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On Pairs of Non Measurable Linear Varieties in A_n

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Abstract. We consider a family of varieties, where each variety is a pair consisting of a hyperplane and a straight line in n-dimensional affine space A_n, where $n \geq 3$. Using Stoka's second condition, we show that this family is not measurable, therefore it is an example of a family of varietes in the sense of Dulio's classification [6] .

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1. Introduction

A measure on a family of geometric objects can be introduced by assigning to each object a point of an auxiliary space and considering a suitable measure on that space. In general the dimension of the auxiliary space is equal to the number of parameters on which the geometric objects depend. A basic problem is to specify measures which are invariant with respect to a given group of transformations which map the family onto itself.

This problem was first considered by Crofton [3] who specified the invariant measure on the family of all straight lines in Euclidean 2-space E^2. This was extended to E^3 by Deltheil [5] and Chern [1] first considered families of geometric objects in projective space.

Santalò [12] calculated measures of certain families of varieties with respect to three different groups and found that these were equal. Stoka [13] studied the family of parabolas. He proved that a family is measurable if it is measurable with respect to its maximal group of invariance

However Cirlincione [2] found a measurable family of varieties even though the family was not measurable with respect to the maximal group of invariance. This proves that the Stoka's condition is not necessary.

In Section 2 we provide background and definitions and in Section 3 we prove that the family of varieties, where each variety is a pair consisting of a hyperplane and a straight line in n-dimensional affine space A_n is not measurable.

2. Background

Let $\mathcal{H}_n$ be an $n-$dimensional space with coordinates $x_1, x_2, \dots, x_n$ in which a Lie group of transformations acts.

Let G_r be one of its subgroups defined by the equations

$$y_i = f_i(x_{1,}x_2, \dots, x_n; a_1, a_2, \dots, a_r) \qquad (i = 1, 2, \dots, n)$$

where $a_1, a_2, \dots, a_r$ are basic parameters.

Definition 1. *The function $F(x_{1,}x_2, \dots, x_n)$ is an integral invariant function of the group G_r, if*

$$\int_{\mathcal{A}_x} F(x_{1,}x_2, \dots, x_n)dx_1dx_2 \cdots dx_n = \int_{\mathcal{A}_y} F(y_1, y_2, \dots, y_n)dy_1dy_2 \cdots d_n$$

for each measurable set of points $\mathcal{A}_x$ of the space $\mathcal{H}_n$, where $\mathcal{A}_y$ is the image of $\mathcal{A}_x$ by the group G_r.

Theorem 1. *The integral invariant functions of the group G_r are the solutions of the following Deltheil's system of partial differential equations:*

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left[\xi_h^i(x)F(x)\right] = 0 \quad (h = 1, 2, \dots, r),$$

where $\xi_h^i(x)$ are the coefficients of the infinitesimal transformations of the group G_r (see [5], *p.* 28 *and* [15], *p.* 4*).*

Definition 2. *A measurable Lie group of transformations is a group which admits only one integral invariant function (up to a multiplicative constant).*

Let G be a group which leaves globally invariant a family $\Im$ of varietes in $\mathcal{H}_n$. To G there is associated a group H (isomorphic to G) of transformations acting on the (auxiliary) space of parameters of the family.

Definition 3. *A family $\Im$ is measurable with respect to G if H is measurable in the sense of Definition* 2*. If Φ is its integral invariant function, then the measure of $\Im$ with respect to the group G is given by*

$$\mu_G = \int_{\mathcal{A}_\alpha} \Phi(\alpha_{1,}\alpha_2, \dots, \alpha_q)d\alpha_1 d\alpha_2 \cdots d\alpha_q,$$

where $\mathcal{A}_\alpha$ *is the set of points of the auxiliary space which corresponds to the family* $\Im$.

Definition 4. *A family* $\Im$ *of varieties is measurable if the measures with respect to every group of invariance of the family are equal, if they exist.*

Theorem 2. (Stoka's first condition) *If the group* $\overline{H}$ *associated to the maximal group of invariance of* $\Im$ *(where the only transformation, which leaves invariant each element of the family, is the identity) is measurable, the family is measurable.*

Theorem 3. (Stoka's second condition) *If* $\overline{H}$ *is not measurable and there are two measurable subgroups with different integral invariant functions, then* $\Im$ *is not measurable.*

3. Non-measurability of the family $\Im_{3n-2}$

Theorem 4. *The family of varieties, where each variety is a pair consisting of a hyperplane and a straight line in n-dimensional affine space* A_n*, is not measurable.*

We use of the following notation

$$\begin{aligned} X^T &= (x_1, x_2, \ldots, x_n), \qquad B^T = (b_1, b_2, \ldots, b_n), \quad L^T = (l_1, l_2, \ldots, l_{n-1}, 1), \\ Q^T &= (q_1, q_2, \ldots, q_{n-1}, 0), \quad A^T = (a_1, a_2, \ldots, a_n). \end{aligned}$$

$\overline{X}^T$ is obtained from X by deleting the last coordinate and similary in other cases.

Let $\Im_{3n-2}$ be the family of all pairs, each consisting of a hyperplane and a straight line in A_n in general position. The hyperplane and the line depend on parameters $b_1, b_2, \ldots, b_n, l_1, l_2, \ldots, l_{n-1}, q_1, q_2, \ldots, q_{n-1}$, respectively, and are represented in the following form

$$\begin{aligned} &\Sigma_{i=1}^n b_i x_i = 1 \\ &x_i = l_i x_n + q_i \quad i = 1, 2, n-1. \end{aligned}$$

The affine group G_{n^2+n} is given by

$$x_i = \sum_{j=1}^{n} p_{ij} x'_j + a_i, \quad i, j = 1, 2, \ldots, n,$$

where $\det(p_{ij}) \neq 0$ and $\Sigma_{i=1}^n b_i a_i \neq 1$.

For the $n \times n$ matrix $P = (p_{ij})$ we write also $P = \begin{pmatrix} P_1 & P_2 & \cdots & P_n \end{pmatrix}$, where P_j, $j = 1, 2, \ldots, n$, is the *j-th* column of P.

For the proof of the Theorem 4 we are proving the Lemmas 1, 2, 3.

Lemma 1. *The group associated to maximal group of invariance of the family* $\Im_{3n-2}$ *is not measurable.*

Proof. The family $\Im_{3n-2}$ and the group G_{n^2+n}, can be written in the form

$$\begin{aligned} B^T \cdot X &= 1, \\ X &= Lx_n + Q, \end{aligned} \tag{1}$$

$$X = P \cdot X' + A. \tag{2}$$

Applying G_{n^2+n} to $\Im_{3n-2}$ we obtain that

$$\begin{aligned} B'^T \cdot X' &= 1, \\ X' &= L'x'_n + Q'. \end{aligned} \tag{3}$$

According to (2), from the first equality (1), we find that

$$\frac{1}{1 - B^T \cdot A}(B^T \cdot P)X' = 1 \tag{4}$$

and the second equality in (1) implies that

$$PX' = L \cdot (p_{n1}x'_1 + p_{n2}x'_2 + \cdots + p_{nn}x'_n + a_n) + Q - A. \tag{5}$$

Considering the first $n-1$ rows, (5) can be written as follows:

$$R\overline{X'} = (\overline{L}p_{nn} - \overline{P})x'_n + \overline{L}\alpha_n + \overline{Q} - \overline{A},$$

where

$$R = \begin{pmatrix} p_{11} - l_1 p_{n1} & p_{12} - l_1 p_{n2} & \dots & p_{1n-1} - l_1 p_{nn-1} \\ p_{21} - l_2 p_{n1} & p_{22} - l_2 p_{n2} & \dots & p_{2n-1} - l_2 p_{nn-1} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n-11} - l_{n-1} p_{n1} & p_{n-12} - l_{n-1} p_{n2} & \dots & p_{n-1n-1} - l_{n-1} p_{nn-1} \end{pmatrix}, \ |R| \neq 0.$$

This implies that

$$\overline{X}' = R^{-1}(\overline{L}p_{nn} - \overline{P_n})x'_n + R^{-1}\overline{L}a_n + R^{-1}(\overline{Q} - \overline{A}). \tag{6}$$

According to $X' = \begin{pmatrix} \overline{X}' \\ x'_n \end{pmatrix}$ and by comparing (3) with (4) and (6), respectively, we obtain the following relations between the new parameters of the family $\Im_{3n-2}$ and the original ones:

$$\begin{aligned} B'^T &= \tfrac{1}{1 - {}^tB \cdot A \cdot} B^T \cdot P, \\ \overline{L}' &= R^{-1} \cdot (\overline{L}p_{nn} - \overline{P_n}), \\ \overline{Q'} &= R^{-1} \cdot (a_n \overline{L} + \overline{Q} - \overline{A}). \end{aligned} \tag{7}$$

These are the equations of group H_{n^2+n} associated to G_{n^2+n} in the $(3n-2)$-dimensional space $\mathcal{A}_{3n-2}$. The unit $e \in H_{n^2+n}$ is obtained by

$$p_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad \text{and } a_i = 0 \quad (i, j = 1, 2, \dots, n).$$

The matrix of coefficients of the infinitesimal transformations of H_{n^2+n}, which has as columns the partial derivates of

$$b'_1, b'_2, \ldots, b'_n, l'_1, l'_2, \ldots, l'_{n-1}, q'_1, q'_2, \ldots, q'_{n-1}$$

with respect to the parameters

$$p_{11}, p_{21}, \ldots, p_{n1}, p_{12}, p_{22}, \ldots, p_{n2}, \ldots, p_{1n}, p_{2n}, \ldots, p_{nn}, a_1, a_2, \ldots, a_n,$$

is given by

$$\xi = \begin{pmatrix} B & O & O & \ldots & O & -l_1 H & -q_1 H \\ O & B & O & \ldots & O & -l_2 H & -q_2 H \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ O & O & O & \ldots & O & -l_{n-1} H & -q_{n-1} H \\ O & O & O & \ldots & B & -H & O \\ & & BB^T & & & O & -H \end{pmatrix},$$

where

$$H = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & \ldots & 0 & 1 \\ -l_1 & -l_2 & \ldots & -l_{n-2} & -l_{n-1} \end{pmatrix}$$

has n rows and $n-1$ columns and O is the $(n,1)$ zero matrix.

To indicate the type of a matrix, we denote it by capital letters with indices, if necessary.

In order to calculate the rank of the matrix ξ, we select the matrix M of order $3n-2$ which consists of the first two block rows and the first row of each of the following $n-2$ block-rows, i.e.

$$M = \begin{pmatrix} B & O & O & \ldots & O & -l_1 H & -q_1 H \\ O & B & O & \ldots & O & -l_2 H & -q_2 H \\ O_{n-2,1} & O_{n-2,1} & & b_1 I_{n-2} & & U & V \end{pmatrix}, \tag{8}$$

where

$$U^T = \begin{pmatrix} & & O_{n-2,n-2} & & \\ -l_3 & -l_4 & \ldots & -l_{n-1} & -1 \end{pmatrix},$$

$$V^T = \begin{pmatrix} & & O_{n-2,n-2} & & \\ -q_3 & -q_4 & \ldots & -q_{n-1} & 0 \end{pmatrix}.$$

By developing the determinant of the matrix M with respect to the third block column, we have

$$\det M = (b_1)^{n-2} \det \begin{pmatrix} B & O & -l_1 H & -q_1 H \\ O & B & -l_2 H & -q_2 H \end{pmatrix}. \tag{9}$$

Let

$$N = \begin{pmatrix} B & O & -l_1H & -q_1H \\ O & B & -l_2H & -q_2H \end{pmatrix}.$$

We consider the diagonal block matrix

$$\Delta = \begin{pmatrix} I_2 & O_{2,n-1} & O_{2,n-1} \\ O_{n-1,2} & q_1 I_{n-1} & O_{n-1,n-1} \\ O_{n-1,2} & O_{n-1,n-1} & -l_1 I_{n-1} \end{pmatrix}$$

Then

$$N \cdot \Delta = \begin{pmatrix} B & O & -l_1q_1H & l_1q_1H \\ O & B & -l_2q_1H & l_1q_2H \end{pmatrix}.$$

Adding together the last but one column and the last column and substitute the sum for the last column (so that $\det(N \cdot \Delta)$ does not change). This gives

$$\det(N \cdot \Delta) = \det \begin{pmatrix} B & O & -l_1q_1H & O_{n,n-1} \\ O & B & -l_2q_1H & (l_1q_2 - l_2q_1)H \end{pmatrix}.$$

Let

$$K = \begin{pmatrix} B & O & -l_1q_1H & O_{n,n-1} \\ O & B & -l_2q_1H & (l_1q_2 - l_2q_1)H \end{pmatrix}.$$

We put the second column of K in the $(n+1)-th$ position .

This gives the block matrix

$$\widetilde{K} = \begin{pmatrix} B & -l_1q_1H & O & O_{n,n-1} \\ O & -l_2q_1H & B & (l_1q_2 - l_2q_1)H \end{pmatrix}$$

or, more simply

$$\widetilde{K} = \left(\begin{array}{c|c} R_n & O_{n,n} \\ \hline O \quad -l_2q_1H & S_n \end{array} \right),$$

where

$$R_n = \begin{pmatrix} B & -l_1q_1H \end{pmatrix} \qquad S_n = \begin{pmatrix} B & (l_1q_2 - l_2q_1)H \end{pmatrix}.$$

Note that

$$\det K = (-1)^{n-1} \det \widetilde{K} = (-1)^{n-1} \det R_n \cdot \det S_n. \tag{10}$$

We have to calculate $\det R_n$ and $\det S_n$.

We first consider the matrix R_n and prove that

$$\det R_n = (l_1q_1)^{n-1}(l_1b_1 + l_2b_2 + \cdots + l_{n-2}b_{n-2} + l_{n-1}b_{n-1} + b_n), \tag{$*$}$$

where $\sigma = (l_1b_1 + l_2b_2 + \cdots + l_{n-2}b_{n-2} + l_{n-1}b_{n-1} + b_n)$.

R_n can be written as follows:

$$R_n = \begin{pmatrix} \alpha & l_1q_1\beta \\ \gamma & -l_1q_1\delta \end{pmatrix},$$

where

$$\alpha = (b_1), \qquad \beta = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \text{ is a } (1, n-1) \text{ matrix,}$$

$$\gamma^T = \begin{pmatrix} b_2 & b_3 & \dots & b_{n-1} & b_n \end{pmatrix},$$

$$\delta = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -l_1 & -l_2 & -l_3 & \cdots & -l_{n-1} \end{pmatrix} \text{ is a matrix of order } n-1$$

Applying the generalized Gauss algorithm to the matrix R_n, we obtain the matrix

$$\widetilde{K} = \begin{pmatrix} \alpha & \beta \\ O & -l_1 q_1(\delta - \gamma\alpha^{-1}\beta) \end{pmatrix}$$

Then

$$\det R_n = \det \widetilde{R}_n = \det \alpha \cdot \det(-l_1 q_1(\delta - \gamma\alpha^{-1}\beta)) = b_1(-l_1 q_1)^{n-1} \det(\delta - \gamma\alpha^{-1}\beta),$$

where

$$\delta - \gamma\alpha^{-1}\beta = \begin{pmatrix} -\frac{b_2}{b_1} & 1 & 0 & \dots & 0 \\ -\frac{b_3}{b_1} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{b_{n-1}}{b_1} & 0 & 0 & \dots & 1 \\ -l_1 - \frac{b_n}{b_1} & -l_1 & -l_2 & \dots & -l_{n-1} \end{pmatrix}.$$

Next put

$$R_{n-1} = \delta - \gamma\alpha^{-1}\beta = \begin{pmatrix} -\frac{b_2}{b_1} & \beta^* \\ \gamma^* & \delta^* \end{pmatrix},$$

where $\gamma^* = \begin{pmatrix} -\frac{b_3}{b_1} \\ -\frac{b_4}{b_1} \\ \vdots \\ -\frac{b_{n-1}}{b_1} \\ -l_1 - \frac{b_n}{b_1} \end{pmatrix}$, $\beta^* = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}$ is a $(1, n-2)$ matrix and

$$\delta^* = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -l_2 & -l_3 & -l_4 & \dots & -l_{n-1} \end{pmatrix} \text{ is a matrix of order } n-2.$$

The matrix R_{n-1} is the matrix R_n with n replaced by $n-1$. Thus, applying Gauss algorithm again, we obtain

$$\widetilde{R}_{n-1} = \begin{pmatrix} -\frac{b_2}{b_1} & \beta \\ O_{n-2} & \delta^* - \gamma^*(-\frac{b_1}{b_2})\beta^* \end{pmatrix},$$

so that

$$\det R_{n-1} = \det \widetilde{R}_{n-1} = -\frac{b_2}{b_1} \det(\delta^* - \gamma^*(-\frac{b_1}{b_2})\beta^*),$$

with

$$\gamma^*(-\frac{b_1}{b_2})\beta^* = \begin{pmatrix} \frac{b_3}{b_2} & 0 & \dots & 0 & 0 \\ \frac{b_4}{b_2} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{b_{n-1}}{b_2} & 0 & \dots & 0 & 0 \\ \frac{l_1 b_1 + b_n}{b_2} & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Let $R_{n-2} = \delta^* - \gamma^*(-\frac{b_1}{b_2})\beta^*$, where

$$\delta^* - \gamma^*(-\tfrac{b_1}{b_2})\beta^* = \begin{pmatrix} -\frac{b_3}{b_2} & 1 & 0 & \dots & 0 \\ -\frac{b_4}{b_2} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{b_{n-1}}{b_2} & 0 & 0 & \dots & 1 \\ -l_2 - \frac{l_1 b_1 + b_n}{b_2} & -l_3 & -l_4 & \dots & -l_{n-1} \end{pmatrix}$$

is a matrix of order $n-2$.

We have $\det R_{n-1} = (-\frac{b_2}{b_1}) \det R_{n-2}$.

Applying the generalized Gauss algorithm to the matrix R_{n-2}, we obtain the matrix $\widetilde{R}_{n-2}$. Thus $\det R_{n-2} = \det \widetilde{R}_{n-2} = (-\frac{b_3}{b_2}) \det R_{n-3}$, where

$$R_{n-3} = \begin{pmatrix} -\frac{b_4}{b_3} & 1 & 0 & 0 & \dots & 0 \\ -\frac{b_5}{b_3} & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{b_{n-1}}{b_3} & 0 & 0 & 0 & \cdots & 1 \\ -\frac{l_1 b_1 + l_2 b_2 + l_3 b_3 + b_n}{b_3} & -l_4 & -l_5 & -l_6 & \dots & -l_{n-1} \end{pmatrix},$$

etc. In this way we get in finitely many steps the matrix

$$\begin{pmatrix} -\frac{b_{n-1}}{b_{n-2}} & 1 \\ -\frac{l_1 b_1 + l_2 b_2 + \cdots + l_{n-2} b_{n-2} + b_n}{b_{n-2}} & -l_{n-1} \end{pmatrix},$$

which has rank 2. Therefore we have that

$\det R_n = \det \widetilde{R}_n = b_1(-l_1 q_1)^{n-1} \det R_{n-1} = b_1(-l_1 q_1)^{n-1} \det \widetilde{R}_{n-1}$
$= b_1(-l_1 q_1)^{n-1}(-\frac{b_2}{b_1}) \det R_{n-2} = (-l_1 q_1)^{n-1} b_1(-\frac{b_2}{b_1}) \det \widetilde{R}_{n-2}$

$= (-l_1q_1)^{n-1}b_1(-\frac{b_2}{b_1})(-\frac{b_3}{b_2}) \det R_{n-3} = (-l_1q_1)^{n-1}b_1(-\frac{b_2}{b_1})(-\frac{b_3}{b_2}) \det \widetilde{R}_{n-3}$
$= (-l_1q_1)^{n-1}b_1(-\frac{b_2}{b_1})(-\frac{b_3}{b_2})(-\frac{b_4}{b_3}) \det R_{n-4}$
$= (-l_1q_1)^{n-1}b_1(-\frac{b_2}{b_1})(-\frac{b_3}{b_2})(-\frac{b_4}{b_3}) \det \widetilde{R}_{n-4}$
$= \cdots = (-l_1q_1)^{n-1}b_1(-\frac{b_2}{b_1})(-\frac{b_3}{b_2})(-\frac{b_4}{b_3}) \cdots - (\frac{b_{n-2}}{b_{n-1}}) \det R_2$.
As

$$\det R_2 = \begin{vmatrix} -\frac{b_{n-1}}{b_{n-2}} & 1 \\ -\frac{l_1b_1+l_2b_2+\cdots+l_{n-2}b_{n-2}+b_n}{b_{n-2}} & -l_{n-1} \end{vmatrix}$$

$$= -\frac{l_1b_1 + l_2b_2 + \cdots + l_{n-2}b_{n-2} + l_{n-1}b_{n-1} + b_n}{b_{n-2}},$$

it follows that
$\det R_n = (-l_1q_1)^{n-1}(-1)^{n-3}(l_1b_1 + l_2b_2 + \cdots + l_{n-2}b_{n-2} + l_{n-1}b_{n-1} + b_n)$
$= (-1)^{n-1}(l_1q_1)^{n-1}(-1)^{n-3}(l_1b_1 + l_2b_2 + \cdots + l_{n-2}b_{n-2} + l_{n-1}b_{n-1} + b_n)$
$= (l_1q_1)^{n-1}(l_1b_1 + l_2b_2 + \cdots + l_{n-2}b_{n-2} + l_{n-1}b_{n-1} + b_n)$,
concluding the proof of $(*)$.

Now we consider the matrix

$$S_n = \begin{pmatrix} B & (l_1q_2 - l_2q_1)H \end{pmatrix}.$$

The matrix S_n is similar to R_n where the factor $-l_1q_1$ is replaced by $l_1q_2 - l_2q_1$ so that, repeating the previous procedure, we obtain that

$$\det S_n = (-l_1q_2 - l_2q_1)^{n-1}(l_1b_1 + l_2b_2 + \cdots + l_{n-2}b_{n-2} + l_{n-1}b_{n-1} + b_n). \qquad (**)$$

$(*)$ and $(**)$ imply that

$$\det \widetilde{K} = \det R_n \cdot \det S_n =$$

$$l_1q_1)^{n-1}(-l_1q_2 + l_2q_1)^{n-1}(l_1b_1 + l_2b_2 + \cdots + l_{n-2}b_{n-2} + l_{n-1}b_{n-1} + b_n)^2 =$$
$$= (l_1q_1)^{n-1}(-l_1q_2 + l_2q_1)^{n-1}\sigma^2,$$

where $\sigma = (l_1b_1 + l_2b_2 + \cdots + l_{n-2}b_{n-2} + l_{n-1}b_{n-1} + b_n)$.
(10) yields the following:

$$\det K = \det(N \cdot \Delta) = (-1)^{n-1}(l_1q_1)^{n-1}(-l_1q_2 + l_2q_1)^{n-1}\sigma^2 \cdots$$

As $\det \Delta = \det I_2 \cdot \det(q_1I_{n-1}) \cdot \det(-l_1I_{n-1}) = q_1^{n-1}(-l_1)^{n-1} = (-1)^{n-1}(l_1q_1)^{n-1}$, we obtain that

$$\det N = \frac{(-1)^{n-1}(l_1q_1)^{n-1}(-l_1q_2 + l_2q_1)^{n-1}\sigma^2}{(-1)^{n-1}(l_1q_1)^{n-1}} = (-l_1q_2 + l_2q_1)^{n-1}\sigma^2.$$

It follows from (9) that

$$\det M = b_1^{n-2} \det N = b_1^{n-2}(-l_1q_2 + l_2q_1)^{n-1}\sigma^2.$$

Thus rank $\xi = 3n-2$.

Our aim is to find functions $\Phi(b_1, b_2, \dots, b_n, l_1, l_2, \dots, l_{n-1}, q_1, q_2, \dots, q_{n-1})$ satisfying the Deltheil system (see Theorem 1) which has ξ as matrix.

In other words, we look for possible non-zero solutions of the (linear non-homogeneous) system

$$\xi \cdot Y = \nu \tag{11}$$

consisting of n^2+n equations in $3n-2$ unknowns

$$y_1, y_2, \dots, y_n, y_{n+1}, \dots, y_{2n-1}, y_{2n}, \dots, y_{3n-2}$$

with

$$\begin{aligned} y_i &= \frac{\partial ln\Phi}{\partial b_i} \quad i=1,\dots,n \\ y_{n+j} &= \frac{\partial ln\Phi}{\partial l_j} \quad j=1,\dots,n-1 \\ y_{2n-1+h} &= \frac{\partial ln\Phi}{\partial q_h} \quad h=1,\dots,n-1 \end{aligned}$$

and

$$\nu^T = \left(\nu_1^T, \nu_2^T, \nu_3^T, \dots, \nu_0^T, -(n+1)B^T\right),$$

where

$\nu_i^T = \begin{pmatrix} 1 & 0 & \dots & 0 & -(n+1)l_i \end{pmatrix}$ $i=1,\dots,n-1$ and
$\nu_0^T = \begin{pmatrix} 0 & 0 & & 1 & -n \end{pmatrix}$ are row vectors.

As we have previously determined rank ξ, now we are calculating the rank of the complete block matrix

$$\xi' = (\xi, \nu).$$

Consider the following $(3n-1)$x$(3n-1)$ matrix

$$\begin{pmatrix} & & & & M & & & & & \nu_1 \\ & & & & & & & & & \nu_2 \\ & & & & & & & & & O_{n-2,1} \\ 0 & 0 & \dots & b_2 & 0 & -1 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Its determinant is

$$(-b_1)^{n-3}(l_1q_2 - l_2q_1)^{n-2}(q_1b_1 + q_2b_2)(l_1b_1 + l_2b_2 + \dots + l_{n-1}b_{n-1} + b_n)^2.$$

Consequently, the rank of the complete matrix is $3n-1$. This shows that the system (11) is not solvable.

Then group H_{n^2+n} associated to G_{n^2+n} is not measurable. According to Theorem 2 (see Section 1) the family $\Im_{3n-2}$ can be measurable or not.

Lemma 2. *The group associated to the subgroup*

$$G_{3n-2} : X = \widetilde{P}X'$$

where

$$\widetilde{P}=\begin{pmatrix} p_{11} & p_{12} & & & & \\ p_{21} & p_{22} & & & O & \\ p_{31} & p_{32} & p_{33} & & & \\ p_{41} & p_{42} & & p_{44} & & \\ \vdots & \vdots & O & & \ddots & \\ p_{n1} & p_{n2} & & & & p_{nn} \end{pmatrix}$$

of G_{n^2+n} *is measurable and the integral invariant function is*

$$\Phi = k\frac{b_3b_4\cdots b_n}{\sigma^n\tau}$$

where $\sigma = b_1l_1+b_2l_2+\cdots+b_{n-1}l_{n-1}+b_n,\ \tau = l_1q_2-l_2q_1.$

Proof. By applying the subgroup G_{3n-2} to $\Im_{3n-2}$, we obtain the equations of group H_{3n-2} associated to G_{3n-2}.
The matrix of the coefficients of the infinitesimal transformations of group H_{3n-2} as follows

$$\widetilde{\eta}=\begin{pmatrix} B & O & O_{n,n-2} & -l_1H & -q_1H \\ O & B & O_{n,n-2} & -l_2H & -q_2H \\ O_{n-2,1} & O_{n-2,1} & D & C & F \end{pmatrix},$$

where $D=\begin{pmatrix} b_3 & 0 & \dots & 0 & 0 \\ 0 & b_4 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b_{n-1} & 0 \\ 0 & 0 & \dots & 0 & b_n \end{pmatrix}$ is a diagonal matrix of order $n-2$, and

$$C=\begin{pmatrix} 0 & 0 & -l_3 & 0 & \cdots & 0 \\ 0 & 0 & 0 & -l_4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -l_{n-1} \\ l_1 & l_2 & l_3 & l_4 & \dots & l_{n-1} \end{pmatrix}, \quad F=\begin{pmatrix} 0 & 0 & -q_3 & 0 & \dots & 0 \\ 0 & 0 & 0 & -q_4 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -q_{n-1} \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

are $(n-2, n-1)$ matrices.
Then

$$\det\widetilde{\eta} = b_3b_4\cdots b_n \det N = b_3b_4\cdots b_n(-l_1q_2+l_2q_1)^{n-1}(b_1l_1+b_2l_2+\cdots+b_{n-1}l_{n-1}+b_n)^2,$$

the computation being similar to that for $\det M$ (see (8) and (9)).

The Deltheil system of the subgroup H_{3n-2}, associated to G_{3n-2}, is solvable because its incomplete matrix $\widetilde{\eta}$ has maximal rank. Then it admits only one solution (up to a multiplicative constant)

$$\Phi(b_1, b_2, \dots, b_n, l_1, l_2, \dots, l_{n-1}, q_1, q_2, \dots, q_{n-1}).$$

We will show that Φ is given by

$$\Phi = k\frac{b_3b_4\cdots b_n}{\sigma^n\tau}. \tag{12}$$

From the definition of measurability it follows that the group H_{3n-2}, associated to G_{3n-2}, is measurable, but we cannot assert yet that the family $\Im_{3n-2}$ is measurable (see Theorem 3).

Lemma 3. *The group associated to subgroup*

$$G_{n^2+n-1} : X = PX' + A$$

with $\det P = 1$ *is measurable and the integral invariant function is*

$$\Phi = k\sigma^{-(n+1)},$$

where $\sigma = b_1 l_1 + b_2 l_2 + \cdots + b_{n-1} l_{n-1} + b_n$

Proof. From $\det P = 1$ it follows

$$p_{11} = \frac{1 + p_{12} p_{21} (p_{nn} \cdots p_{33})}{(p_{nn} \cdots p_{33}) p_{22}}.$$

Repeating for this subgroup the whole procedure as for subgroups considered above, we obtain the matrix of the coefficients of the infinitesimal transformations of the associated group H_{n^2+n-1} and then we reach the following system of n^2+n-1 linear equations in $3n-2$ unknowns:

$$\eta \cdot Y = \varepsilon, \tag{13}$$

where

$$\eta = \begin{pmatrix} \gamma & O & \dots & O & \Lambda & \Psi \\ -b_1 E^2 & B & \dots & O & -l_2 H & -q_2 H \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ -b_1 E^{n-1} & O & \dots & O & -l_{n-1} H & -q_{n-1} H \\ -b_1 E^n & O & \dots & B & \Gamma & \Theta \\ & BB^T & & & O & -H \end{pmatrix}, \quad \gamma = \begin{pmatrix} b_2 \\ b_3 \\ \vdots \\ b_{n-1} \\ b_n \end{pmatrix},$$

$$\Lambda = \begin{pmatrix} 0 & -l_1 & 0 & \dots & 0 \\ 0 & 0 & -l_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -l_1 \\ l_1^2 & l_1 l_2 & l_1 l_3 & \dots & l_1 l_{n-1} \end{pmatrix}, \quad \Psi = \begin{pmatrix} 0 & -q_1 & 0 & \dots & 0 \\ 0 & 0 & -q_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -q_1 \\ l_1 q_1 & l_2 q_1 & l_3 q_1 & \dots & l_{n-1} q_1 \end{pmatrix},$$

$$\Gamma = \begin{pmatrix} -1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -1 \\ 2l_1 & l_2 & l_3 & \dots & l_{n-1} \end{pmatrix}, \qquad \Theta = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ q_1 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

E^i is the $(n, 1)$ matrix with 1 at the *i-th* place $(i = 2, 3, \dots, n)$, and 0 at all other places, $\varepsilon^T = \begin{pmatrix} \varepsilon_1^T & \varepsilon_2^T & \dots & \varepsilon_{n-1}^T & \varepsilon_0^T & -(n+1) B^T \end{pmatrix}$, where

$\varepsilon_1^T = \begin{pmatrix} 0 & 0 & \dots & 0 & -(n+1)l_1 \end{pmatrix}$ is a $1, n-1)$ matrix and
$\varepsilon_i^T = \begin{pmatrix} 0 & 0 & \dots & 0 & -(n+1)l_i \end{pmatrix}$, $i = 2, \dots, n-1$,
$\varepsilon_0^T = \begin{pmatrix} 0 & 0 & \dots & 0 & -(n+1) \end{pmatrix}$
are $(1, n)$ matrices.

It is easy to see that both η and (η, ε) have rank $3n-2$. This condition ensures that the system (13) is solvable and admits the unique solution

$$\begin{pmatrix} -\frac{n+1}{\sigma} L^T & -\frac{n+1}{\sigma} \overline{B}^T & O_{1,n-1} \end{pmatrix}.$$

It is equally easy to see that the non-trivial solution of the Deltheil system, which has η as matrix [15], is

$$\Phi = k\sigma^{-(n+1)}. \tag{14}$$

In conclusion the solution (14) is independent from the solution (12) so that the family $\Im_{3n-2}$ is not measurable by Theorem 3.

The proof of Theorem 4 is complete.

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