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New Extensions of Napoleon's Theorem to Higher Dimensions

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Abstract. If equilateral triangles are erected outwardly on the sides of any given triangle, then the circumcenters of the three erected triangles form an equilateral triangle. This statement, known as Napoleon's theorem, and the configuration involved, usually called the Torricelli configuration of the initial triangle, were generalized to d-dimensional simplices ($d \geq 3$) in [12]. It is obvious that for $d \geq 3$ regular d-simplices cannot be erected on the facets of an arbitrary initial d-simplex S. Thus, instead of erecting such simplices, the authors of [12] used a related sphere configuration which also occurs in the planar situation. In the present paper, we give new d-dimensional analogues, mainly based on a higher dimensional Torricelli configuration constructed with the help of segments on lines through isogonal points and vertices of S. Interesting further properties of d-dimensional Torricelli configurations are obtained, too.

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1. Introduction

Napoleon's Theorem states that if equilateral triangles ABC', BCA', and CAB' are erected outwardly on the sides of any triangle ABC, then the centroids C^*, A^*, and B^* of these three triangles form an equilateral triangle; see [11] and [7, Corollary 4.1]. Also, the three triangles $A^*B^*C^*$ and $A'B'C'$, and ABC turn out to have the same centroid [7, Corollary 4.2]. The configuration involved is shown in Figure 1 and has several other interesting properties that are described in [11]. It is often referred to as the *Torricelli configuration* of ABC because E. Torricelli used it to answer a question of P. de Fermat regarding the point whose distances from the vertices of a given triangle ABC have minimal sum. Such a point is known since then as the *Fermat-Torricelli point* of the triangle ABC. The *Fermat-Torricelli point* of a d-simplex is defined similarly for every $d \geq 3$.

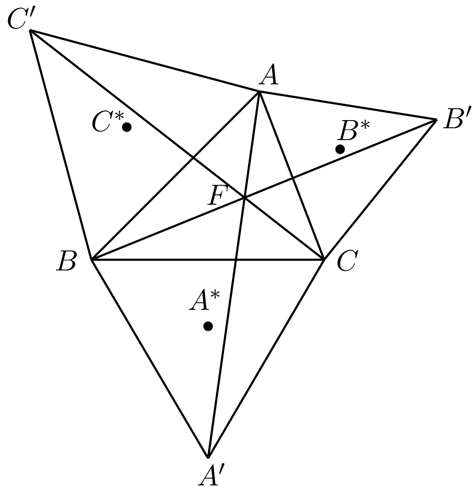

Figure 1.

Referring to the Torricelli configuration in Figure 1, Torricelli and, independently, B. Cavalieri proved that the circumcircles of ABC', BCA', and CAB' are concurrent, and Th. Simpson showed that the lines AA', BB', and CC' (also called Simpson lines) are concurrent and that these two points of concurrence coincide; see [1, Chapter II] for these and more historical details. Letting F be the common point of concurrence, Torricelli showed that if the angles of ABC are less than $120°$ each, then F is the Fermat-Torricelli point of ABC, and that

$$AA' = BB' = CC' = FA + FB + FC. \tag{1}$$

If one of the angles of ABC equals or exceeds $120°$, then the Fermat-Torricelli point coincides with the vertex that holds that angle. Thus if one denotes the Fermat-Torricelli point by $\mathcal{F}$ and the point of intersection of the lines AA', BB', and CC' by F, then $\mathcal{F}$ coincides with F if none of the angles of ABC equals or exceeds $120°$; otherwise $\mathcal{F}$ coincides with the vertex that holds the largest angle.

To extend Napoleon's theorem and related results to higher dimensions in a straightforward manner, one is faced with the obvious difficulty that it is impossible to erect regular d-simplices on the facets of an arbitrary d-simplex when $d \geq 3$;

see Remark 2 for questions and issues that arise in this connection. Thus we move in a completely different direction closely related to that initiated in [12]. It is based on constructing the Torricelli configuration in a reverse manner starting with the Fermat-Torricelli point of the triangle.

Starting with a triangle ABC having an interior Fermat-Torricelli point $\mathcal{F}$, one easily sees that the unit vectors emanating from $\mathcal{F}$ towards the vertices of ABC add up to the zero vector $\mathbb{O}$; see, e.g., [1, Theorem 18.3]. The points A', B', and C' of Figure 1 can now be obtained by either (i) extending $\mathcal{F}A$, $\mathcal{F}B$, and $\mathcal{F}C$ to meet the circumcircles of $\mathcal{F}BC$, $\mathcal{F}CA$, and $\mathcal{F}AB$, respectively, or (ii) extending $\mathcal{F}A$, $\mathcal{F}B$, and $\mathcal{F}C$ to points A', B', and C' so that (1) is satisfied. The first point of view has been taken up in [12] and has led to interesting higher dimensional analogues of Napoleon's theorem. In this paper, we will take up the second point of view.

2. Weighted pre-Torricelli configurations and their properties in dimension two

In this section, we introduce certain d-dimensional configurations that we have chosen to refer to as *pre-Torricelli*, and we summarize their properties for the case $d = 2$ in Theorem 1. This theorem encompasses Napoleon's theorem and the other statements pertaining to the Torricelli configuration, together with similar statements about configurations in which the equilateral triangles are drawn inwardly on the sides of ABC. It is designed as a preparation for the higher dimensional analogues given in Theorems 2 and 4, and its particular statements can, for example, be found in [2], [11], [1, Chapter II], and [7].

Definition 1 below is motivated by the fact that when the Fermat-Torricelli point $\mathcal{F}$ of a d-simplex $S = [A_1, \ldots, A_{d+1}]$ is interior, then the unit vectors emanating from $\mathcal{F}$ towards the vertices of S add up to the zero vector $\mathbb{O}$; see again [1, Theorem 18.3]. Moreover, when $\mathcal{F}$ is the generalized Fermat-Torricelli point that minimizes the sum $\sum w_i \|\mathcal{F} - A_i\|$ for given positive weights w_i, then the unit vectors $\mathbf{u}_1, \ldots, \mathbf{u}_{d+1}$ emanating from $\mathcal{F}$ towards the vertices of S have the property that $\sum w_i \mathbf{u}_i = \mathbb{O}$; cf. also [1, Theorem 18.37]. The converse is also true: If P is a point that is not a vertex of S and if the unit vectors $\mathbf{u}_1, \ldots, \mathbf{u}_{d+1}$ emanating from P towards the vertices of S have the property that $\sum w_i \mathbf{u}_i = \mathbb{O}$, then P is the generalized Fermat-Torricelli point corresponding to the weights $w_1, \ldots, w_{d+1}$. This is proved in [8, Theorem 1.1] for equal weights, but the proof applies word for word to any positive weights.

Definition 1. *A d-dimensional weighted pre-Torricelli configuration $\{\mathbf{u}, w, a, a'\}$ consists of affinely independent unit vectors $\mathbf{u}_1, \ldots, \mathbf{u}_{d+1}$, positive weights $w_1, \ldots, w_{d+1}$, and non-zero real numbers $a_1, \ldots, a_{d+1}$, $a'_1, \ldots, a'_{d+1}$ such that*

$$w_1\mathbf{u}_1 + \cdots + w_{d+1}\mathbf{u}_{d+1} = \mathbb{O}, \text{ where } \mathbb{O} \text{ is the zero vector,} \tag{2}$$

$$a_i > a'_i \text{ for every } i. \tag{3}$$

When the weights are equal, the configuration is called a pre-Torricelli configuration and is denoted by $\{\mathbf{u}, a, a'\}$.

The base or initial d-simplex in the configuration is the d-simplex $S = [A_1, \ldots, A_{d+1}]$, *where* $A_i = a_i \mathbf{u}_i$.
The Torricelli d-simplex is the d-simplex $S' = [A'_1, \ldots, A'_{d+1}]$, *where* $A'_i = a'_i \mathbf{u}_i$, *and the i-th ear d-simplex* S_i *is the d-simplex obtained from* S *by replacing* A_i *with* A'_i.
The centroidal Napoleon d-simplex is the d-simplex $S^* = [A^*_1, \ldots, A^*_{d+1}]$, *where* A^*_i *is the centroid of the i-th ear d-simplex* S_i.
The circumcentral Napoleon d-simplex is the d-simplex $S^\circ = [A^\circ_1, \ldots, A^\circ_{d+1}]$, *where* A°_i *is the circumcenter of the i-th ear d-simplex* S_i.

The use of the prefix *pre* in the term *pre-Torricelli* reflects the fact that no conditions on how a and a' are related are imposed. It also gives the false impression that we will later say what *Torricelli* configurations are. We expect this to be done in later papers where we would explore what minimal conditions a pre-Torricelli configuration has to obey in order that it maximally resembles the ordinary planar Torricelli configuration. A Torricelli configuration would then be a pre-Torricelli configuration that obeys those conditions.

The circumcentral Napoleon d-simplex will never be referred to except in Remark 7 at the end of the paper. Note that in the ordinary planar Torricelli configuration where the ear triangles are equilateral, the centroidal and circumcentral Napoleon triangles coincide.

Note also that the length $\|A_iA'_i\|$ of the line segment $A_iA'_i$ is $a_i - a'_i$, and that (2) is essentially the only dependence relation among the $\mathbf{u}_i$ since they are affinely independent.

In Theorem 1, as well as in Theorems 2 and 4, S, S', and S^* refer to the base, Torricelli, and centroidal Napoleon d-simplices described in Definition 1.

Theorem 1. *Let* $\mathbb{T} = \{\mathbf{u}, a, a'\}$ *be a pre-Torricelli configuration, and let* $s = a_1 + \cdots + a_{d+1}$. *If* $d = 2$, *then*

(A) *the following four statements are equivalent:*

- (A1) *the centroids* G, G', *and* G^* *of* S, S', *and* S^* *coincide,*
- (A2) *any two of the centroids* G, G', *and* G^* *of* S, S', *and* S^* *coincide,*
- (A3) *the centroidal Napoleon triangle* S^* *is equilateral,*
- (A4) $a_i - a'_i = a_j - a'_j$ *for all* i *and* j *in* $\{1, \ldots, d+1\}$,

(B) *the following three statements are equivalent:*

- (B1) *the circumcircles of the* S_i *all pass through* $\mathbb{O}$,
- (B2) $a_i - a'_i = s$ *for all* i *in* $\{1, \ldots, d+1\}$,
- (B3) *the ear triangles are equilateral,*

(C) *the base or initial triangle is arbitrary in the sense that every triangle (including degenerate triangles) is the base triangle of a (not necessarily unique) pre-Torricelli configuration,*

(D) *the Torricelli triangle is arbitrary in the sense that every triangle (including degenerate triangles) is the Torricelli triangle of a (not necessarily unique) pre-Torricelli configuration.*

As mentioned earlier, the particular statements of this theorem can, for example, be found in [2], [11], [1, Chapter II], and [7].

3. Remarks pertaining to weighted pre-Torricelli configurations in higher dimensions

Before extending Theorem 1 to higher dimensions, we find it helpful to give a few remarks.

Remark 1. In Napoleon's theorem stated at the very beginning of this article, the vertices A^*, B^*, and C^* of the Napoleon triangle were taken to be the centroids of the ear triangles. Since the ear triangles are equilateral, these points could as well have been taken to be the circumcenters of these triangles. This explains why it is natural to introduce both the centroidal and circumcentral Napoleon triangles in our definition of the pre-Torricelli configuration.

Remark 2. A main difficulty in extending Napoleon's theorem to higher dimensions in a straightforward manner lies in the impossibility of erecting regular d-simplices on the facets of an arbitrary initial d-simplex when $d \geq 3$. One wonders here whether things would work if the d-simplices erected on the facets are required merely to be regular in some weak sense. This very natural question is related to the different weaker degrees of simplex regularity that abound in the existing literature; see [4]. This type of approach will be taken up in another project and is expected to lead to interesting questions. However, equilaterality (or regularity) of a triangle can be characterized in many different ways that, although equivalent for triangles, are completely different for d-simplices when $d \geq 3$, and especially when $d \geq 4$. A natural type of regularity is obtained by considering d-simplices in which two or more centers coincide. Recalling that the *centroid* $\mathcal{G}$ of a d-simplex $[A_1, \ldots, A_{d+1}]$ is the average $\frac{A_1+\cdots+A_{d+1}}{d+1}$ of its vertices and that the *circumcenter* $\mathcal{C}$ is the d-sphere that passes through the vertices, one may refer to a d-simplex in which $\mathcal{G}$ and $\mathcal{C}$ coincide as a $(\mathcal{G}, \mathcal{C})$-regular d-simplex. Similar definitions are obtained by taking other centers such as the *incenter* $\mathcal{I}$, i.e., the center of the d-sphere that touches all the facets, and the Fermat-Torricelli point $\mathcal{F}$. These degrees of regularity and their relations to facial structures of a d-simplex have been investigated earlier in [4], where it is proved that

$$\begin{aligned} S \text{ is } (\mathcal{G},\mathcal{C})\text{-regular} &\iff S \text{ is } (\mathcal{F},\mathcal{C})\text{-regular} \\ &\iff S \text{ is } (\mathcal{F},\mathcal{G})\text{-regular} \\ &\iff S \text{ has well-distributed edge-lengths in the sense} \\ &\qquad \text{that the sum of squares of the edge-lengths of a} \\ &\qquad \text{facet is the same for all facets,} \end{aligned}$$

S is $(\mathcal{C}, \mathcal{I})$-regular $\iff$ $\mathcal{C}$ is interior and S is equiradial in the sense that its facets have equal circumradii,

S is $(\mathcal{I}, \mathcal{G})$-regular $\iff$ S is equiareal in the sense that its facets have equal volumes.

Note that for triangles all these degrees of regularity collapse to simple equilaterality; see [15, Exercise 1, p. 37], [9, pp. 78–79], and [6]. However, it is seen in [4] that even a $(\mathcal{G}, \mathcal{C}, \mathcal{F})$-regular d-simplex is far from being regular.

In view of this, one would expect the statements (A3) and (B3) in Theorem 1 to take other forms in higher dimensions.

Remark 3. Consider the Torricelli configuration shown in Figure 1 and the point F of concurrence of AA', BB', CC'. If no angle of ABC is larger than 120°, then F is the point P that minimizes $PA + PB + PC$. This is not true if one of the angles of ABC exceeds 120°; see Figure 2. In this case, one wonders whether the point F has any other significance, and it is natural to expect that F is the point that minimizes a quantity of the type $\mp PA \mp PB \mp PC$. This false expectation is asserted in the famous book [3, Chapter VII, § 5.2, pp. 356–358] and it was later corrected in [2], [13], and [5]; see also [10] and [1, § 23.4]. In any case, the unit vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ emanating from F to the vertices of ABC have the property that $\mp\mathbf{u} \mp \mathbf{v} \mp \mathbf{w} = \mathbb{O}$. Such points are usually referred to as isogonal points and will be discussed in the next remark.

Remark 4. Statement (C) in Theorem 1 follows from the fact that every triangle has at least one isogonal point, where a point P is said to be an *isogonal* point of a d-simplex S if the unit vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{d+1}$ emanating from P towards the vertices of S satisfy $\pm\mathbf{v}_1 \pm \cdots \pm \mathbf{v}_{d+1} = \mathbb{O}$ for some choice of $\pm$ signs. For a triangle ABC, the isogonal points are described in [5]. Assuming that the angles B and C of the triangle ABC satisfy ($B \geq 60°$ and $C \geq 60°$) or ($B \leq 60°$ and $C \leq 60°$), we draw an equilateral triangle BCX and we let P be where the line AX meets the circumcircle of BCX. Then P is an isogonal point. Since there are two equilateral triangles BCX and BCX', we obtain two isogonal points. In fact, it is shown in the last theorem of [5] that every non-equilateral triangle has exactly two isogonal points, and that all the points on the circumcircle of an equilateral triangle are isogonal points. The question whether similar statements about the existence and significance of isogonal points hold in higher dimensions will not be addressed here. Thus we will not seek a higher dimensional analogue of statement (C) in this paper.

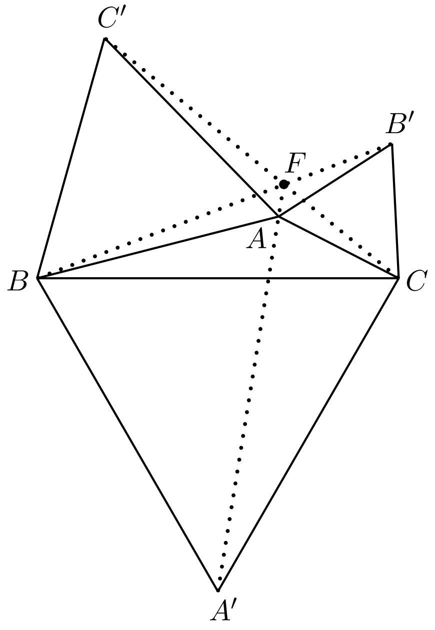

Figure 2.

Remark 5. Statement (D) of Theorem 1 states that the shape of the triangle $A'B'C'$ in the Torricelli configuration is arbitrary in the sense that given any triangle UVW, there exists a triangle ABC in whose Torricelli configuration the triangle $A'B'C'$ is similar to UVW. This is proved in [7]. Higher dimensional analogues of this are expected to be very difficult and will also not be addressed in this paper.

Remark 6. In the Torricelli configuration in Figure 1, the triangles $A^*B^*C^*$ and $A'B'C'$ are referred to as the *negative Napoleon* and *negative Torricelli triangles*, respectively; see [7]. The corresponding positive triangles are the triangles obtained when the equilateral triangles ABC', BCA', and CAB' are erected inwardly; for analogous properties of the obtained configuration see once more the survey [11].

4. Properties of weighted pre-Torricelli configurations in higher dimensions

In view of Theorem 1 and the remarks after it, we formulate and prove the following theorem. It complements, and naturally has non-empty intersection with, the results in [12], where similar generalizations of Napoleon's theorem are obtained.

Theorem 2. *Let* $\mathbb{T} = \{\mathbf{u}, w, a, a'\}$ *be a weighted pre-Torricelli configuration and let* $s = w_1a_1 + \cdots + w_{d+1}a_{d+1}$. *Then*

(A) *the following four statements are equivalent:*

(A1) *the centroids* G, G', *and* G^* *of* S, S', *and* S^* *coincide,*

(A2) *any two of the centroids* G, G', *and* G^* *of* S, S', *and* S^* *coincide,* (A3) *the Fermat-Torricelli point, with respect to corresponding assumed weights* w_i, *of the centroidal Napoleon* d-*simplex* S^* *coincides with its centroid,*

(A4) $\frac{a_i-a_i'}{w_i} = \frac{a_j-a_j'}{w_j}$ *for all* i *and* j *in* $\{1,\ldots,d+1\}$,

(B) *the following two statements are equivalent:*

(B1) *the circumspheres of the* $d+1$ *ear* d-*simplices* S_i *all pass through* $\mathbb{O}$,

(B2) $w_i(a_i - a_i') = s$ *for all* i *in* $\{1,\ldots,d+1\}$.

Proof. (A) Let G, G', and G^* be the centroids of S, S', and S^*, respectively, and let G_j be the centroid of S_j, $j = 1,\ldots,d+1$. Then

$$
\begin{aligned}
G &= \frac{1}{d+1}\sum_{i=1}^{d+1} a_i\mathbf{u}_i, && (4)\\
G' &= \frac{1}{d+1}\sum_{i=1}^{d+1} a_i'\mathbf{u}_i, && (5)\\
G_j &= \frac{1}{d+1}\left[\left(\sum_{i=1}^{d+1} a_i\mathbf{u}_i\right) - (a_j - a_j')\mathbf{u}_j\right] \;=\; G - \frac{1}{d+1}(a_j-a_j')\mathbf{u}_j, && (6)\\
G^* &= \frac{1}{d+1}\sum_{i=1}^{d+1} G_i \;=\; G - \frac{1}{(d+1)^2}\sum_{i=1}^{d+1}(a_i-a_i')\mathbf{u}_i && (7)\\
&= G - \frac{1}{d+1}\left(G - G'\right) \;=\; \frac{1}{d+1}\left(dG + G'\right). && (8)
\end{aligned}
$$

It follows from (8) that $(d+1)G^* = dG + G'$ and thus if any two of G, G', and G^* coincide, then the three do. It also follows from (4) and (5) that

$$
\begin{aligned}
G = G' \iff& \sum_{i=1}^{d+1}(a_i' - a_i)\mathbf{u}_i = \mathbb{O}\\
\iff& \frac{a_i'-a_i}{w_i} = \frac{a_j'-a_j}{w_j} \text{ for all } i \text{ and } j \text{ in } \{1,\ldots,d+1\},
\end{aligned}
$$

because $w_1\mathbf{u}_1 + \cdots + w_{d+1}\mathbf{u}_{d+1} = \mathbb{O}$ is essentially the only dependence relation among the $\mathbf{u}_i$. This shows that (A1), (A2), and (A4) are equivalent. To deal with (A3), note that it follows from (6) that the d-simplex S^* is similar to the d-simplex $S_0 = [b_1w_1\mathbf{u}_1,\ldots,b_{d+1}w_{d+1}\mathbf{u}_{d+1}]$, where $b_j = (a_j - a_j')/w_j$. This is seen by translating each vertex G_i of S^* by $-G$ and then multiplying by $-(d+1)$. Since $b_j > 0$, it follows that $\mathbb{O}$ is the Fermat-Torricelli point of S_0 with weights $w_1,\ldots,w_{d+1}$. If the b_j are all equal, then $b_1w_1\mathbf{u}_1 + \cdots + b_{d+1}w_{d+1}\mathbf{u}_{d+1} = \mathbb{O}$, and $\mathbb{O}$ is the centroid of S_0. Conversely, if $\mathbb{O}$ is the centroid of S_0, then $b_1w_1\mathbf{u}_1 + \cdots + b_{d+1}w_{d+1}\mathbf{u}_{d+1} = \mathbb{O}$, and therefore the b_i are all equal, since $w_1\mathbf{u}_1 + \cdots + w_{d+1}\mathbf{u}_{d+1} = \mathbb{O}$ is essentially the only dependence relation among the $\mathbf{u}_i$. Thus (A3) and (A4) are equivalent, and the proof of (A) is complete.

(B) This follows from the following equivalences:
The circumsphere of S_{d+1} passes through $\mathbb{O}$

$$\begin{aligned}
&\iff \text{there exists an } X \text{ such that} \\
&\qquad \|X\| = \|X - a_1\mathbf{u}_1\| = \cdots = \|X - a_d\mathbf{u}_d\| = \|X - a'_{d+1}\mathbf{u}_{d+1}\| \\
&\iff \text{there exists an } X \text{ such that} \\
&\qquad 2X \cdot \mathbf{u}_1 = a_1, \ldots, 2X \cdot \mathbf{u}_d = a_d, 2X \cdot \mathbf{u}_{d+1} = a'_{d+1} \\
&\iff \text{the system } MX = C \text{ has a solution, where } M \text{ is the matrix whose rows} \\
&\qquad \text{are } \mathbf{u}_1, \ldots, \mathbf{u}_{d+1}, \text{ and where } C = [a_1, \ldots, a_d, a'_{d+1}]^T \\
&\iff w_1a_1 + \cdots + w_da_d + w_{d+1}a'_{d+1} = 0 \text{ because } w_1\mathbf{u}_1 + \cdots + w_{d+1}\mathbf{u}_{d+1} = \mathbb{O} \\
&\qquad \text{is essentially the only dependence relation among the rows of } M \\
&\iff w_{d+1}(a_{d+1} - a'_{d+1}) = s.
\end{aligned}$$

This proves that (B1) and (B2) are equivalent. □

With the exception of (B3), Theorem 1 follows from Theorem 2 by taking the weights to be equal and by observing that if two centers of a triangle coincide then it is equilateral. To cover (and prove an analogue of) (B3), we introduce the notion of Torricelli regularity. Then Theorem 4 follows immediately.

Definition 2. *A d-simplex is said to be Torricelli regular if it is the base d-simplex in a pre-Torricelli configuration* $\{\mathbf{u}, a, a'\}$ *in which* $a_1 + \cdots + a_{d+1} = 0$.

In view of the the investigation in [4], it would be interesting to explore how Torricelli regularity is related to other degrees of regularity, such as equiareality, equiradiality, etc. For triangles, the following theorem gives the answer.

Theorem 3. *A triangle is Torricelli regular if and only if it is equilateral.*

Proof. Let ABC be an equilateral triangle. To show that it is Torricelli regular, we take the origin $\mathbb{O}$ to be any point on the smaller arc A_1A_2 of the circumcircle of $A_1A_2A_3$, and we let

$$\mathbf{u}_1 = \frac{\mathbb{O}A_1}{\|\mathbb{O}A_1\|},\ \mathbf{u}_2 = \frac{\mathbb{O}A_2}{\|\mathbb{O}A_2\|},\ \mathbf{u}_3 = -\frac{\mathbb{O}A_3}{\|\mathbb{O}A_3\|},$$
$$a_1 = \|\mathbb{O}A_1\|,\ a_2 = \|\mathbb{O}A_2\|,\ a_3 = -\|\mathbb{O}A_3\|;$$

see Figure 3. Then $A_i = a_i\mathbf{u}_i$, $i = 1, 2, 3$. Also, $\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3 = \mathbb{O}$ because the $\mathbf{u}_i$ are equally inclined, and $a_1 + a_2 + a_3 = 0$ by Ptolemy's theorem; see [14, p. 90].

Conversely, suppose that $\mathbf{u}_1, \mathbf{u}_2$, and $\mathbf{u}_3$ are unit vectors whose sum is $\mathbb{O}$ and that a_1, a_2, a_3 are real numbers whose sum is 0. We are to show that $a_1\mathbf{u}_1, a_2\mathbf{u}_2, a_3\mathbf{u}_3$ are the vertices of an equilateral triangle. It follows from $\|\mathbf{u}_1\| = \|\mathbf{u}_2\| = \|\mathbf{u}_3\| = 1$ and $\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3 = \mathbb{O}$ that $\mathbf{u}_i \cdot \mathbf{u}_j = -1/2$ for all $i \neq j$.

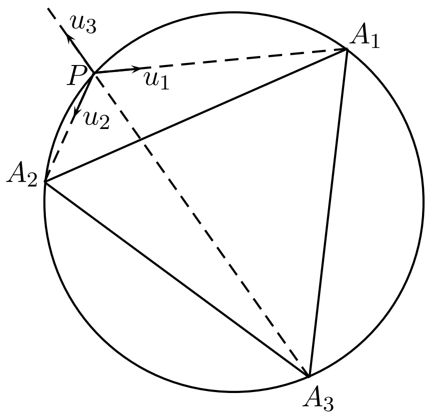

Figure 3.

Therefore

$$\begin{aligned} \|a_1\mathbf{u}_1 - a_2\mathbf{u}_2\|^2 - \|a_1\mathbf{u}_1 - a_3\mathbf{u}_3\|^2 &= \left(a_1^2 + a_2^2 + a_1a_2\right) - \left(a_1^2 + a_3^2 + a_1a_3\right) \\ &= (a_2 - a_3)\,(a_2 + a_3 + a_1) \\ &= 0. \end{aligned}$$

Therefore $\|a_1\mathbf{u}_1 - a_2\mathbf{u}_2\| = \|a_1\mathbf{u}_1 - a_3\mathbf{u}_3\|$ and hence the triangle whose vertices are $a_1\mathbf{u}_1, a_2\mathbf{u}_2, a_3\mathbf{u}_3$ is equilateral, as desired. □

Now we are ready to present the announced Theorem 4.

Theorem 4. *Let $\mathbb{T} = \{\mathbf{u}, a, a'\}$ be a pre-Torricelli configuration and let $s = a_1 + \cdots + a_{d+1}$. Then*

(A) *the following four statements are equivalent:*

(A1) *the centroids G, G', and G^* of S, S', and S^* coincide,*

(A2) *any two of the centroids G, G', and G^* of S, S', and S^* coincide,*

(A3) *the Fermat-Torricelli point of the centroidal Napoleon d-simplex S^* coincides with its centroid,*

(A4) $a_i - a_i' = a_j - a_j'$ *for all i and j in* $\{1, \ldots, d+1\}$,

(B) *the following four statements are equivalent:*

(B1) *the circumspheres of the $d+1$ ear d-simplices S_i all pass through $\mathbb{O}$,*

(B2) $a_i - a_i' = s$ *for all i in* $\{1, \ldots, d+1\}$,

(B3) *the ear d-simplices are Torricelli regular.*

Remark 7. Let $\mathbb{T} = \{\mathbf{u}, a, a'\}$ be a d-dimensional weighted pre-Torricelli configuration. Statements (A4) and (B2) in Theorems 1, 2, and 4 show that much of the geometry of the configuration can be stated in terms of the tuples a and a', or rather the tuple $a - a'$. One of the geometric properties that was highlighted in [12] pertains to what was referred to (in Definition 1) as the circumcentral

Napoleon d-simplex S°. We call this property (B4) and we write it down in three forms that suit Theorems 1, 2, and 4, respectively.

(B4-1) The circumcentral Napoleon triangle S° is equilateral.

(B4-2) The $(d-1)$-volumes V_i of the facets of the circumcentral Napoleon d-simplex S° are proportional to the weights w_i.

(B4-4) The circumcentral Napoleon d-simplex S° is equiareal (or, equivalently, $(\mathcal{I},\mathcal{G})$-regular in the sense that its incenter $\mathcal{I}$ and centroid $\mathcal{G}$ coincide).

A main theorem in [12] says that if (B1) (or, equivalently, any of the statements in (B)) holds in any of Theorems 1, 2, or 4, then the corresponding (B4) holds also. On the other hand, the converse of this is not true even in dimension 2. To see this, consider the pre-Torricelli configuration with equal weights and with $a_1 = a_2 = a_3$ and $a_1' = a_2' = a_3'$. This is the configuration with equilateral base triangle and with isosceles ear triangles. It is clear that the circumcentral Napoleon triangle is equilateral for all such a and a'. However, (B2) is satisfied if and only if $a_1' = 4a_1$. Note that in this particular example, (A4) (and hence all the statements in (A)) hold.

It would be interesting to write in terms of a and a' a condition that is equivalent to (B4). It would also be interesting to explore whether (B4) implies the statements in (A).

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