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Frenet Formulas and Geodesics in *Sol* Geometry

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Abstract. In this paper we deal with one of the homogeneous 3-geometries, the *Sol* geometry. The Frenet frame and the curvature and torsion of a curve has been determined, moreover, we have computed the parametric form of geodesics, their curvatures and torsions in Theorem 4.1.

1. Introduction

Let (M, g) be a Riemannian manifold. If for any $x, y \in M$ there does exist an isometry $\Phi : M \to M$ such that $y = \Phi(x)$, then the Riemannian manifold is called *homogeneous*.

Homogeneous geometries have main roles in the modern theory of three-manifolds. Homogeneous spaces are, in a sense, the nicest examples of Riemannian manifolds and have applications in physics (e.g. the *Sol* geometry is useful for studying holography, Yang-Mills theory) [1].

To underline their importance from mathematical point of view we roughly cite the famous Thurston conjecture stating that a compact three-manifold with a given topology has a canonical decomposition into a sum of 'simple three-manifolds',

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whose interiors each admits one, and only one metric, of eight homogeneous Riemann geometries: E^3, H^3, S^3, $S^2 \times \mathbb{R}$, $H^2 \times \mathbb{R}$, $\widetilde{SL(2, \mathbb{R})}$, *Nil* and *Sol* [5].

We do not intend to prove or disprove the Thurston conjecture, our aim is just to go a bit closer to the *Sol* geometry.

Sol geometry can be obtained by giving a group structure to be the semi-direct product $\mathbb{R} \ltimes \mathbb{R}^2$ as follows:

$$\begin{pmatrix} 1 & a & b & c \end{pmatrix} \begin{pmatrix} 1 & x & y & z \\ 0 & e^{-z} & 0 & 0 \\ 0 & 0 & e^{z} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x + ae^{-z} & y + be^{z} & z + c \end{pmatrix} \tag{1.1}$$

is the right action by a translation $(1, x, y, z)$ on a point $(1, a, b, c)$ yielding also a point of *Sol* expressed in homogeneous coordinates, for (x, y, z) and (a, b, c), after choosing a fixed origin $O(1, 0, 0, 0)$. Then an invariant metric on $Sol(O, T)$ is given by

$$(ds)^2 = (dx)^2 e^{2z} + (dy)^2 e^{-2z} + (dz)^2 , \tag{1.2}$$

as arc length square, now in any point $(1, x, y, z)$ [5], [2].

2. Model of *Sol* geometry

The illustration of *Sol* geometry is really a hard task. On J. F. Weeks's opinion:

> *"This (the Sol geometry) is the real weird. Unlike the previous geometries, solve geometry isn't even rotationally symmetric. I don't know any good intrinsic way to understand it."* [7]

Firstly for making the reader more familiar to the topic we briefly cite a good model of *Sol* from [2], where the projective interpretation of all the eight homogeneous geometries are carried out in details by Emil Molnár.

The forthcoming computations, however, are not based on that paper.

At the beginning the intuitive idea of a manifold, having certain geometry, can now be refreshed [5], [3]:

A model geometry (G, X) is a manifold X, with a Lie Group G of diffeomorphism of X such that:

1. X is connected and simply connected,
2. G acts transitively on X with compact point stabilizers,
3. G is not contained in any larger groups of diffeomorphisms of X with compact point stabilizers,
4. there exists at least one compact (G, X)-manifold.

The short description of the above mentioned model is the following [2]:

Consider a symmetric linear mapping (polarity) $(_*) : V_4 \to \mathbf{V}^4$, $u \mapsto u_* =: \mathbf{u}$ and the induced scalar product $u_* v = v_* u =: \langle u, v \rangle \in \mathbb{R}$. Here $\mathbf{V}^4$ denotes the space of four-dimensional vectors for projective points, as its 1-spaces or rays, and the subspace structure for lines, planes, in general. V_4 is its dual for planes, similarly. In an appropriate dual basis pair each polarity has a diagonal matrix form, where the diagonal entries are all 0's, 1's and -1's, providing the signature of the scalar product.

In our case it is $(0 - + +)$ with respect to our standard basis pair $\{\mathbf{f}_i\}, \{f^j\}$ with $\mathbf{f}_i f^j = \delta_i^j$ (the Kronecker symbol) $i, j = 1, \ldots, 4$ of $\mathbf{V}^4$, V_4, respectively.

The polarity is given by $f_*^0 = \mathbf{0}$, $f_*^1 = \mathbf{f}_2$, $f_*^2 = \mathbf{f}_1$, $f_*^3 = \mathbf{f}_3$.

Consider a collineation group, leaving invariant the above polarity, thus acting on the points of the three dimensional projective space $\mathcal{P}^3(\mathbf{V}^4, V_4, \mathbb{R})$ by our matrix (1.1) in the introduction:

$$\hat{\alpha}: \begin{pmatrix} \mathbf{f}_0 \\ \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \end{pmatrix} \mapsto \begin{pmatrix} 1 & x & y & z \\ 0 & e^{-z} & 0 & 0 \\ 0 & 0 & e^z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{f}_0 \\ \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \end{pmatrix} \tag{2.1}$$

and with the inverse matrix, acting on the planes of $\mathcal{P}^3$ above [2]:

$$\alpha: \begin{pmatrix} f^0 & f^1 & f^2 & f^3 \end{pmatrix} \mapsto \begin{pmatrix} f^0 & f^1 & f^2 & f^3 \end{pmatrix} \begin{pmatrix} 1 & -xe^z & -ye^{-z} & -z \\ 0 & e^z & 0 & 0 \\ 0 & 0 & e^{-z} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.2}$$

We obtain the full isometry group of *Sol* by adding two new generators

$$\begin{pmatrix} \mathbf{f}_0 \\ \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{f}_0 \\ \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{f}_0 \\ \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \mathbf{f}_0 \\ \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \end{pmatrix} \tag{2.3}$$

at the origin.

The model consists of fibres of different Minkowski planes along the z-axis. This model shows that the *Sol* geometry has a distinguished direction line (we denote it by z). It is well-known that there exist only three homogeneous two-dimensional geometries, namely the elliptic, the Euclidean and the hyperbolic geometry. They are also isotropic, i.e. every direction can be equivalent for descriptions. Homogeneous, but non isotropic geometries can be found in three- or higher dimensional manifolds. (Note that the word "geometry" means certain family of geometries.)

3. Frenet formulas in *Sol* geometry

One goal of our work is the computation of the Frenet frame $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ and the presentation of the curvature and the torsion of a curve in *Sol* geometry by the classical Frenet formulas

$$\begin{aligned} \mathbf{t}'(s) &= \phantom{-\kappa(s)\mathbf{t}(s)} \kappa(s)\mathbf{n}(s)\,, \\ \mathbf{n}'(s) &= -\kappa(s)\mathbf{t}(s) \phantom{\kappa(s)\mathbf{n}(s)} + \tau(s)\mathbf{b}(s)\,, \\ \mathbf{b}'(s) &= \phantom{-\kappa(s)\mathbf{t}(s)} -\tau(s)\mathbf{n}(s)\,. \end{aligned} \tag{3.1}$$

Let $\mathbf{r}(t) = (1, x(t), y(t), z(t))$ be a curve with a real parameter t in *Sol*. The arc length s is defined by (1.2) and so by

$$\frac{ds}{dt} = \sqrt{e^{2z}\left(\frac{dx}{dt}\right)^2 + e^{-2z}\left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = |\dot{\mathbf{r}}| \text{ and}$$

$$\frac{d}{ds} = \frac{dt}{ds}\frac{d}{dt} = \frac{1}{|\dot{\mathbf{r}}|}\frac{d}{dt}\,, \quad \text{in general}\,. \tag{3.2}$$

Denote the assumed linearly independent vectors by

$$\frac{d\mathbf{r}(t)}{dt} =: \dot{\mathbf{r}}(t), \quad \frac{d^2\mathbf{r}(t)}{dt^2} =: \ddot{\mathbf{r}}(t), \quad \frac{d^3\mathbf{r}(t)}{dt^3} =: \dddot{\mathbf{r}}(t)\,.$$

Applying the Gram-Schmidt orthonormalization and derivatives by s above $\left(\mathbf{r}'(s) = \frac{d\mathbf{r}(s)}{ds},\ \mathbf{r}''(s) = \frac{d^2\mathbf{r}(s)}{ds^2},\ \mathbf{r}'''(s) = \frac{d^3\mathbf{r}(s)}{ds^3}\right)$, we find first the following orthogonal vector system:

$$\begin{aligned}
\mathbf{e}_1(t) &= |\dot{\mathbf{r}}|\mathbf{r}'(s)\,,\\
\mathbf{e}_2(t) &= |\dot{\mathbf{r}}|\begin{vmatrix} \mathbf{r}'(s) & \mathbf{r}''(s)\\ \langle\mathbf{r}'(s),\mathbf{r}'(s)\rangle & \langle\mathbf{r}'(s),\mathbf{r}''(s)\rangle\end{vmatrix}\,,\\
\mathbf{e}_3(t) &= |\dot{\mathbf{r}}|\frac{\begin{vmatrix} \mathbf{r}'(s) & \mathbf{r}''(s) & \mathbf{r}'''(s)\\ \langle\mathbf{r}'(s),\mathbf{r}'(s)\rangle & \langle\mathbf{r}'(s),\mathbf{r}''(s)\rangle & \langle\mathbf{r}'(s),\mathbf{r}'''(s)\rangle\\ \langle\mathbf{r}''(s),\mathbf{r}'(s)\rangle & \langle\mathbf{r}''(s),\mathbf{r}''(s)\rangle & \langle\mathbf{r}''(s),\mathbf{r}'''(s)\rangle\end{vmatrix}}{\begin{vmatrix}\langle\mathbf{r}'(s),\mathbf{r}'(s)\rangle & \langle\mathbf{r}'(s),\mathbf{r}''(s)\rangle\\ \langle\mathbf{r}''(s),\mathbf{r}'(s)\rangle & \langle\mathbf{r}''(s),\mathbf{r}''(s)\rangle\end{vmatrix}}\,.
\end{aligned}$$

The tangent, normal and binormal unit vectors are then consequently:

$$\begin{aligned}
\mathbf{t}(s) &= \mathbf{r}'(s)\,,\\
\mathbf{n}(s) &= -\frac{\begin{vmatrix} \mathbf{r}'(s) & \mathbf{r}''(s)\\ \langle\mathbf{r}'(s),\mathbf{r}'(s)\rangle & \langle\mathbf{r}'(s),\mathbf{r}''(s)\rangle\end{vmatrix}}{\sqrt{\begin{vmatrix}\langle\mathbf{r}'(s),\mathbf{r}'(s)\rangle & \langle\mathbf{r}'(s),\mathbf{r}''(s)\rangle\\ \langle\mathbf{r}''(s),\mathbf{r}'(s)\rangle & \langle\mathbf{r}''(s),\mathbf{r}''(s)\rangle\end{vmatrix}}}\,, \qquad (3.3)\\
\mathbf{b}(s) &= \boxed{\square}\cdot\frac{\begin{vmatrix} \mathbf{r}'(s) & \mathbf{r}''(s) & \mathbf{r}'''(s)\\ \langle\mathbf{r}'(s),\mathbf{r}'(s)\rangle & \langle\mathbf{r}'(s),\mathbf{r}''(s)\rangle & \langle\mathbf{r}'(s),\mathbf{r}'''(s)\rangle\\ \langle\mathbf{r}''(s),\mathbf{r}'(s)\rangle & \langle\mathbf{r}''(s),\mathbf{r}''(s)\rangle & \langle\mathbf{r}''(s),\mathbf{r}'''(s)\rangle\end{vmatrix}}{\sqrt{\begin{vmatrix}\langle\mathbf{r}'(s),\mathbf{r}'(s)\rangle & \langle\mathbf{r}'(s),\mathbf{r}''(s)\rangle & \langle\mathbf{r}'(s),\mathbf{r}'''(s)\rangle\\ \langle\mathbf{r}''(s),\mathbf{r}'(s)\rangle & \langle\mathbf{r}''(s),\mathbf{r}''(s)\rangle & \langle\mathbf{r}''(s),\mathbf{r}'''(s)\rangle\\ \langle\mathbf{r}'''(s),\mathbf{r}'(s)\rangle & \langle\mathbf{r}'''(s),\mathbf{r}''(s)\rangle & \langle\mathbf{r}'''(s),\mathbf{r}'''(s)\rangle\end{vmatrix}}}\\
&\quad\cdot\sqrt{\begin{vmatrix}\langle\mathbf{r}'(s),\mathbf{r}'(s)\rangle & \langle\mathbf{r}'(s),\mathbf{r}''(s)\rangle\\ \langle\mathbf{r}''(s),\mathbf{r}'(s)\rangle & \langle\mathbf{r}''(s),\mathbf{r}''(s)\rangle\end{vmatrix}}
\end{aligned}$$

with formal determinants in the numerators and where $\boxed{\square}$ denotes the sign of the numerical determinant

$$\begin{vmatrix} \dot{x}(t) & \dot{y}(t) & \dot{z}(t) \\ \ddot{x}(t) & \ddot{y}(t) & \ddot{z}(t) \\ \dddot{x}(t) & \dddot{y}(t) & \dddot{z}(t) \end{vmatrix} . \tag{3.4}$$

We note, that the considered numerical determinants under the roots are always positive because of positive definiteness of the scalar product.

We consider the classical Frenet formulas as definitions for curvature and torsion of a curve. For the curvature given with arc length parametrization we have found by (3.3):

$$\kappa(s) = \langle \mathbf{t}'(s), \mathbf{n}(s) \rangle = \sqrt{\begin{vmatrix} \langle \mathbf{r}'(s), \mathbf{r}'(s) \rangle & \langle \mathbf{r}'(s), \mathbf{r}''(s) \rangle \\ \langle \mathbf{r}''(s), \mathbf{r}'(s) \rangle & \langle \mathbf{r}''(s), \mathbf{r}''(s) \rangle \end{vmatrix}} . \tag{3.5}$$

For arbitrary parameter t the formula above turns to

$$\kappa(t) = |\dot{\mathbf{r}}(t)|^{-3} \sqrt{\begin{vmatrix} \langle \dot{\mathbf{r}}(t), \dot{\mathbf{r}}(t) \rangle & \langle \dot{\mathbf{r}}(t), \ddot{\mathbf{r}}(t) \rangle \\ \langle \ddot{\mathbf{r}}(t), \dot{\mathbf{r}}(t) \rangle & \langle \ddot{\mathbf{r}}(t), \ddot{\mathbf{r}}(t) \rangle \end{vmatrix}} . \tag{3.6}$$

The computation of the torsion τ is rather lengthy and based by (3.3) on the scalar product:

$$\tau(s) = \langle \mathbf{n}'(s) + \kappa(s)\mathbf{t}(s), \mathbf{b}(s) \rangle = \langle \mathbf{n}'(s), \mathbf{b}(s) \rangle .$$

The final solution is the following:

$$\tau(s) = \boxed{\square} \cdot \sqrt{\begin{vmatrix} \langle \mathbf{r}'(s), \mathbf{r}'(s) \rangle & \langle \mathbf{r}'(s), \mathbf{r}''(s) \rangle & \langle \mathbf{r}'(s), \mathbf{r}'''(s) \rangle \\ \langle \mathbf{r}''(s), \mathbf{r}'(s) \rangle & \langle \mathbf{r}''(s), \mathbf{r}''(s) \rangle & \langle \mathbf{r}''(s), \mathbf{r}'''(s) \rangle \\ \langle \mathbf{r}'''(s), \mathbf{r}'(s) \rangle & \langle \mathbf{r}'''(s), \mathbf{r}''(s) \rangle & \langle \mathbf{r}'''(s), \mathbf{r}'''(s) \rangle \end{vmatrix}}$$
$$\cdot \begin{vmatrix} \langle \mathbf{r}'(s), \mathbf{r}'(s) \rangle & \langle \mathbf{r}'(s), \mathbf{r}''(s) \rangle \\ \langle \mathbf{r}''(s), \mathbf{r}'(s) \rangle & \langle \mathbf{r}''(s), \mathbf{r}''(s) \rangle \end{vmatrix} . \tag{3.7}$$

If we use arbitrary parameter t the solution will be

$$\tau(t) = \boxed{\square} \cdot |\dot{\mathbf{r}}(t)|^{-12} \sqrt{\begin{vmatrix} \langle \dot{\mathbf{r}}(t), \dot{\mathbf{r}}(t) \rangle & \langle \dot{\mathbf{r}}(t), \ddot{\mathbf{r}}(t) \rangle & \langle \dot{\mathbf{r}}(t), \dddot{\mathbf{r}}(t) \rangle \\ \langle \ddot{\mathbf{r}}(t), \dot{\mathbf{r}}(t) \rangle & \langle \ddot{\mathbf{r}}(t), \ddot{\mathbf{r}}(t) \rangle & \langle \ddot{\mathbf{r}}(t), \dddot{\mathbf{r}}(t) \rangle \\ \langle \dddot{\mathbf{r}}(t), \dot{\mathbf{r}}(t) \rangle & \langle \dddot{\mathbf{r}}(t), \ddot{\mathbf{r}}(t) \rangle & \langle \dddot{\mathbf{r}}(t), \dddot{\mathbf{r}}(t) \rangle \end{vmatrix}}$$
$$\cdot \begin{vmatrix} \langle \dot{\mathbf{r}}(t), \dot{\mathbf{r}}(t) \rangle & \langle \dot{\mathbf{r}}(t), \ddot{\mathbf{r}}(t) \rangle \\ \langle \ddot{\mathbf{r}}(t), \dot{\mathbf{r}}(t) \rangle & \langle \ddot{\mathbf{r}}(t), \ddot{\mathbf{r}}(t) \rangle \end{vmatrix} , \tag{3.8}$$

where $\boxed{\square}$ denotes the sign of the numerical determinant above (3.4). These can be extended for $\kappa = 0$ curve if $|\dot{\mathbf{r}}| > 0$, but $\dot{\mathbf{r}}$ and $\ddot{\mathbf{r}}$ are dependent vectors, moreover for $\tau = 0$ curve where $|\dot{\mathbf{r}}| > 0$, but $\dot{\mathbf{r}}, \ddot{\mathbf{r}}, \dddot{\mathbf{r}}$ are linearly dependent vectors.

We summarize this section with

Theorem 3.1. *Let* $\mathbf{r} : [a, b] \to Sol$ *be a* C^3 *curve given with arc length and arbitrary parametrization, respectively. Then the curvature and the torsion of the curve fulfill* (3.5) *and* (3.6), (3.7) *and* (3.8), *respectively.*

Remark. Of course, the formulas above are similar to those in the other homogeneous Riemann geometries as well with appropriate $|\dot{\mathbf{r}}|$.

4. Geodesics in *Sol* geometry

Now we intend to describe the geodesics of *Sol*. First, consider the former fundamental (metric) tensor (1.2)

$$(g_{ij}) = \begin{pmatrix} e^{2z} & 0 & 0 \\ 0 & e^{-2z} & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{4.1}$$

Compute now the Christoffel symbols in a usual way (applying the Einstein-Schouten conventions; $i, j, k, l \in \{1, 2, 3\}$ and $u^1 = x,\ u^2 = y,\ u^3 = z$):

$$\Gamma^k_{ij} = \frac{1}{2}\left(\frac{\partial g_{jl}}{\partial u^i} + \frac{\partial g_{li}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^l}\right) g^{lk} , \tag{4.2}$$

where (g^{lk}) denotes the inverse of (g_{ij}).

The non-vanishing components are:

$$\begin{aligned} \Gamma^1_{13} &= \Gamma^1_{31} = 1, \\ \Gamma^2_{23} &= \Gamma^2_{32} = -1, \\ \Gamma^3_{11} &= -e^{2z}, \\ \Gamma^3_{22} &= e^{-2z}. \end{aligned}$$

Therefore, the well-known equation of geodesics

$$\frac{d^2u^k}{dt^2} + \Gamma^k_{ij}\frac{du^i}{dt}\frac{du^j}{dt} = 0 \tag{4.3}$$

turns to

$$\begin{aligned} &\frac{d^2x(t)}{dt^2} + 2\frac{dx(t)}{dt}\frac{dz(t)}{dt} = 0 , \\ &\frac{d^2y(t)}{dt^2} - 2\frac{dy(t)}{dt}\frac{dz(t)}{dt} = 0 , \\ &\frac{d^2z(t)}{dt^2} - e^{2z(t)}\frac{dx(t)}{dt}\frac{dx(t)}{dt} + e^{-2z(t)}\frac{dy(t)}{dt}\frac{dy(t)}{dt} = 0 . \end{aligned}$$

As usual we abbreviate the notation:

$$\begin{aligned} &\ddot{x} + 2\dot{x}\dot{z} = 0 \\ &\ddot{y} - 2\dot{y}\dot{z} = 0 \\ &\ddot{z} - e^{2z}(\dot{x})^2 + e^{-2z}(\dot{y})^2 = 0. \end{aligned} \tag{4.4}$$

We solve this differential equation system as a Cauchy problem:

$$\begin{aligned} x(0) &= 0 & \qquad & & \dot{x}(0) &= u \\ y(0) &= 0 & & \text{and} & \dot{y}(0) &= v \\ z(0) &= 0 & & & \dot{z}(0) &= w\,. \end{aligned}$$

We start with the most general case, $u \neq 0$, $v \neq 0$, $w \neq 0$. By elimination

$$\frac{\ddot{x}}{\dot{x}} = -2\dot{z} \quad \text{and} \quad \frac{\ddot{y}}{\dot{y}} = 2\dot{z} \tag{4.5}$$

hold in a neighborhood of $t = 0$. (The case $u = 0$ or $v = 0$ will be treated later.) The first equation of (4.5) leads to

$$\dot{x} = ue^{-2z} \tag{4.6}$$

implying

$$x(t) = u\int_0^t e^{-2z(\tau)}d\tau\,. \tag{4.7}$$

Similarly we get

$$\dot{y} = ve^{2z} \tag{4.8}$$

and

$$y(t) = v\int_0^t e^{2z(\tau)}d\tau\,. \tag{4.9}$$

Substituting (4.6) and (4.8) into the last equation of (4.4) we obtain

$$\ddot{z} - u^2e^{-2z} + v^2e^{2z} = 0\,. \tag{4.10}$$

We multiply this equation by $2\dot{z}(\neq 0)$ and get

$$(\dot{z})^2 = -u^2e^{-2z} - v^2e^{2z} + c^2\,. \tag{4.11}$$

Using the initial conditions we have

$$w^2 = -u^2 - v^2 + c^2\,.$$

Without loss of generality we can take $c^2 = 1$, i.e. arc length parametrization at the origin $t = 0$, and so at other t's as well. From (4.11) it follows the separable differential equation

$$\frac{dz}{\pm\sqrt{1 - u^2e^{-2z} - v^2e^{2z}}} = dt \tag{4.12}$$

whose solution with elliptic integral cannot be expressed in terms of a finite number of elementary functions. Depending on the sign of w, we take the square root either with $+$ or $-$, respectively.

If $u \neq 0$, $v \neq 0$ and $w = 0$, then by (4.11) and $u^2 + v^2 = 1$ it follows $\dot{z} = 0$, and the solution

$$\begin{aligned} x(t) &= ut\,, \\ y(t) &= vt\,, \\ z(t) &= 0\,. \end{aligned} \tag{4.13}$$

Examine now the cases when the division by $\dot{x}$ or $\dot{y}$ is not possible in (4.5).

Next consider $u \neq 0$ and $v = 0$ $(u^2 = 1 - w^2)$. We look for $\dot{y}(t)$ in the form $c(t)e^{2z(t)}$. Then the second equation of (4.4) provides $c(t) = \text{constant} = 0$ by $v = 0$ and (4.6)

$$\ddot{z} - u^2 e^{-2z} = 0\,.$$

Solving this in a standard way, we get a parametric system of the geodesics:

$$\begin{aligned} x(t) &= u\frac{\sinh t}{\cosh t + w\sinh t}\,, \\ y(t) &= 0\,, \\ z(t) &= \ln\left(\cosh t + w\sinh t\right)\,. \end{aligned} \tag{4.14}$$

Similarly, if $u = 0$ and $v \neq 0$ $(1 = v^2 + w^2)$ we obtain

$$\begin{aligned} x(t) &= 0\,, \\ y(t) &= v\,\frac{\sinh t}{\cosh t - w\sinh t}\,, \\ z(t) &= -\ln\left(\cosh t - w\sinh t\right)\,. \end{aligned} \tag{4.15}$$

This corresponds to the second symmetric property of (2.3). Namely the change $(u, v, w) \leftrightarrow (v, u, -w)$ leads to the isometry of the corresponding geodesic curves.

Finally consider the case $u = 0$, $v = 0$ and $w = 1$ (e.g.). We look for the solution in the form

$$\begin{aligned} \dot{x}(t) &= a(t)e^{-2z(t)}\,, \\ \dot{y}(t) &= b(t)e^{2z(t)}\,. \end{aligned}$$

Then the first and second equation of (4.4) yield $a(t) = 0 = b(t)$ and so $x(t) = y(t) = 0$.

Then from the third equation of (4.4) it is easy to see that $\ddot{z} = 0$, so $z(t) = t$. The system of geodesics in this case is

$$\begin{aligned} x(t) &= 0 \\ y(t) &= 0 \\ z(t) &= t\,, \end{aligned} \tag{4.16}$$

which is not surprising at all.

Theorem 4.1. *The geodesic curves in Sol geometry have the equations in Table* 1, *depending on the initial conditions:*

$$\begin{aligned}
&0 = x(0) = y(0) = z(0)\\
&\dot{x}(0) = u\\
&\dot{y}(0) = v\\
&\dot{z}(0) = w\\
&u^2 + v^2 + w^2 = 1:
\end{aligned}$$

(1) $u \neq 0,\ v \neq 0,\ 0 < |w| = \sqrt{1-u^2-v^2} < 1$ (4.12)*;*
(2) $u \neq 0,\ v \neq 0,\ w = 0$ (4.13)*;*
(3) $v = 0,\ 0 < |w| = \sqrt{1-u^2} < 1$ (4.14)*;*
(4) $u = 0,\ 0 < |w| = \sqrt{1-v^2} < 1$ (4.15)*;*
(5) $u = 0,\ v = 0,\ |w| = 1$ (4.16).

We summarize the derivatives of $(x(t), y(t), z(t))$ *from which the curvature* $\kappa(t)$ *and torsion* $\tau(t)$ *can be computed by* (3.6) *and* (3.8) *and by computer (because of length of formulas):*

$$\begin{aligned}
&\dot{x} = ue^{-2z}\\
&\ddot{x} = -2ue^{-2z}\dot{z} = \mp 2ue^{-2z}\sqrt{1-u^2e^{-2z}-v^2e^{2z}}\\
&\dddot{x} = 4ue^{-2z}(\dot{z})^2 - 2ue^{-2z}\ddot{z} = -6u^3e^{-4z} + 4ue^{-2z} - 2uv^2
\end{aligned}$$

$$\begin{aligned}
&\dot{y} = ve^{2z}\\
&\ddot{y} = 2ve^{2z}\dot{z} = \pm 2ve^{2z}\sqrt{1-u^2e^{-2z}-v^2e^{2z}}\\
&\dddot{y} = 4ve^{2z}(\dot{z})^2 + 2ve^{2z}\ddot{z} = -2u^2v + 4ve^{2z} - 6v^3e^{4z}
\end{aligned}$$

$$\begin{aligned}
&\dot{z} = \pm\sqrt{1-u^2e^{-2z}-v^2e^{2z}}\\
&\ddot{z} = u^2e^{-2z} - v^2e^{2z}\\
&\dddot{z} = -2u^2e^{-2z}\dot{z} - 2v^2e^{2z}\dot{z} = \mp 2(u^2e^{-2z} + v^2e^{2z})\sqrt{1-u^2e^{-2z}-v^2e^{2z}}
\end{aligned}$$

for example

$$\langle \dot{\mathbf{r}}, \ddot{\mathbf{r}} \rangle = -\dot{z}\ddot{z} = \mp\left(u^2e^{-2z} - v^2e^{2z}\right)\sqrt{1-u^2e^{-2z}-v^2e^{2z}}\,.$$

(1)	$x(t) = u\int_0^t e^{-2z(\tau)}d\tau$ $y(t) = v\int_0^t e^{2z(\tau)}d\tau$ $z(t)$ comes from the separable differential equation $\dfrac{dz}{\pm\sqrt{1-u^2e^{-2z}-v^2e^{2z}}=dt}$, iff $w \gtreqless 0$ whose solution is non-elementary function.	$\kappa(0) =$ $\sqrt{(1-w^2)((1+w^2)^2-4u^2v^2)}$ $\tau(0) = \boxed{*}16u^2v^2(1-w^2)\cdot$ $((1+w^2)^2-4u^2v^2)\cdot$ $(3(u^2+v^2)(u^2-v^2-2)+4+2u^2v^2)^2$
(2)	$x(t) = ut$ $y(t) = vt$ $z(t) = 0$	$\kappa(t) \equiv 0$ $\tau(t) \equiv 0$
(3)	$x(t) = u\dfrac{\sinh t}{\cosh t + w\sinh t}$ $y(t) = 0$ $z(t) = \ln(\cosh t + w\sinh t)$	$\kappa(0) = (1+w^2)\sqrt{1-w^2}$ $\tau(0) = 0$ (moreover $\tau(t) \equiv 0$)
(4)	$x(t) = 0$ $y(t) = v\ \dfrac{\sinh t}{\cosh t - w\sinh t}$ $z(t) = -\ln(\cosh t - w\sinh t)$	$\kappa(0) = (1+w^2)\sqrt{1-w^2}$ $\tau(0) = 0$ (moreover $\tau(t) \equiv 0$)
(5)	$x(t) = 0$ $y(t) = 0$ $z(t) = \pm t$, iff $w = \pm 1$	$\kappa(t) \equiv 0$ $\tau(t) \equiv 0$

Table 1. Curvature and torsion of geodesics in *Sol* geometry, depending on the initial velocity parameters (u, v, w), $u^2 + v^2 + w^2 = 1$. The curvature κ and torsion τ are taken at the origin $(t = 0)$. $\boxed{*}$ denotes the sign of $[-4uv(-2w^2u^2 + 4v^2u^2 + 4v^2w^2 + u^4 + u^4v^4 - 2v^4 + 4w^2)]$.

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