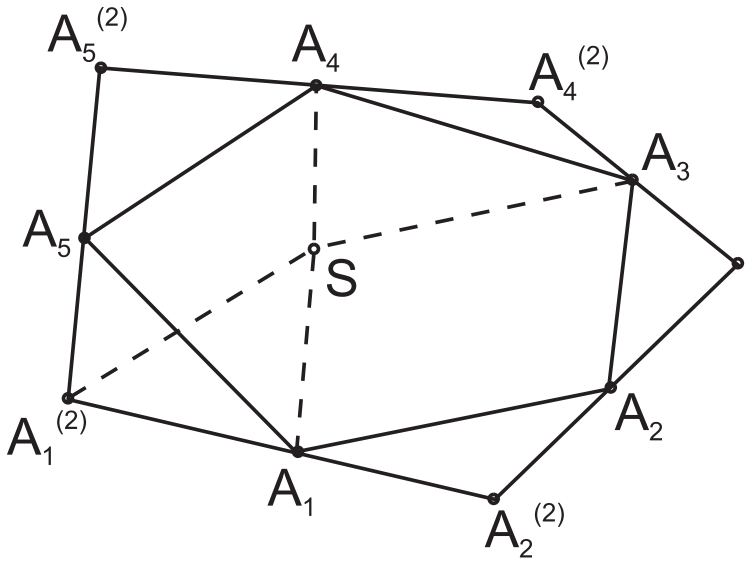

Figure 2.1

The point $A_2^{(2)}$ can be constructed such that A_1 be midpoint of $A_1^{(2)}A_2^{(2)}$, the point $A_3^{(2)}$ can be constructed such that A_2 be midpoint of $A_2^{(2)}A_3^{(2)}$, and so on.

If $n = 7$, the point $A_1^{(2)}$ can be constructed as shown in Figure 2.2. The quadrilaterals $A_1A_2A_3S_1, S_1A_4A_5S_2, S_2A_6A_7A_1^{(2)}$ are parallelograms. It follows that $A_1^{(2)} = A_1 - A_2 + A_3 - A_4 + A_5 - A_6 + A_7$.

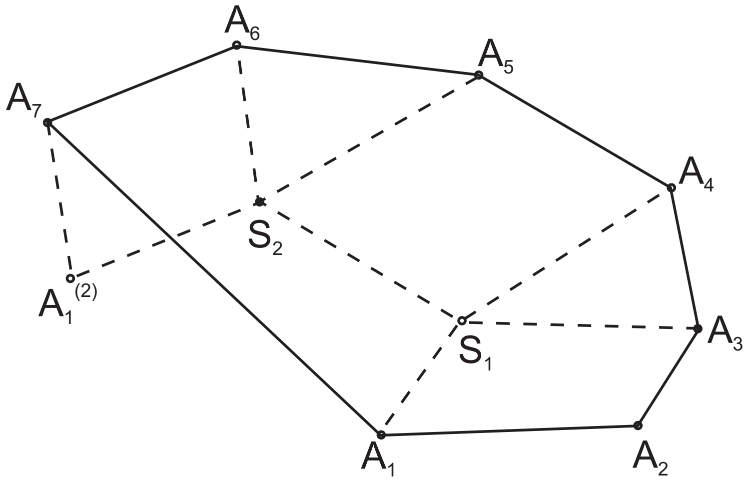

Figure 2.2

Theorem 4. *Let $A_1 \cdots A_n$ be a polygon in R^2 and let n be an odd integer. Then*

$$\text{area of } A_1^{(2)} \cdots A_n^{(2)} = 2|A_1, \ldots, A_n|, \tag{2.11}$$

where $A_i^{(2)}$, $i = 1, \ldots, n$, are given by (2.10).

Proof. Follows from Theorem 3 since $A_i^{(2)} + A_{i+1}^{(2)} = 2A_i,\ i = 1, \ldots, n.$ $\square$

Theorem 5. *Let $A_1 \cdots A_n$ be a polygon in R^2 with even n and let*

$$A_1 + A_3 + \cdots + A_{n-1} = A_2 + A_4 + \cdots + A_n$$

or

$$\sum_{i=1}^{n} (-1) A_i = 0. \tag{2.12}$$

Then there are infinitely many polygons which are 2-outscribed to the polygon $A_1 \cdots A_n$ and all of them have the same area given by

$$\text{area of } Z_1 \cdots Z_n = 2|A_1, \ldots, A_n| \tag{2.13}$$

where $Z_1 \cdots Z_n$ is any arbitrary polygon which is 2-outscribed to the polygon $A_1 \cdots A_n$.

In the case when (2.12) *is not fulfilled, then there is no polygon which is 2-outscribed to the polygon $A_1 \cdots A_n$.*

Proof. We shall first consider the case when $n = 4$. If

$$A_1 - A_2 + A_3 - A_4 = 0 \quad \text{or} \quad \frac{A_1 + A_3}{2} = \frac{A_2 + A_4}{2},$$

then quadrilateral $A_1A_2A_3A_4$ is a parallelogram. It is easy to see that in this case the following system

$$Z_1 + Z_2 = 2A_1, \quad Z_2 + Z_3 = 2A_2, \quad Z_3 + Z_4 = 2A_3, \quad Z_4 + Z_1 = 2A_4$$

has infinitely many solutions. Namely, we can write

$$\begin{aligned} Z_2 &= 2A_1 - Z_1, \\ Z_3 &= 2A_2 - Z_2 = 2A_2 - 2A_1 + Z_1, \\ Z_4 &= 2A_3 - Z_3 = 2A_3 - 2A_2 + 2A_1 - Z_1, \end{aligned} \tag{2.14}$$

where Z_1 may be chosen arbitrarily in R^2.

Let us remark that from (2.14), since $Z_4 + Z_1 = 2A_4$, it follows

$$2A_4 - 2A_3 + 2A_2 - 2A_1 = 0 \quad \text{or} \quad A_1 - A_2 + A_3 - A_4 = 0.$$

Generally, let $A_1 \cdots A_n$ be a polygon in R^2 with even n and let (2.12) be fulfilled. Then the system

$$Z_1 + Z_2 = 2A_1, \quad Z_2 + Z_3 = 2A_2, \quad \ldots, \quad Z_n + Z_1 = 2A_n$$

has infinitely many solutions since for every Z_1 in R^2 there are $Z_2, Z_3, \ldots, Z_n$ given by

$$\begin{aligned} Z_2 &= 2A_1 - Z_1, \\ Z_3 &= 2A_2 - 2A_1 + Z_1, \\ Z_4 &= 2A_3 - 2A_2 + 2A_1 - Z_1, \\ &\cdots\cdots\cdots\cdots \\ Z_n &= 2A_{n-1} - 2A_{n-2} + \cdots - 2A_2 + 2A_1 - Z_1. \end{aligned}$$

Let us remark that from the last equation, since $Z_n + Z_1 = 2A_n$, it follows that

$$2A_n - 2A_{n-1} + 2A_{n-2} - \cdots + 2A_2 - A_1 = 0.$$

Also, it is clear that, if (2.12) is not fulfilled, then there are no Z_1 and Z_n such that $Z_n + Z_1 = 2A_n$.

Concerning area of $Z_1 \cdots Z_n$, we can write

$$\begin{aligned} 2 \text{ area of } Z_1 \cdots Z_n &= |Z_1 + Z_2, Z_2 + Z_3, \ldots, Z_n + Z_1| \\ &= |2A_1, 2A_2, \ldots, 2A_n|, \end{aligned}$$

from which follows (2.13) □

In connection with even n let us point out the following.

In order to obtain a polygon $A_1 \cdots A_n$ which can be 2-outscribed, we can take an arbitrary polygon $A_1, \ldots, A_{n-1}$ in R^2, but then A_n must be chosen so that holds (2.12). It may be interesting that such obtained A_n is equal to $A_1^{(2)}$, where $A_1^{(2)}$ is the first vertex of the polygon $A_1^{(2)} \cdots A_{n-1}^{(2)}$ which is 2-outscribed to the polygon $A_1 \cdots A_{n-1}$. So, for example, the hexagon $A_1 \cdots A_5 A_1^{(2)}$ shown in Figure 2.1 can be 2-outscribed since

$$A_1 - A_2 + A_3 - A_4 + A_5 - A_1^{(2)} = 0.$$

The same holds for octagon $A_1 \cdots A_7 A_1^{(2)}$ shown in Figure 2.2 since

$$\sum_{i=1}^{7} (-1)^{i+1} A_i = A_i^{(2)}.$$

Also let us remark that for any given polygon $A_1 \cdots A_n$ in R^2 with even n it holds

$$\sum_{i=1}^{n} (-1)^{i+1} (A_i + A_{i+1}) = 0.$$

Thus, the polygon $B_1 \cdots B_n$, where $B_i = A_i + A_{i+1},\ i = 1, \ldots, n$, can be 2-outscribed.

Here are some examples.

Example 1. Let $A_1A_2A_3A_4$ be a quadrilateral shown in Figure 2.3 and let $B_i = \frac{A_i + A_{i+1}}{2}$, $i = 1, 2, 3, 4$. Then for every point Z_1 in R^2 there are Z_2, Z_3, Z_4 such that $\frac{Z_i + Z_{i+1}}{2} = B_i$, $i = 1, 2, 3, 4$.

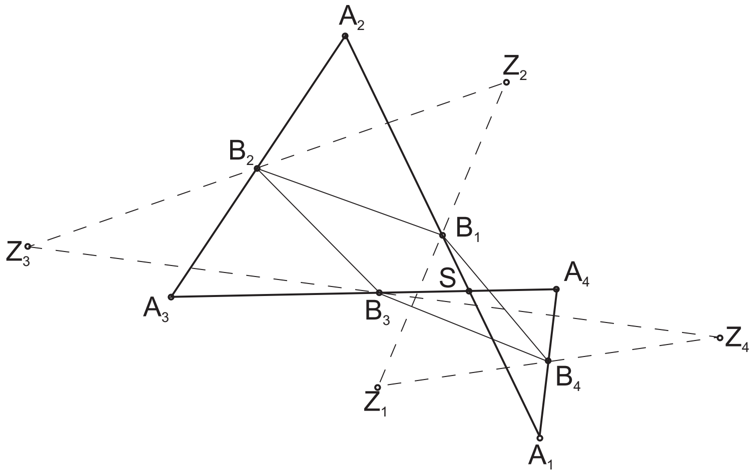

Figure 2.3

Every quadrilateral $Z_1Z_2Z_3Z_4$ which is 2-outscribed to the quadrilateral $B_1B_2B_3B_4$ has area equal to the area of the quadrilateral $A_1A_2A_3A_4$. Thus, it holds

$$\begin{aligned} 2 \text{ area of } Z_1Z_2Z_3Z_4 &= |A_1 + A_2, A_2 + A_3, A_3 + A_4, A_4 + A_1| \\ &= 4|B_1, B_2, B_3, B_4| \end{aligned}$$

or

$$\text{area of } Z_1Z_2Z_3Z_4 = 2|B_1B_2B_3B_4|.$$

Here let us remark that it is (by definition of area of an oriented polygon which has intersecting sides)

$$|A_1 + A_2, A_2 + A_3, A_3 + A_4, A_4 + A_1| = 2 \text{ area of } \Delta SA_2A_3 - 2 \text{ area of } \Delta SA_4A_1.$$

Example 2. Let $A_1 \cdots A_6$ be a hexagon shown in Figure 2.4 and let $B_i = \frac{A_i + A_{i+1}}{2}$, $i = 1, \dots, 6$. Then for every point Z_1 in R^2 there are $Z_2, \dots, Z_6$ such that $\frac{Z_i + Z_{i+1}}{2} = B_i$, $i = 1, \dots, 6$.

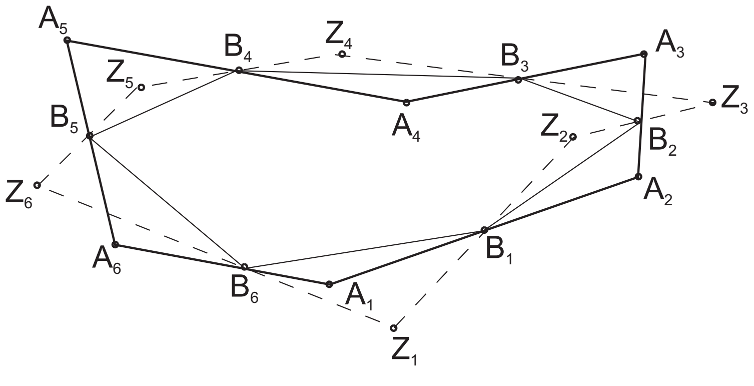

Figure 2.4

For every hexagon $Z_1 \cdots Z_6$ which is 2-outscribed to the hexagon $B_1 \cdots B_6$ it holds

$$\text{area of } Z_1 \cdots Z_6 = 2|B_1, \dots, B_6|.$$

Example 3. A polygon $A_1A_2A_3A_4$ with $A_2 = A_3$ will be 2-outscribed if $A_1 - A_2 + A_3 - A_4 = 0$, that is, if $A_1 = A_4$. (See Figure 2.5.) For every point Z_1 in R^2 there is a polygon $Z_1Z_2Z_3Z_4$ which is 2-outscribed to the polygon $B_1B_2B_3B_4$, where $B_i = \frac{A_i + A_{i+1}}{2}$, $i = 1, 2, 3, 4$.

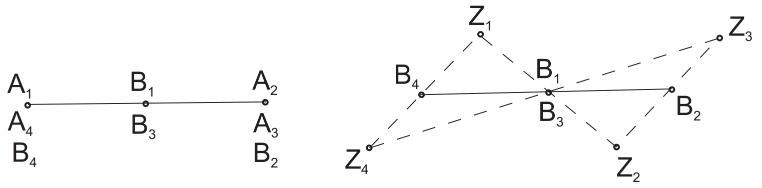

Figure 2.5

Of course, here we have

$$\begin{aligned} |A_1 + A_2, A_2 + A_3, A_3 + A_4, A_4 + A_1| &= |A_1, A_2| + |A_2, A_3| + |A_3, A_4| + |A_4, A_1| \\ &= |A_1, A_2| + |A_2, A_2| + |A_2, A_1| + |A_1, A_1| = 0. \end{aligned}$$

In this connection let us remark that triangles $B_1Z_2Z_3$ and $B_1Z_4Z_1$ are congruent and oppositely oriented.

Example 4. Let $A_1A_2A_3A_4$ be a quadrilateral such that $A_2 = A_3$ (Figure 2.6). Then

$$\begin{aligned} 2 \text{ area of } A_1A_2A_3A_4 &= |A_1 + A_2, A_2 + A_3, A_3 + A_4, A_4 + A_1| \\ &= |A_1, A_2| + |A_2, A_3| + |A_3, A_4| + |A_4, A_1| \\ &= |A_1, A_2| + |A_2, A_4| + |A_4, A_1| \\ &= 2 \text{ area of } \Delta A_1A_2A_4. \end{aligned}$$

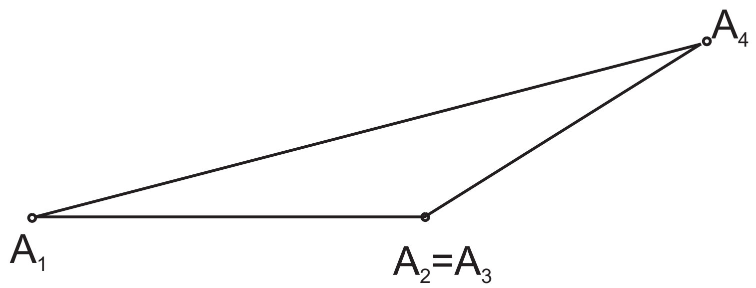

Figure 2.6

In connection with the case when n is even and holds (2.12), the following question arises: If $Z_1 \cdots Z_n$ is a polygon which is 2-outscribed to the polygon $A_1 \cdots A_n$, is there a polygon which is 2-outscribed to the polygon $Z_1 \cdots Z_n$? It is not difficult to show that there is only one polygon $Z_1 \cdots Z_n$ which is 2-outscribed to the polygon $A_1 \cdots A_n$ and has the property that there is a polygon which is 2-outscribed to the polygon $Z_1 \cdots Z_n$.

We begin with a quadrilateral $A_1A_2A_3A_4$ which is a parallelogram, that is, $A_1 - A_2 + A_3 - A_4 = 0$. Since

$$\begin{aligned} Z_2 &= 2A_1 - Z_1, \\ Z_3 &= 2A_2 - Z_2 = 2A_2 - 2A_1 + Z_1, \\ Z_4 &= 2A_3 - Z_3 = 2A_3 - 2A_2 + 2A_1 - Z_1 = 2A_4 - Z_1, \end{aligned}$$

the condition $Z_1 - Z_2 + Z_3 - Z_4 = 0$ will be satisfied only if

$$Z_1 = \frac{3A_1 - 2A_2 + A_3}{2}. \tag{2.15}$$

Thus, Z_1 is uniquely determined by A_1, A_2, A_3. We get

$$\begin{aligned} Z_2 &= \frac{3A_2 - 2A_3 + A_4}{2}, \\ Z_3 &= \frac{3A_3 - 2A_4 + A_1}{2}, \\ Z_4 &= \frac{3A_4 - 2A_1 + A_2}{2}. \end{aligned}$$

According to relation (2.13) we have

$$\text{area of } Z_1Z_2Z_3Z_4 = 2|A_1, A_2, A_3, A_4|.$$

Using Theorem 2 we can write

$$\begin{aligned} |A_1, A_2, A_3, A_4| &= |A_1, A_2, A_3| + |A_1 - A_2 + A_3, A_4| \\ &= |A_1, A_2, A_3| \text{ (since } A_4 = A_1 - A_2 + A_3) \\ &= |A_1, A_2| - |A_1, A_3| + |A_2, A_3| \\ &= |A_1, A_2| + |A_2, A_3| + |A_3, A_1| \\ &= 2 \text{ area of } \Delta A_1A_2A_3. \end{aligned}$$

Thus, area of $Z_1Z_2Z_3Z_4 = 2$ area of parallellogram $A_1A_2A_3A_4$. The parallelogram $Z_1Z_2Z_3Z_4$ is shown in Figure 2.7.

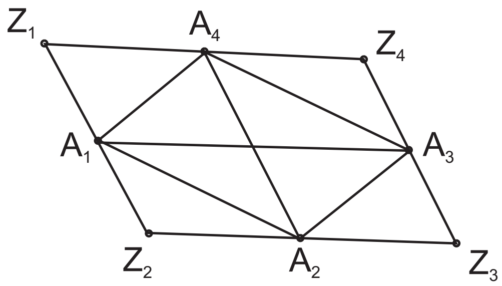

Figure 2.7

In the same way we find that for hexagon $A_1 \cdots A_6$, where $\sum_{i=1}^{6}(-1)^i A_i = 0$, will be satisfied $\sum_{i=1}^{6}(-1)^i Z_i = 0$ only if

$$Z_1 = \frac{5A_1 - 4A_2 + 3A_3 - 2A_4 + A_5}{3}.$$

Generally, if $A_1 \cdots A_n$ is a polygon with even n and (2.12) is satisfied, then $\sum_{i=1}^{n}(-1)^i Z_i = 0$ only if

$$Z_1 = \frac{(n-1)A_1 - (n-2)A_2 + (n-3)A_3 - \cdots + A_{n-1}}{\frac{n}{2}}.$$

Example 5. Let $A_1 \cdots A_6$ be a hexagon such that $\sum_{i=1}^{6}(-1)^i A_i = 0$. Then for every hexagon $Z_1 \cdots Z_6$ which is 2-outscribed to the hexagon $A_1 \cdots A_6$ it holds

$$\begin{aligned}\text{area of } Z_1 \cdots Z_6 &= 4 \text{ area of pentagon } A_1A_2A_3A_4A_5 \\ &\quad -4 \text{ area of pentagon } A_1A_3A_5A_2A_4.\end{aligned}$$

The vertex A_6 can be omitted since $A_6 = \sum_{i=1}^{5}(-1)^{i+1}A_i$. The proof is as follows. Since, by Theorem 5, area of $Z_1 \cdots Z_6 = 2|A_1, \ldots, A_6|$, we can write

$$\begin{aligned}|A_1, \ldots, A_6| &= |A_1, \ldots, A_5| + |A_1 - A_2 + A_3 - A_4 + A_5, A_6| \\ &= |A_1, \ldots, A_5| \\ &= |A_1, A_2| - |A_1, A_3| + |A_1, A_4| - |A_1, A_5| \\ &\qquad + |A_2, A_3| - |A_2, A_4| + |A_2, A_5| \\ &\qquad\qquad + |A_3, A_4| - |A_3, A_5| \\ &\qquad\qquad\qquad + |A_4, A_5| \\ &= |A_1, A_2| + |A_2, A_3| + |A_3, A_4| + |A_4, A_5| + |A_5, A_1| \\ &\quad -(|A_1, A_3| + |A_3, A_5| + |A_5, A_2| + |A_2, A_4| + |A_4, A_1|).\end{aligned}$$

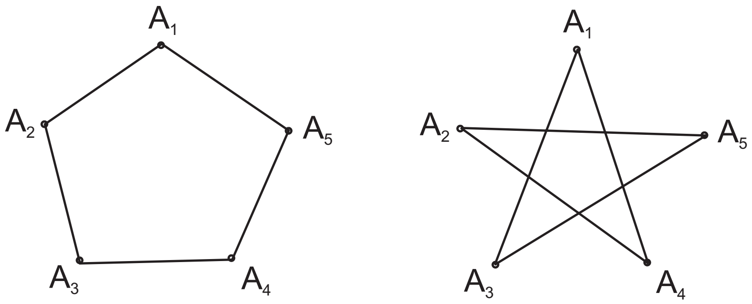

Figure 2.8

Example 6. Let $A_1 \cdots A_8$ be an octagon such that $\sum_{i=1}^{8}(-1)^i A_i = 0$. Then for every octagon $Z_1 \cdots Z_8$ which is 2-outscribed to the octagon $A_1 \cdots A_8$ it holds

$$\begin{aligned}\text{area of } Z_1 \cdots Z_8 &= 4 \text{ area of heptagon } A_1A_2A_3A_4A_5A_6A_7 \\ &\quad -4 \text{ area of heptagon } A_1A_3A_5A_7A_2A_4A_6 \\ &\quad +4 \text{ area of heptagon } A_1A_4A_7A_3A_6A_2A_5.\end{aligned}$$

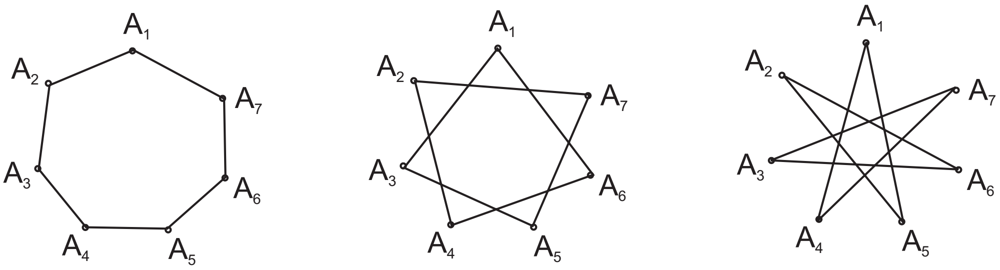

Figure 2.9

The proof is in the same way as that in Example 5.

Theorem 6. *Let $A_1 \cdots A_n$ be a polygon in R^2 and let n be an odd integer. Then*

$$2|A_1, \dots, A_n| = \sum_{j=1}^{n} |A_j, \sum_{i=1}^{n-1} (-1)^{i+1} A_{i+j}|. \tag{2.16}$$

Proof. Follows directly from the definition given by (1.2). □

Corollary 6.1. *If n is odd, then for every point X in R^2 it holds*

$$|A_1 + X, \dots, A_n + X| = |A_1, \dots, A_n|. \tag{2.17}$$

Proof. By Theorem 6 it holds

$$\begin{aligned} 2|A_1 + X, \dots, A_n + X| &= |A_1 + X, A_2 - A_3 + \cdots - A_n| + \\ &\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\quad |A_n + X, A_1 - A_2 + \cdots - A_{n-1}|. \end{aligned}$$

But

$$|X, (A_2 - A_3 + \cdots - A_n) + (A_3 - A_4 + \cdots - A_1) + \cdots + (A_1 - A_2 + \cdots - A_{n-1})| = 0$$

since

$$(A_2 - A_3 + \cdots - A_n) + (A_3 - A_4 + \cdots - A_1) + \cdots + (A_1 - A_2 + \cdots - A_{n-1}) = 0.$$

Here let us remark that (2.17) can also be proved using Laplace's expansion. So, for example, we can write (since $y - y + y - y = 0$)

$$\begin{aligned} &\begin{vmatrix} a_1 + x & a_2 + x & a_3 + x & a_4 + x & a_5 + x \\ b_1 + y & b_2 + y & b_3 + y & b_4 + y & b_5 + y \end{vmatrix} = \\ &(a_1 + x)(b_2 - b_3 + b_4 - b_5) - (a_2 + x)(b_1 - b_3 + b_4 - b_5) + \\ &(a_3 + x)(b_1 - b_2 + b_4 - b_5) - (a_4 + x)(b_1 - b_2 - b_3 - b_5) + \\ &(a_5 + x)(b_1 - b_2 + b_3 - b_4) = \\ &\begin{vmatrix} a_1 + x & a_2 + x & a_3 + x & a_4 + x & a_5 + x \\ b_1 & b_2 & b_3 & b_4 & b_5 \end{vmatrix} = \\ &-b_1(a_2 - a_3 + a_4 - a_5) + b_2(a_1 - a_3 + a_4 - a_5) - b_3(a_1 - a_2 + a_4 - a_5) \\ &+b_4(a_1 - a_2 + a_3 - a_5) - b_5(a_1 - a_2 + a_3 - a_4) = \\ &\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \end{vmatrix}. \end{aligned}$$

□

Theorem 7. *Let $A_1 \cdots A_n$ be a polygon in R^2 and let n be an even integer. Then for any point X in R^2 it holds*

$$|A_1 + X, A_2 + X, \dots, A_n + X| = |A_1, A_2, \dots, A_n| \tag{2.18}$$

only if $\sum_{i=1}^{n} (-1)^i A_i = 0$.

Proof. First it is clear that for any point P in R^2 it holds

$$|A_1, A_2, \ldots, A_n, P| = |A_1, A_2, \ldots, A_n| + |A_1 - A_2 + \cdots - A_n, P|$$

and that will be

$$|A_1, A_2, \ldots, A_n, P| = |A_1, A_2, \ldots, A_n| \tag{2.19}$$

only if $A_1 - A_2 + \cdots - A_n = 0$.

Now, by Corollary 6.1 (since $n+1$ is odd), taking $X = -P$, we can write

$$\begin{aligned} |A_1, \ldots, A_n, P| &= |A_1 + (-P), \ldots, A_n + (-P), P - P| \\ &= |A_1 + (-P), \ldots, A_n + (-P)| \end{aligned}$$

or, since (2.19) holds,

$$|A_1, \ldots, A_n| = |A_1 + (-P), \ldots, A_n + (-P)|.$$

Putting $X = -P$ we get (2.18). □

Corollary 7.1. *It holds*

$$|A_1 + A_2 + X, \ldots, A_n + A_1 + X| = |A_1 + A_2, \ldots, A_n + A_1|,$$

where $\sum_{i=1}^{n}(-1)^i A_i = 0$ *need not be fulfilled. (Only* n *must be even.)*

Proof. It holds $\sum_{i=1}^{n}(-1)^i (A_i + A_{i+1}) = 0$. □

Theorem 8. *Let* $A_1 \cdots A_n$ *be a polygon in* R^2 *with even* n *and let* $\sum_{i=1}^{n}(-1)^i A_i = 0$. *Then*

$$|A_1, \ldots, A_n| = |A_1, \ldots, A_{n-1}|.$$

Proof. It holds

$$|A_1, \ldots, A_n| = |A_1, \ldots, A_{n-1}| + |A_1 - A_2 + \cdots + A_{n-1}, A_n| = |A_1, \ldots, A_{n-1}|$$

since $A_1 - A_2 + \cdots + A_{n-1} = A_n$. □

Corollary 8.1. *Let* $A_1 \cdots A_n$ *be a polygon as stated in Theorem* 8. *Then the area of every polygon which is* 2*-outscribed to the polygon* $A_1 \cdots A_n$ *is given by* $2|A_1, \ldots, A_{n-1}|$.

For example, if $A_1A_2A_3A_4$ *is a parallelogram, then*

$$|A_1, A_2, A_3, A_4| = |A_1, A_2, A_3|\,.$$

Theorem 9. *Let* $A_1 \cdots A_n$ *be a polygon in* R^2 *with even* n *and let* $\sum_{i=1}^{n}(-1)^i A_i = 0$. *Then*

$$|A_1, \ldots, A_n| = |A_1, \ldots, A_k| + |A_{k+1}, \ldots, A_n|, \tag{2.20}$$

where k *may be any integer such that* $1 < k < n$.

Proof. It holds

$$|A_1, \dots, A_n| = |A_1, \dots, A_k| + |A_{k+1}, \dots, A_n| + \Delta,$$

where

$$\Delta = \Big|\sum_{i=1}^{k}(-1)^{i+1}A_i, \sum_{i=k+1}^{n}(-1)^i A_i\Big|.$$

But, if $\sum_{i=1}^{n}(-1)^i A_i = 0$, then $\Delta = 0$. □

For example, if $A_1A_2A_3A_4$ is a parallelogram, then

$$|A_1, A_2, A_3, A_4| = |A_1, A_2| + |A_3, A_4|.$$

Let us remark that by Theorem 8 it holds

$$|A_1, A_2, A_3, A_4| = |A_1, A_2, A_3|$$

and that

$$\begin{aligned}
|A_1, A_2, A_3| &= |A_1, A_2| - |A_1, A_3| + |A_2, A_3| \\
&= |A_1, A_2| + |-A_1 + A_2, A_3| \\
&= |A_1, A_2| + |A_3 - A_4, A_3| \text{ (since } -A_1 + A_2 = A_3 - A_4) \\
&= |A_1, A_2| + |-A_4, A_3| = |A_1, A_2| + |A_3, A_4|.
\end{aligned}$$

Theorem 10. *Let $A_1 \cdots A_n$ be a polygon in R^2 with odd n. Then*

$$|A_1, \dots, A_n| = |A_n, A_1, A_2, \dots, A_{n-1}|. \tag{2.21}$$

Proof. First it is clear that

$$|0, A_1, \dots, A_n| = |A_1, \dots, A_n, 0| = |A_1, \dots, A_n|$$

since

$$\begin{aligned}
|0, A_1, \dots, A_n| &= |0, A_1 - A_2 + \cdots + A_n| + |A_1, \dots, A_n|, \\
|A_1, \dots, A_n, 0| &= |A_1, \dots, A_n| + |A_1 - A_2 + \cdots + A_n, 0|.
\end{aligned}$$

Now, using Corollary 6.1, taking $X = -A_n$, we can write

$$\begin{aligned}
|A_1, \dots, A_n| &= |A_1 - A_n, A_2 - A_n, \dots, A_{n-1} - A_n, A_n - A_n| \\
&= |A_1 - A_n, A_2 - A_n, \dots, A_{n-1} - A_n| \\
&= |0, A_1 - A_n, A_2 - A_n, \dots, A_{n-1} - A_n|,
\end{aligned}$$

from which, adding A_n to each column in $|0, A_1 - A_n, A_2 - A_n, \dots, A_{n-1} - A_n|$ we get (2.21). □

Corollary 10.1. *If n is odd then for each $i \in \{1, \dots, n\}$ holds (cyclic)*

$$|A_i, \dots, A_n, A_1, \dots, A_{i-1}| = |A_1, \dots, A_n|.$$

Theorem 11. *Let $A_1 \cdots A_n$ be a polygon in R^2 with even n and let $\sum_{i=1}^n (-1)^i A_i = 0$. Then*

$$|A_1, \dots, A_n| = |A_n, A_1, \dots, A_{n-1}|.$$

Proof. By Theorem 7, taking $X = -A_n$, we can write

$$|A_1, \dots, A_n| = |A_1 - A_n, \dots, A_{n-1} - A_n, A_n - A_n|$$

and so on as in the proof of Theorem 10. □

Theorem 12. *Let $A_1 \cdots A_n$ be a polygon in R^2 with odd n and let k be an integer such that $1 \le k < n$. Then*

$$\left|\sum_{i=1}^{k} A_i, \sum_{i=2}^{k+1} A_i, \dots, \sum_{i=n}^{n+k-1} A_i\right| = \left|\sum_{i=1}^{n-k} A_i, \sum_{i=2}^{n-k+1} A_i, \dots, \sum_{i=n}^{2n-k-1} A_i\right|. \tag{2.22}$$

Proof. Follows from Corollary 6.1 and Corollary 10.1. Namely, if we add $X = -\sum_{i=1}^n A_i$ to each column of the g-determinant on the left of (2.22) we shall get the g-determinant on the right side. □

For example, if $k = 2$, we can write

$$\begin{aligned} |A_1 + A_2, \dots, A_5 + A_1| &= |A_1 + A_2 - \sum_{i=1}^{5} A_i, \dots, A_5 + A_1 - \sum_{i=1}^{5} A_i| \\ &= |-(A_3 + A_4 + A_5), \dots, -(A_2 + A_3 + A_4)| \\ &= |A_1 + A_2 + A_3, \dots, A_5 + A_1 + A_2|. \end{aligned}$$

So, for pentagon we have the following equalities:

$$|A_1 + A_2, \dots, A_5 + A_1| = |A_1 + A_2 + A_3, \dots, A_5 + A_1 + A_2|,$$
$$|A_1, \dots, A_5| = |A_1 + A_2 + A_3 + A_4, \dots, A_5 + A_1 + A_2 + A_3|.$$

In this connection let us remark that (since $m = 2$) it holds

$$\begin{vmatrix} -a_1, & \dots, & -a_n \\ -b_1, & \dots, & -b_n \end{vmatrix} = \begin{vmatrix} a_1, & \dots, & a_n \\ b_1, & \dots, & b_n \end{vmatrix}.$$

Corollary 12.1. *Let $B_1 \cdots B_n$ and $C_1 \cdots C_n$ be polygons in R^2 such that*

$$\begin{aligned} B_j &= \sum_{i=j}^{j+k-1} A_i, \ j = 1, \dots, n \\ C_j &= \sum_{i=j}^{j+n-k-1} A_i, \ j = 1, \dots, n. \end{aligned}$$

Then the area of the 2-outscribed polygon to the polygon $B_1 \cdots B_n$ is equal to the area of the 2-outscribed polygon to the polygon $C_1 \cdots C_n$.

Proof. Clear from Theorem 4. □

3. Some generalizations of similarity in the set of the polygons in a plane; pseudosimilarities

First about notation which will be used. For convenience in the following we shall, instead of the plane R^2 use the Gauss plane. So, if $z_1, \dots, z_n$ are given complex numbers, then by $z_1 \cdots z_n$ will be denoted the polygon whose vertices correspond to the complex numbers $z_1, \dots, z_n$ (in this order).

Let $\underline{A} = a_1 \cdots a_n$, $\underline{B} = b_1 \cdots b_n$ be given polygons in Gauss plane. Then will be written

$$\underline{A}(\mathrm{dir} \sim)\underline{B}$$

if there are complex numbers $\alpha \neq 0$, β such that

$$a_j = \alpha b_j + \beta,\ j = 1, \dots, n.$$

(One says: $\underline{A}$ is directly similar to $\underline{B}$.)

If $a_j = \alpha \bar{b}_j + \beta,\ j = 1, \dots, n$, then will be written $\underline{A}(\mathrm{ind} \sim)\underline{B}$.

For a given polygon $\underline{A} = a_1 \cdots a_n$, the polygon $\mathrm{Ker}^{(k)}\underline{A} = b_1 \cdots b_n$, so-called k-kernel of $\underline{A}$, is defined by

$$b_j = \frac{a_j + a_{j+1} + \cdots + a_{j+k-1}}{k},\ j = 1, \dots, n. \tag{3.1}$$

If there exists a unique polygon whose k-kernel is the polygon $\underline{A}$, let it be denoted by $\underline{A}^{(k)} = a_1^{(k)}, \dots, a_n^{(k)}$. For example, if $\underline{A} = a_1 a_2 a_3$, then $\underline{A}^{(2)} = a_1^{(2)} a_2^{(2)} a_3^{(2)}$, where

$$a_1^{(2)} = a_1 - a_2 + a_3,\ a_2^{(2)} = a_2 - a_3 + a_1,\ a_3^{(2)} = a_3 - a_1 + a_2. \tag{3.2}$$

(It is easy to show that for a given polygon $\underline{A} = a_1 \cdots a_n$ there exists polygon $A^{(k)}$ and is unique iff $GCD(k, n) = 1$.)

Sometimes it is convenient to write $(a_1, \dots, a_n)$ instead of $a_1 \cdots a_n$. So, the polygon given by (3.2) can be written as

$$(a_1 - a_2 + a_3, a_2 - a_3 + a_1, a_3 - a_1 + a_2).$$

We commence with triangle. The following theorem can be easily proved.

Theorem 13. *Two triangles $\underline{A} = a_1 a_2 a_3$, $\underline{B} = b_1 b_2 b_3$ are directly similar iff*

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0. \tag{3.3}$$

Proof. First let us suppose that $\underline{A}(\mathrm{dir} \sim)\underline{B}$, that is, there are $\alpha \neq 0$ and β such that

$$b_i = \alpha a_i + \beta,\ i = 1, 2, 3. \tag{3.4}$$

We have to prove that in this case holds (3.3). The proof is easy. Namely, we can use Corollary 6.1, taking $X = (0, -\beta)$, and write

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ \alpha a_1 + \beta & \alpha a_2 + \beta & \alpha a_3 + \beta \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ \alpha a_1 & \alpha a_2 & \alpha a_3 \end{vmatrix} = 0.$$

Conversely, if (3.3) is fulfilled, we can write

$$0 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_1 - a_3 & a_2 - a_3 & 0 \\ b_1 - b_3 & b_2 - b_3 & 0 \end{vmatrix} = \begin{vmatrix} a_1 - a_3 & a_2 - a_3 \\ b_1 - b_3 & b_2 - b_3 \end{vmatrix}$$
$$= (a_1 - a_3)(b_2 - b_3) - (a_2 - a_3)(b_1 - b_3).$$

Putting

$$\alpha = \frac{b_1 - b_3}{a_1 - a_3} = \frac{b_2 - b_3}{a_2 - a_3}$$

we get

$$b_1 = \alpha a_1 + b_3 - \alpha a_3,\ b_2 = \alpha a_2 + b_3 - \alpha a_3$$

or

$$b_1 = \alpha a_1 + \beta,\ b_2 = \alpha a_2 + \beta,\ b_3 = \alpha a_3 + \beta,$$

where $\beta = b_3 - \alpha a_3$. (So, $b_3 = \alpha a_3 + \beta$.) Theorem 13 is proved. □

In the same way it can be proved that two triangles $\underline{A} = a_1 a_2 a_3$ and $\underline{B} = b_1 b_2 b_3$ are oppositely similar iff

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ \overline{b}_1 & \overline{b}_2 & \overline{b}_3 \end{vmatrix} = 0. \tag{3.5}$$

Now we shall define some relations between polygons in the Gauss plane. With $\mathcal{P}$ we shall denote the set of all polygons in the Gauss plane, and by $\mathcal{P}_2$ will be denoted the set of all polygons in the Gauss plane which have even number of vertices. The set $\mathcal{P} \backslash \mathcal{P}_2$ will be shorter written as $\mathcal{P}_1$.

We may proceed in two ways.

First way: Let R_1 and R_2 be binary relations defined in $\mathcal{P}_1$ and $\mathcal{P}_2$ respectively, such that holds:

If $\underline{A} = a_1 \cdots a_n$ and $B = b_1 \cdots b_n$ are polygons in $\mathcal{P}_1$, then

$$\underline{A} R_1 \underline{B} \iff \begin{vmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{vmatrix} = 0.$$

But if $\underline{A} = a_1 \cdots a_n$ and $\underline{B} = b_1 \cdots b_n$ are polygons in $\mathcal{P}_2$, then

$$\underline{A} R_2 \underline{B} \iff \begin{vmatrix} a_1 + a_2 & a_2 + a_3 & \cdots & a_n + a_1 \\ b_1 + b_2 & b_2 + b_3 & \cdots & b_n + b_1 \end{vmatrix} = 0.$$

Second way: Let R be the binary relation defined in $\mathcal{P}$ such that holds: If $\underline{A} = a_1 \cdots a_n$ and $\underline{B} = b_1 \cdots b_n$ are polygons in $\mathcal{P}$, then

$$\underline{A} R \underline{B} \iff \begin{vmatrix} a_1 + a_2, & a_2 + a_3, & \cdots, & a_n + a_1 \\ b_1 + b_2, & b_2 + b_3, & \cdots, & b_n + b_1 \end{vmatrix} = 0. \tag{3.6}$$

As can be seen, the second way has an advantage in relation to the first since (3.6) can be used for even n and odd n. In this connection let us remark that Theorem 7 can be used since for even n

$$\sum_{i=1}^{n}(-1)^i \binom{a_i + a_{i+1}}{b_i + b_{i+1}} = \binom{0}{0}.$$

So, we shall proceed in the second way. The relation R in the following will be given by (3.6).

First it is easy to see that R is reflexive and symmetric relation, but is not transitive. Also it is clear that this relation contains the relation "direct similarity" in the set $\mathcal{P}$ since

$$\begin{vmatrix} a_1 + a_2 & , & \cdots, & a_n + a_1 \\ \alpha(a_1 + a_2) + 2\beta & , & \cdots, & \alpha(a_n + a_1) + 2\beta \end{vmatrix} = \begin{vmatrix} a_1 + a_2 & , & \cdots, & a_n + a_1 \\ \alpha(a_1 + a_2) & , & \cdots, & \alpha(a_n + a_1) \end{vmatrix} = 0.$$

So the relation R is a generalization of the relation "direct similarity" in the set $\mathcal{P}$. We shall investigate it in more detail. The following definition will be used.

Definition 2. *Let R_2 be the binary relation defined in $\mathcal{P}_2$ such that holds: If $\underline{A} = a_1 \cdots a_n$ and $\underline{B} = b_1 \cdots b_n$ are polygons in $\mathcal{P}_2$ then $\underline{A}R_2\underline{B}$ iff there are numbers $\alpha \neq 0$, β, γ so that*

$$b_i = \alpha a_i + \beta,\ i = 1, 3, \ldots, n-1 \tag{3.7}$$
$$b_j = \alpha a_j + \gamma,\ j = 2, 4, \ldots, n \tag{3.8}$$

where may be $\beta = \gamma$ but it is not necessary.

From (3.7) and (3.8) it follows that R_2 is an equivalence relation in $\mathcal{P}_2$. This relation is a subrelation of the relation R. The proof is as that for "direct similarity".

Theorem 14. *Let $\underline{A} = a_1 \cdots a_n$ and $\underline{B} = b_1 \cdots b_n$ be any given polygon in $\mathcal{P}_2$. Then*

$$\mathrm{Ker}^{(2)}\underline{A}(\mathrm{dir} \sim)\mathrm{Ker}^{(2)}\underline{B} \Longleftrightarrow \underline{A}R_2\underline{B}.$$

Proof. As already said, R_2 is an equivalence relation in $\mathcal{P}_2$. From (3.7) and (3.8) follows that

$$b_i + b_{i+1} = \alpha(a_i + a_{i+1}) + \beta + \gamma,\ i = 1, \ldots, n$$

which means that $\mathrm{Ker}^{(2)}\underline{A}(\mathrm{dir} \sim)\mathrm{Ker}^{(2)}\underline{B}$.

That also $\mathrm{Ker}^{(2)}\underline{A}(\mathrm{dir} \sim)\mathrm{Ker}^{(2)}\underline{B} \Longrightarrow \underline{A}R_2\underline{B}$ it can be seen that from

$$b_j + b_{j+1} = \alpha(a_j + a_{j+1}) + \delta,\ j = 1, \ldots, n$$

it follows

$$b_1 - \alpha a_1 = b_3 - \alpha a_3 = \cdots = b_{n-1} - \alpha a_{n-1},$$
$$b_2 - \alpha a_2 = b_4 - \alpha a_4 = \cdots = b_n - \alpha a_n,$$

where can be put

$$b_1 - \alpha a_1 = b_3 - \alpha a_3 = \cdots = b_{n-1} - \alpha a_{n-1} = \beta$$
$$b_2 - \alpha a_2 = b_4 - \alpha a_4 = \cdots = b_n - \alpha a_n = \gamma.$$

Theorem 14 is proved. □

An important property of the relation R given by (3.6) will be stated by the following theorem.

Theorem 15. *Let $\underline{A} = a_1 \cdots a_n$ and $\underline{B} = b_1 \cdots b_n$ be polygons in $\mathcal{P}$ such that for every $\underline{Z}$ for which $\underline{B}R\underline{Z}$ be also $\underline{A}R\underline{Z}$. Then $\underline{A}\rho\underline{B}$, where ρ is the union of the relation R_2 in $\mathcal{P}_2$ and the relation "direct similaritiy" (dir $\sim$) in $\mathcal{P}_1$.*

The converse is also true, namely, if $\underline{A}\rho\underline{B}$ then for every $\underline{Z}$ for which $\underline{B}R\underline{Z}$ is also $\underline{A}R\underline{Z}$.

Briefly expressed: $(\forall \underline{Z} \in \mathcal{P} : \underline{A}R\underline{Z} \Longleftrightarrow \underline{Z}R\underline{B}) \Longleftrightarrow \underline{A}\rho\underline{B}$.

Proof. First it is clear from Theorem 3 that the equation

$$\begin{vmatrix} x_1 + x_2, & x_2 + x_3, & \cdots, & x_n + x_1 \\ y_1 + y_2, & y_2 + y_3, & \cdots, & y_n + y_1 \end{vmatrix} = 0$$

can be written as

$$\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} + \cdots + \begin{vmatrix} x_n & x_1 \\ y_n & y_1 \end{vmatrix} = 0 \tag{3.9}$$

or

$$x_1(y_2 - y_n) + x_2(y_3 - y_1) + \cdots + x_{n-1}(y_n - y_{n-2}) + x_n(y_1 - y_{n-1}) = 0.$$

Now let $\underline{Z} = z_1 \cdots z_n$ be any polygon in $\mathcal{P}$ such that $\underline{Z}R\underline{B}$, that is

$$\begin{vmatrix} z_1 + z_2, & z_2 + z_3, & \cdots, & z_n + z_1 \\ b_1 + b_2, & b_2 + b_3, & \cdots, & b_n + b_1 \end{vmatrix} = 0$$

or

$$z_1(b_2 - b_n) + z_2(b_3 - b_1) + \cdots + z_{n-1}(b_n - b_{n-2}) + z_n(b_1 - b_{n-1}) = 0. \tag{3.10}$$

We have to find the condition that $\underline{A}R\underline{Z}$ be also valid. For this purpose we shall use the fact that every polygon $\underline{Z}$ which satisfies the condition $\underline{Z}R\underline{B}$ must be in the general solution of the equation (3.10) which may be expressed as

$$\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \alpha_1 \begin{bmatrix} b_1 - b_{n-1} \\ 0 \\ \vdots \\ b_n - b_2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ b_1 - b_{n-1} \\ \vdots \\ b_1 - b_3 \end{bmatrix} + \cdots + \alpha_{n-1} \begin{bmatrix} 0 \\ \vdots \\ b_1 - b_{n-1} \\ b_{n-2} - b_n \end{bmatrix}.$$

Using the above equations in $z_1 \cdots z_n$ we have to find the condition that for all complex numbers $\alpha_1, \ldots, \alpha_{n-1}$ is also valid $\underline{A}R\underline{Z}$, that is

$$a_1(z_2 - z_n) + a_2(z_3 - z_1) + \cdots + a_{n-1}(z_n - z_{n-2}) + a_n(z_1 - z_n) = 0$$

or

$$\begin{aligned} &\alpha_1[-a_1(b_n - b_2) - a_2(b_1 - b_{n-1}) + a_{n-1}(b_n - b_2) + a_n(b_1 - b_{n-1})] \\ &+\alpha_2[a_1(b_1 - b_{n-1}) - a_2(b_1 - b_3) - a_3(b_1 - b_{n-1}) + a_{n-1}(b_1 - b_3)] \\ &\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &+\alpha_{n-1}[-a_1(b_{n-2} - b_n) + a_{n-2}(b_1 - b_{n-1}) + a_{n-1}(b_{n-2} - b_n) - a_n(b_1 - b_{n-1})] = 0. \end{aligned}$$

The required condition we shall obtain by equalizing the expressions in the square brackets with zero. From these equations it follows

$$\frac{a_2 - a_n}{b_2 - b_n} = \frac{a_3 - a_1}{b_3 - b_1} = \cdots = \frac{a_n - a_{n-2}}{b_n - b_{n-2}} = \frac{a_1 - a_{n-1}}{b_1 - b_{n-1}}. \tag{3.11}$$

If the common value of the above fractions is denoted by α, we have the following equalities

$$\begin{aligned} &a_1 - \alpha b_1 = a_3 - \alpha b_3, \\ &a_2 - \alpha b_2 = a_4 - \alpha b_4, \\ &\dots\dots\dots\dots\dots\dots\dots \\ &a_{n-2} - \alpha b_{n-2} = a_n - \alpha b_n. \end{aligned}$$

Hence, if n is odd, it follows

$$\begin{aligned} &a_1 - \alpha b_1 = a_3 - \alpha b_3 = \cdots = a_n - \alpha b_n = \\ &a_2 - \alpha b_2 = a_4 - \alpha b_4 = \cdots = a_{n-1} - \alpha b_{n-1}. \end{aligned}$$

But, if n is even, then

$$\begin{aligned} &a_1 - \alpha b_1 = a_3 - \alpha b_3 = \cdots = a_{n-1} - \alpha b_{n-1} = \beta, \\ &a_2 - \alpha b_2 = a_4 - \alpha b_4 = \cdots = a_n - \alpha b_n = \gamma, \end{aligned}$$

where may not be $\beta = \gamma$.

It can be easily seen that the converse is also true, namely, inspection of the proof shows that it also works in the other direction. Theorem 15 is proved. $\square$

So we have that $R \supset \rho \supset R_2$. The relation R may be called *direct weak pseudosimilarity* in $\mathcal{P}$, and the relation ρ may be called *direct strong pseudosimilarity* in $\mathcal{P}$.

Concerning geometrical interpretation of the relation R we have the following theorem.

Theorem 16. *Let a_j and b_j in* (3.6) *be given by $a_j = x_j + iy_j$, $b_j = u_j + iv_j$, $j = 1, \dots, n$, and let the polygons $C_1 \cdots C_n$, $D_1 \cdots D_n$, $E_1 \cdots E_n$, $F_1 \cdots F_n$ be given by*

$$C_j(x_j, u_j),\ D_j(y_j, v_j),\ E_j(x_j, v_j),\ F_j(y_j, u_j),\ j = 1, \dots, n.$$

Then $\underline{A}R\underline{B}$ iff

$$\begin{aligned} &\text{area of } C_1 \cdots C_n = \text{area of } D_1 \cdots D_n, \\ &\text{area of } E_1 \cdots E_n = \text{area of } F_1 \cdots F_n. \end{aligned}$$

Proof. According to (2.4), the equality

$$\begin{vmatrix} a_1 + a_2, & a_2 + a_3, & \cdots, & a_n + a_1 \\ b_1 + b_2, & b_2 + b_3, & \cdots, & b_n + b_1 \end{vmatrix} = 0$$

can be written as

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} + \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} + \cdots + \begin{vmatrix} a_n & a_1 \\ b_n & b_1 \end{vmatrix} = 0$$

or

$$\sum_{j=1}^{n} \begin{vmatrix} x_j + iy_j, & x_{j+1} + iy_{j+1} \\ u_j + iv_j, & u_{j+1} + iv_{j+1} \end{vmatrix} = 0,$$

from which follows

$$\sum_{j=1}^{n} \left(\begin{vmatrix} x_j, & x_{j+1} \\ u_j, & u_{j+1} \end{vmatrix} - \begin{vmatrix} y_j, & y_{j+1} \\ v_j, & v_{j+1} \end{vmatrix} \right) = 0, \quad \sum_{j=1}^{n} \left(\begin{vmatrix} y_j, & y_{j+1} \\ u_j, & u_{j+1} \end{vmatrix} + \begin{vmatrix} x_j, & x_{j+1} \\ v_j, & v_{j+1} \end{vmatrix} \right) = 0,$$

since

$$\begin{vmatrix} x_j + iy_j, & x_{j+1} + iy_{j+1} \\ u_j + iv_j, & u_{j+1} + iv_{j+1} \end{vmatrix} = \begin{vmatrix} x_j, & x_{j+1} \\ u_j, & u_{j+1} \end{vmatrix} - \begin{vmatrix} y_j, & y_{j+1} \\ v_j, & v_{j+1} \end{vmatrix} + i \left(\begin{vmatrix} y_j, & y_{j+1} \\ u_j, & u_{j+1} \end{vmatrix} + \begin{vmatrix} x_j, & x_{j+1} \\ v_j, & v_{j+1} \end{vmatrix} \right).$$

Theorem 16 is proved. □

Here are some examples which may be interesting in itself.

Example 1. Let $\underline{A} = a_1 \cdots a_6$ be any given hexagon in $\mathcal{P}_2$ and let the hexagons $\underline{B} = b_1 \cdots b_6$ and $\underline{C} = c_1 \cdots c_6$ be defined by

$$\begin{aligned} b_i &= a_i + u_1,\ i = 1,3,5, \quad & b_i &= a_i + u_2,\ i = 2,4,6 \\ c_i &= a_i + v_1,\ i = 1,3,5, \quad & c_i &= a_i + v_2,\ i = 2,4,6 \end{aligned}$$

where u_1, u_2, v_1, v_2 are any given complex numbers. Then $\underline{B}R_2\underline{C}$.

It is clear from Definition 2.

Of course, analogously holds for any polygon $a_1 \cdots a_n$ in $\mathcal{P}_2$.

In this connection let us remark that area of $\underline{B}$ = area of $\underline{C}$. It can be seen from Corollary 3.2 since $\sum_{i=1}^{6}(-1)^i \begin{pmatrix} a_i + a_{i+1} + p \\ \overline{a}_i + \overline{a}_{i+1} + \overline{q} \end{pmatrix} = 0$ and Theorem 7 can be used.

Example 2. Let $\underline{A} = a_1a_2a_3a_4$ be any given quadrilateral in $\mathcal{P}_2$ which is not a parallelogram, that is $a_1 - a_2 + a_3 - a_4 \neq 0$, and let quadrilateral $b_1b_2b_3b_4$ be given by

$$b_1 = a_2 - a_3 + a_4,\ b_2 = a_3 - a_4 + a_1,\ b_3 = a_4 - a_1 + a_2,\ b_4 = a_1 - a_2 + a_3.$$

Then $\underline{A}R_2\underline{B}$ and area of $\underline{A}$ = area of $\underline{B}$. The proof is easy since

$$b_1 = a_1 - w,\ b_2 = a_2 + w,\ b_3 = a_3 - w,\ b_4 = a_4 + w$$

where $w = a_1 - a_2 + a_3 - a_4$.

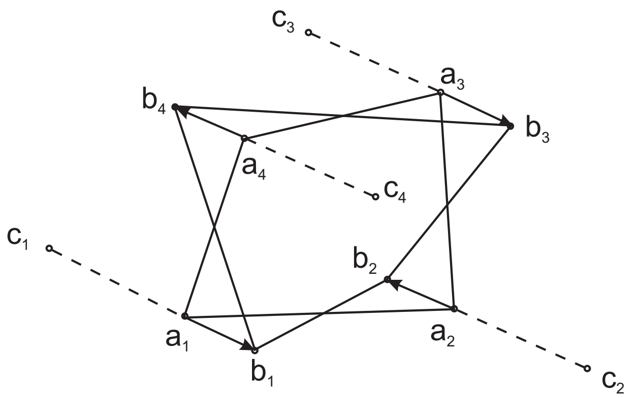

Figure 3.1

In this connection let us remark that

$$b_1a_2a_3a_4,\ b_2a_3a_4a_1,\ b_3a_4a_1a_2,\ b_4a_1a_2a_3$$

are parallelograms and that $\overrightarrow{a_1b_1} = \overrightarrow{b_2a_2} = \overrightarrow{a_3b_3} = \overrightarrow{b_4a_4}$ (see Figure 3.1).

If now, starting from quadrilateral $b_1b_2b_3b_4$, the quadrilateral $c_1c_2c_3c_4$ is given by $c_1 = b_2 - b_3 + b_4$, $c_2 = b_3 - b_4 + b_1$ and so on, and if we so proceed, then every two such obtained quadrilaterals are in relation R. Also may be interesting that vertices $a_i, b_i, c_i, \dots$ lie on the line a_ib_i, $i = 1, 2, 3, 4$.

In the same way can be seen that analogously holds for any polygon $a_1 \cdots a_n$ in $\mathcal{P}_2$ such that $w = \sum_{i=1}^n (-1)^i a_i \neq 0$. (The polygon can not be 2-outscribed.) So, $b_i = a_i - w$, $i = 1, 3, \dots, n-1$, $b_i = a_i + w$, $i = 2, 4, \dots, n$.

Example 3. Let $\underline{A} = a_1a_2a_3$ be any given triangle in $\mathcal{P}$ and let $\underline{B} = (a_2+a_3, a_3+a_1, a_1+a_2)$. Then, as will be shown, $\underline{A}^{(2)} R \underline{B}^{(2)}$, that is

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ a_2+a_3, & a_3+a_1, & a_1+a_2 \end{vmatrix} = 0. \tag{3.12}$$

First let us remark that

$$\underline{A}^{(2)} = (a_1 - a_2 + a_3, a_2 - a_3 + a_1, a_3 - a_1 + a_2),\ \underline{B}^{(2)} = (2a_2, 2a_3, 2a_1)$$

since

$$\mathrm{Ker}^{(2)}\underline{A}^{(2)} = (a_1, a_2, a_3) = \underline{A},\ \mathrm{Ker}^{(2)}\underline{B}^{(2)} = (a_2 + a_3, a_3 + a_1, a_1 + a_2) = \underline{B}.$$

The proof that $\underline{A}^{(2)} R \underline{B}^{(2)}$ is as follows:

$$\begin{aligned}
\begin{vmatrix} a_1 & a_2 & a_3 \\ a_2+a_3, & a_3+a_1, & a_1+a_2 \end{vmatrix} &= \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_1 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & a_3 \\ a_3 & a_1 & a_2 \end{vmatrix} \\
&= \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_1 \end{vmatrix} - \begin{vmatrix} a_3 & a_1 & a_2 \\ a_1 & a_2 & a_3 \end{vmatrix} \\
&= \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_1 \end{vmatrix} - \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_1 \end{vmatrix} = 0
\end{aligned}$$

since by Corollary 10.1 it holds

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_1 \end{vmatrix} = \begin{vmatrix} a_3 & a_1 & a_2 \\ a_1 & a_2 & a_3 \end{vmatrix}.$$

Here let us remark that in this example instead of $\underline{A}^{(2)} R \underline{B}^{(2)}$ can also be written $\underline{A}^{(2)}$ (dir $\sim$)$\underline{B}^{(2)}$.

Example 4. Let $\underline{A} = a_1a_2a_3a_4a_5$ be any given pentagon in $\mathcal{P}$ and let

$$\underline{B} = (a_2 + a_5, a_3 + a_1, a_4 + a_2, a_5 + a_3, a_1 + a_4),$$
$$\underline{C} = (a_3 + a_4, a_4 + a_5, a_5 + a_1, a_1 + a_2, a_2 + a_3).$$

Then

$$\underline{A}^{(2)} R \underline{B}^{(2)},\ \underline{A}^{(2)} R \underline{C}^{(2)}, \underline{B}^{(2)} R \underline{C}^{(2)},$$

that is

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_2 + a_5, & a_3 + a_1, & a_4 + a_2, & a_5 + a_3, & a_1 + a_4 \end{vmatrix} = 0, \tag{3.13}$$

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_3 + a_4, & a_4 + a_5, & a_5 + a_1, & a_1 + a_2, & a_2 + a_3 \end{vmatrix} = 0, \tag{3.14}$$

$$\begin{vmatrix} a_2 + a_5, & a_3 + a_1, & a_4 + a_2, & a_5 + a_3, & a_1 + a_4 \\ a_3 + a_4, & a_4 + a_5, & a_5 + a_1, & a_1 + a_2, & a_2 + a_3 \end{vmatrix} = 0. \tag{3.15}$$

The proof that hold (3.13) and (3.14) is in the same way as that in Example 7. The proof that holds (3.15) is similar. Namely, the determinant in (3.15) can be written as the sum

$$\begin{vmatrix} a_2 & a_3 & a_4 & a_5 & a_1 \\ a_3 & a_4 & a_5 & a_1 & a_2 \end{vmatrix} + \begin{vmatrix} a_2 & a_3 & a_4 & a_5 & a_1 \\ a_4 & a_5 & a_1 & a_2 & a_3 \end{vmatrix} + \begin{vmatrix} a_5 & a_1 & a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 & a_1 & a_2 \end{vmatrix} + \begin{vmatrix} a_5 & a_1 & a_2 & a_3 & a_4 \\ a_4 & a_5 & a_1 & a_2 & a_3 \end{vmatrix}.$$

The sum of the first and the fourth of the above four g-determinants is equal to zero since (by Corollary 10.1)

$$\begin{vmatrix} a_5 & a_1 & a_2 & a_3 & a_4 \\ a_4 & a_5 & a_1 & a_2 & a_3 \end{vmatrix} = - \begin{vmatrix} a_4 & a_5 & a_1 & a_2 & a_3 \\ a_5 & a_1 & a_2 & a_3 & a_4 \end{vmatrix} = - \begin{vmatrix} a_2 & a_3 & a_4 & a_5 & a_1 \\ a_3 & a_4 & a_5 & a_1 & a_2 \end{vmatrix}.$$

In the same way can be seen that the sum of the second and the third is equal to zero.

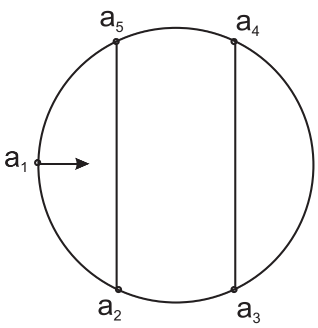

Figure 3.2

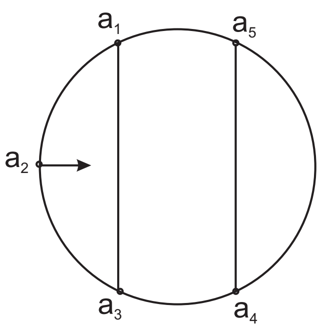

Figure 3.3

In connection with the determinants given by (3.13) and (3.14) let us remark that (where $\to$ can be read correspond to):

$$\begin{aligned} &a_1 \text{ in (3.13) } \to a_2 + a_5, \quad a_1 \text{ in (3.14) } \to a_3 + a_4 \text{ (Figure 3.2)},\\ &a_2 \text{ in (3.13) } \to a_3 + a_1, \quad a_2 \text{ in (3.14) } \to a_4 + a_5 \text{ (Figure 3.3)},\end{aligned}$$

and so on for $a_j,\ j = 3, 4, 5$. (See Figure 3.3. Of course, indices are calculated modulo 5.)

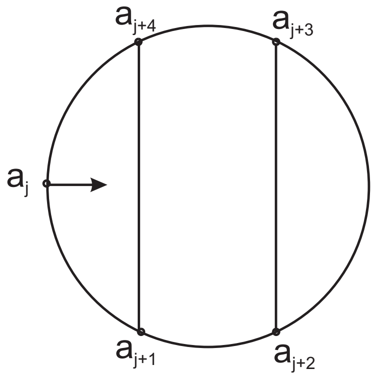

Figure 3.4

In connection with this example it may be interesting that $\underline{B}(\text{dir} \sim)\underline{A}^{(2)}$, namely, it holds

$$\begin{aligned} a_2 + a_5 &= \alpha(a_4 - a_5 + a_1 - a_2 + a_3) + \beta,\\ a_3 + a_1 &= \alpha(a_5 - a_1 + a_2 - a_3 + a_4) + \beta,\\ a_4 + a_2 &= \alpha(a_1 - a_2 + a_3 - a_4 + a_5) + \beta,\\ a_5 + a_3 &= \alpha(a_2 - a_3 + a_4 - a_5 + a_1) + \beta,\\ a_1 + a_4 &= \alpha(a_3 - a_4 + a_5 - a_1 + a_2) + \beta,\end{aligned}$$

where $\alpha = -\frac{1}{2}$, $\beta = \frac{1}{2}(a_1 + a_2 + a_3 + a_4 + a_5)$.

But we have not found that in one of the expressions $\underline{A}^{(2)} R \underline{B}^{(2)}$, $\underline{A}^{(2)} R \underline{C}^{(2)}$, $\underline{B}^{(2)} R \underline{C}^{(2)}$ can be put (dir $\sim$) instead of R (although at first sight may look like similarity).

Example 5. Let $\underline{A} = a_1 \cdots a_7$ be any given heptagon in $\mathcal{P}$ and let

$$\begin{aligned} \underline{B} &= (a_2 + a_7, a_3 + a_1, a_4 + a_2, a_5 + a_3, a_6 + a_4, a_7 + a_5, a_1 + a_6),\\ \underline{C} &= (a_3 + a_6, a_4 + a_7, a_5 + a_1, a_6 + a_2, a_7 + a_3, a_1 + a_4, a_2 + a_5),\\ \underline{D} &= (a_4 + a_5, a_5 + a_6, a_6 + a_7, a_7 + a_1, a_1 + a_2, a_2 + a_3, a_3 + a_4).\end{aligned}$$

Then any two of the heptagons $\underline{A}^{(2)}$, $\underline{B}^{(2)}$, $\underline{C}^{(2)}$, $\underline{D}^{(2)}$ are in relation R. The proof is in the same way as that in Example 4.

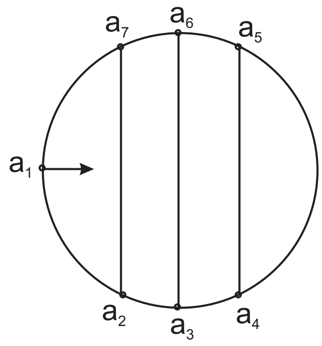

Figure 3.5

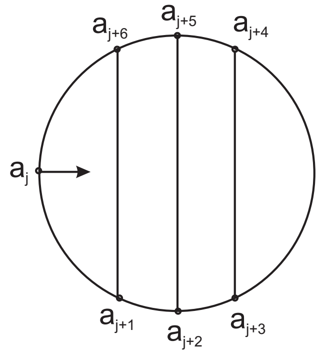

Figure 3.6

Here are some remarks concerning expressions $\underline{A}^{(2)}R\underline{B}^{(2)}$, $\underline{A}^{(2)}R\underline{C}^{(2)}$, $\underline{A}^{(2)}R\underline{D}^{(2)}$ given by

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ a_2+a_7, & a_3+a_1, & a_4+a_2, & a_5+a_3, & a_6+a_4, & a_7+a_5, & a_1+a_6 \end{vmatrix} = 0, \tag{3.16}$$

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ a_3+a_6, & a_4+a_7, & a_5+a_1, & a_6+a_2, & a_7+a_3, & a_1+a_4, & a_2+a_5 \end{vmatrix} = 0, \tag{3.17}$$

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ a_4+a_5, & a_5+a_6, & a_6+a_7, & a_7+a_1, & a_1+a_2 & a_2+a_3, & a_3+a_4 \end{vmatrix} = 0. \tag{3.18}$$

We see that a_1 in (3.16) $\to a_2 + a_7$, a_1 in (3.17) $\to a_3 + a_6$, a_1 in (3.18) $\to a_4 + a_5$ (Figure 3.5), and so on for a_j, $j = 2, \ldots, 7$ (see Figure 3.6).

Briefly told, we see that

$$a_i \to a_{i+k} + a_{i+7-k}, \ i = 1, \ldots, 7 \text{ and } k = 1, 2, 3$$

where $k = 1, 2, 3$ refers to (3.16), (3.17),(3.18) respectively.

In the same way it can be seen that analogously holds for any polygon $a_1 \cdots a_n$ in $\mathcal{P}$ with odd n.

Example 6. Let $\underline{A} = a_1 \ldots a_6$ be any given hexagon in $\mathcal{P}$ and let the hexagon $\underline{B} = b_1 \cdots b_6$ be given by

$$b_1 = a_3 + a_5, \quad b_3 = a_5 + a_1, \quad b_5 = a_1 + a_3,$$
$$b_2 = a_4 + a_6, \quad b_4 = a_6 + a_2, \quad b_6 = a_2 + a_4.$$

Then $\underline{A}R\underline{B}$ and area of $\underline{A}$ = area of $\underline{B}$. The proof is as follows.

Let $s = \sum_{i=1}^{6} a_i$. Then $b_i + b_{i+1} = -(a_i + a_{i+1}) + s$, $i = 1, \ldots, 6$. Using Theorem 7 (putting X with components $0, -s$), we can write

$$\begin{vmatrix} a_1+a_2, & \ldots, & a_6+a_1 \\ b_1+b_2, & \ldots, & b_6+b_1 \end{vmatrix} = - \begin{vmatrix} a_1+a_2, & \ldots, & a_6+a_1 \\ a_1+a_2, & \cdots, & a_6+a_1 \end{vmatrix} = 0.$$

In this connection the following may be interesting. Namely, if now, starting from the hexagon $b_1 \cdots b_6$, the hexagon $c_1 \cdots c_6$ is given by $c_1 = b_3 + b_5$, $c_2 = b_4 + b_6$ and so on, and if we so proceed, then every two such obtained hexagons are in relation R and have the same area.

In the same way it can be seen that analogously holds for any polygon $a_1 \cdots a_n$ in $\mathcal{P}$ with even n.

Remark 2. In this paper we touch only on some problems where the determinant given by (1.2) can be useful. Of course, there are many other such problems. But we hope that from this we have stated here can be seen that this determinant may be an elegant tool in some considerations concerning polygons.

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