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Convex Hulls of Polyominoes

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Abstract. In this article we prove a conjecture of Bezdek, Braß, and Harborth concerning the maximum volume of the convex hull of any facet-to-facet connected system of n unit hypercubes in $\mathbb{R}^d$ [4]. For $d = 2$ we enumerate the extremal polyominoes and determine the set of possible areas of the convex hull for each n.

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1. Introduction

In the legend [1] of the founding of Carthage, Queen Dido purchased the right to get as much land as she could enclose with the skin of an ox. She splitted the skin into thin stripes and tied them together. Using the natural boundary of the sea and by constructing a giant semicircle she enclosed more land than the seller could have ever imagined.

Dido-type problems have been treated by many authors i.e. [2, 4, 5, 6, 9], here we consider the maximum volume of a union of unit hypercubes. A d-dimensional polyomino is a facet-to-facet connected system of d-dimensional unit hypercubes. Examples for 2-dimensional polyominoes are the pieces of the computer game Tetris.

In 1994 Bezdek, Braß, and Harborth conjectured that the maximum volume of the convex hull of a d-dimensional polyomino consisting of n hypercubes is given by

$$\sum_{I\subseteq\{1,\dots,d\}} \frac{1}{|I|!} \prod_{i\in I} \left\lfloor \frac{n-2+i}{d} \right\rfloor,$$

but were only able to prove it for $d = 2$. In Section 3 we prove this conjecture. They also asked for the number $c_2(n)$ of different polyominoes with n cells and maximum area $n + \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor$. In Section 2 we prove

Theorem 1.

$$c_2(n) = \begin{cases} \frac{n^3-2n^2+4n}{16} & \text{if } n \equiv 0 \mod 4, \\ \frac{n^3-2n^2+13n+20}{32} & \text{if } n \equiv 1 \mod 4, \\ \frac{n^3-2n^2+4n+8}{16} & \text{if } n \equiv 2 \mod 4, \\ \frac{n^3-2n^2+5n+8}{32} & \text{if } n \equiv 3 \mod 4. \end{cases}$$

Besides the maximum area $n + \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor$ and the minimum area n of the convex hull of polyominoes with n cells several other values may be attained. For each n we characterize the corresponding sets.

Theorem 2. *A polyomino consisting of n cells with area $\alpha = n + \frac{m}{2}$ of the convex hull exists if and only if $m \in \mathbb{N}_0$, $0 \le m \le \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor$, and $m \neq 1$ if $n + 1$ is a prime.*

2. The planar case

An example which attains the upper bound $n + \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor$ for the area of the convex hull of a polyomino with n cells is quite obvious, see Figure 1. Instead of proving this upper bound by induction over n we specify polyominoes by further parameters and then apply an induction argument.

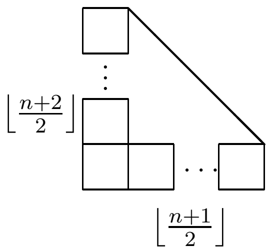

Figure 1. 2-dimensional polyomino with maximum convex hull

We describe these parameters for the more general d-dimensional case and therefore denote the standard coordinate axes of $\mathbb{R}^d$ by $1, \ldots, d$. Every d-dimensional polyomino has a smallest surrounding box with side lengths $l_1, \ldots, l_d$, where l_i is the length in direction i. If we build up a polyomino cell by cell then after adding a cell one of the l_i will increase by 1 or none of the l_i will increase. In the second case we increase v_i by 1, where the new hypercube has a facet-neighbor in direction of axis i. If M is the set of axis-directions of facet-neighbors of the new hypercube, then we will increase v_i by 1 for only one $i \in M$. Since at this position there is the possibility to choose, we must face the fact that there might be different tuples $(l_1, \ldots, l_d, v_1, \ldots, v_d)$ for the same polyomino. We define $v_1 = \cdots = v_d = 0$ for the polyomino consisting of a single hypercube. This definition of the l_i and the v_i leads to

$$n = 1 + \sum_{i=1}^{d} (l_i - 1) + \sum_{i=1}^{d} v_i. \tag{1}$$

Example 1. The possible tuples describing a rectangular 2×3-polyomino are $(2,3,2,0)$, $(2,3,1,1)$, and $(2,3,0,2)$.

Definition 1.

$$\begin{aligned} f_2(l_1, l_2, v_1, v_2) &= 1 + (l_1 - 1) + (l_2 - 1) + \frac{(l_1 - 1)(l_2 - 1)}{2} \\ &\quad + v_1 + v_2 + \frac{v_1(l_2 - 1)}{2} + \frac{v_2(l_1 - 1)}{2} + \frac{v_1 v_2}{2}. \end{aligned}$$

Lemma 1. *The area of the convex hull of a 2-dimensional polyomino with tuple (l_1, l_2, v_1, v_2) is at most $f_2(l_1, l_2, v_1, v_2)$.*

Proof. We prove the statement by induction on n, using equation (1). For $n = 1$ only $l_1 = l_2 = 1$, $v_1 = v_2 = 0$ is possible. With $f_2(1, 1, 0, 0) = 1$ the induction base is done. Now we assume that the statement is true for all possible tuples (l_1, l_2, v_1, v_2) with $1 + \sum_{i=1}^{d} (l_i - 1) + \sum_{i=1}^{d} v_i = n - 1$.

Due to symmetry we consider only the growth of l_1 or v_1, and the area a of the convex hull by adding the n-th square.

(i) l_1 increases by one:

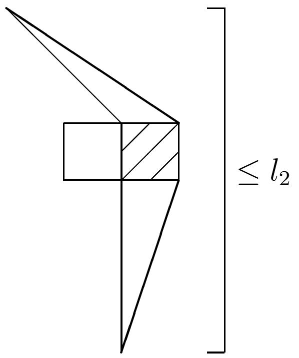

Figure 2. Increasing l_1

We depict (see Figure 2) the new square by 3 diagonal lines. Since l_1 increases the new square must have a left or a right neighbor. Without loss of generality it has a left neighbor. The new square contributes at most 2 (thick) lines to the convex hull of the polyomino. By drawing lines from the neighbor square to the endpoints of the new lines we see that the growth is at most $1 + \frac{l_2 - 1}{2}$, a growth of 1 for the new square and the rest for the triangles. Since $f_2(l_1 + 1, l_2, v_1, v_2) - f_2(l_1, l_2, v_1, v_2) = 1 + \frac{l_2 - 1}{2} + \frac{v_2}{2}$ the induction step follows.

(ii) v_1 increases by one:

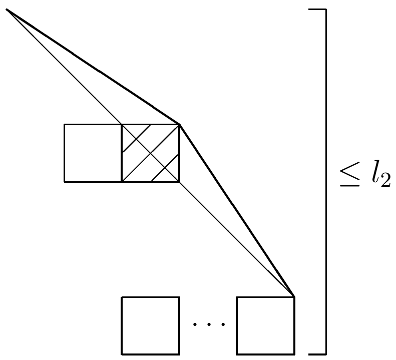

Figure 3. Increasing v_1

In Figure 3 we depict the new square by 3 diagonal lines. Without loss of generality we assume that the new square has a left neighbor, and contributes at most 2 lines to the convex hull of the polyomino. As l_1 is not increased there must be a square in the same column as the new square. Similar to (i) we draw lines from the neighbor square to the endpoints of the new lines and see that the growth of the area of the convex hull is less than $\frac{l_2-1}{2}$. □

Theorem 3. *The area of the convex hull of a 2-dimensional polyomino with n unit squares is at most $n + \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor$.*

Proof. For given n we determine the maximum of $f_2(l_1, l_2, v_1, v_2)$. Since $f_2(l_1 + 1, l_2, v_1 - 1, v_2) - f_2(l_1, l_2, v_1, v_2) = 0$ and due to symmetry we assume $v_1 = v_2 = 0$ and $l_1 \leq l_2$. With

$$f_2(l_1 + 1, l_2 - 1, 0, 0) - f_2(l_1, l_2, 0, 0) = \frac{l_2 - l_1 - 1}{2} > 0$$

we conclude $0 \leq l_2 - l_1 \leq 1$. Using equation (1) gives $l_1 = \left\lfloor \frac{n+1}{2} \right\rfloor$, $l_2 = \left\lfloor \frac{n+2}{2} \right\rfloor$. Thus by inserting in Lemma 1 we receive $f_2(l_1, l_2, v_1, v_2) \leq n + \frac{1}{2} \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor$. This maximum is attained for example by the polyomino in Figure 1. □

In the next lemma we describe the shape of the 2-dimensional polyominoes with maximum area of the convex hull in order to determine their number $c_2(n)$.

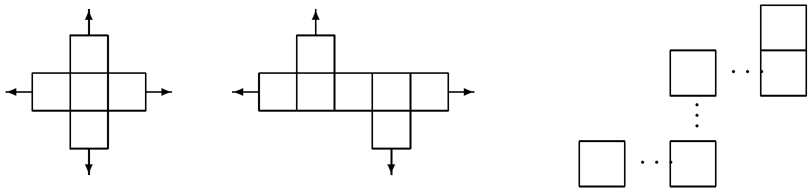

Figure 4. The two shapes of polyominoes with maximum area of the convex hull and a forbidden sub-polyomino

Lemma 2. *Every 2-dimensional polyomino with parameters l_1, l_2, v_1, v_2, and with the maximum area $n + \frac{1}{2} \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor$ of the convex hull consists of a linear strip with at most one orthogonal linear strip on each side (see the left two pictures in Figure 4). Additionally we have $v_1 = v_2 = 0$ and the area of the convex hull is given by $f_2(l_1, l_2, v_1, v_2)$.*

Proof. From the proof of Lemma 1 we deduce $v_1 = v_2 = 0$ and that every sub-polyomino has also the maximum area of the convex hull. Since the area of the polyomino on the right hand side of Figure 4 has an area of the convex hull which is less than $f_2(l_1, l_2, 0, 0)$ it is a forbidden sub-polyomino and only the described shapes remain. All these polyominoes attain the maximum $f_2(l_1, l_2, 0, 0)$. □

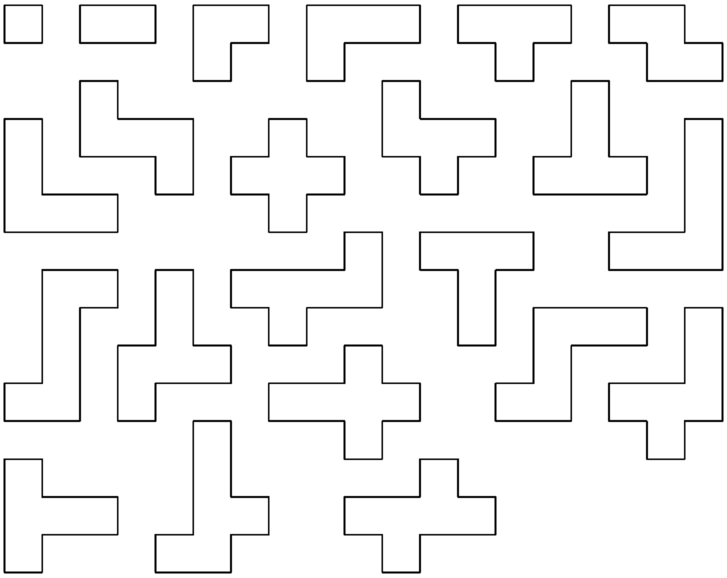

Figure 5. Complete set of extremal polyominoes with $n \le 6$ cells

Theorem 1.

$$c_2(n) = \begin{cases} \frac{n^3-2n^2+4n}{16} & \text{if} \quad n \equiv 0 \mod 4, \\ \frac{n^3-2n^2+13n+20}{32} & \text{if} \quad n \equiv 1 \mod 4, \\ \frac{n^3-2n^2+4n+8}{16} & \text{if} \quad n \equiv 2 \mod 4, \\ \frac{n^3-2n^2+5n+8}{32} & \text{if} \quad n \equiv 3 \mod 4. \end{cases}$$

Proof of Theorem 1. (Formula for $c_2(n)$.) We use Lemma 2 and do a short calculation applying the lemma of Cauchy-Frobenius. □

Corollary 1. *The ordinary generating function for $c_2(n)$ is given by*

$$\frac{1 + x - x^2 - x^3 + 2x^5 + 8x^6 + 2x^7 + 4x^8 + 2x^9 - x^{10} + x^{12}}{(1 - x^2)^2(1 - x^4)^2}.$$

We have depicted the polyominoes with at most 6 cells and maximum area of the convex hull in Figure 5. For more cells we give only a few concrete numbers:

$$(c_2(n))_{n=1,\dots} = 1, 1, 1, 3, 5, 11, 9, 26, 22, 53, 36, 93, 64, 151, 94, 228, 143, 329, 195, 455, 271, 611, 351, 798, 460, 1021, 574, 1281, 722, 1583, 876, 1928, 1069, 2321, 1269, 2763, 1513, 3259, 1765, 3810, 2066, 4421, 2376, 5093, 2740, \dots$$

This is sequence A122133 in the “Online-Encyclopedia of Integer Sequences” [10].

Besides the maximum area $n+\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n}{2}\right\rfloor$ and the minimum area n of the convex hull of polyominoes with n cells several other values may be attained. In Theorem 2 we have completely characterized the set of areas of the convex hull of polyominoes with n cells.

Proof of Theorem 2. Since the vertex points of the convex hull of a polyomino are lattice points on an integer grid the area of the convex hull is an integral multiple of $\frac{1}{2}$. with Theorem 3 we conclude that the desired set is a subset of

$$S=\left\{n+\frac{m}{2}\mid m\le\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n}{2}\right\rfloor,\, m\in\mathbb{N}_0\right\}.$$

A polyomino P consisting of n cells with area $n+\frac{1}{2}$ of the convex hull must contain a triangle of area $\frac{1}{2}$. If we extend the triangle to a square we get a convex polyomino P' consisting of $n+1$ cells. Thus P' is a rectangular $s\times t$-polyomino with $s\cdot t=n+1$ and $s,t\in\mathbb{N}$. If $n+1$ is a prime there exists only the $1\times(n+1)$-polyomino where deleting a square yields an area of n for the convex hull. So we have to exclude this case in the above set S and receive the proposed set.

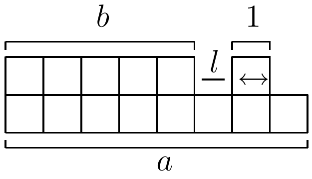

Figure 6. Construction 1: $2\le m\le 2n-8$

For the other direction we give some constructions. For $m=0$ we have the rectangular $1\times n$-polyomino as an example. The above consideration $m=1$ yields a construction if $n+1$ is a composite number. Now we consider Construction 1 depicted in Figure 6. We choose $n=a+b+1$, $\left\lceil\frac{n}{2}\right\rceil\le a\le n-2$, and $0\le l\le a-b-1=2a-n$. Thus $a\ge b+1$ and Construction 1 is possible. If we run through the possible values of a and l we obtain examples for

$$\begin{aligned} m\in\; & \{0\},\{2,3,4\},\ldots,\{2a-n,\ldots,4a-2n\},\ldots,\{n-4,\ldots,2n-8\}\\ =\; & \{0,2,3,\ldots,2n-8\}\end{aligned}$$

if $n\equiv 0\mod 2$ and for

$$\begin{aligned} m\in\; & \{1,2\},\{3,4,5,6\},\ldots,\{2a-n,\ldots,4a-2n\},\ldots,\{n-4,\ldots,2n-8\}\\ =\; & \{1,2,\ldots,2n-8\}\end{aligned}$$

if $n\equiv 1\mod 2$.

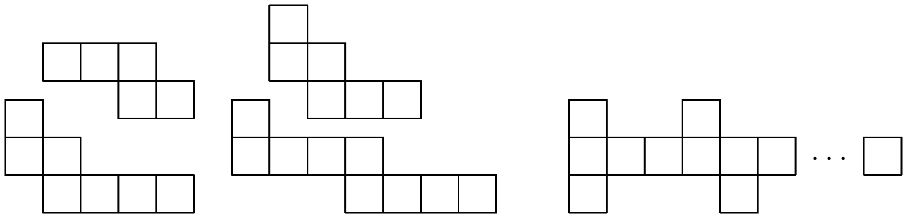

Figure 7. Construction 2: $m = 2n - 7$

In Figure 7 we give a construction for $m = 2n - 7$ and in Figure 8 we give on the left hand side a construction for $2n - 6 \le m \le \left\lceil \frac{n^2-4n}{4} \right\rceil$ with parameters k_1, k_2, and b. The conditions for these parameters are $0 \le k_1, k_2 \le n - 2b - 2$ and $n - 2b - 2 \ge b$. With given k_1, k_2, b, n we have $m = bn - 2b^2 - 2b + k_1 + k_2(b-1)$.

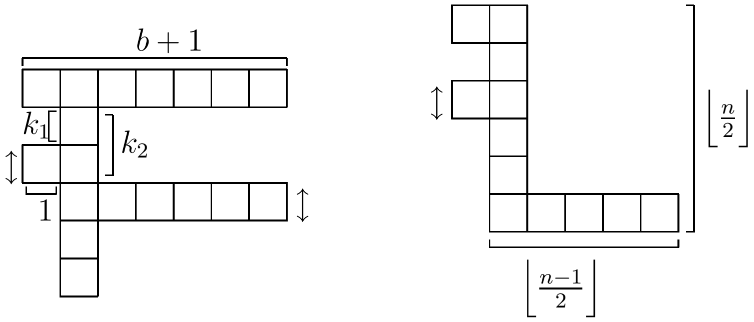

Figure 8. Construction 3 and Construction 4

Since we can vary k_1 at least between 0 and $b - 1$ we can produce for a fix b all values between $b(n - 2b - 2)$ and $2b(n - 2b - 2)$ by varying k_1 and k_2. Now we want to combine those intervals for successive values for b. The assumption that the intervals leave a gap is equivalent to $2(b-1)(n-2(b-1)-2) < b(n-2b-2)$, that is, $n < 2b\frac{b-3}{b-2}$. We choose $2 \le b \le \left\lfloor \frac{n}{4} \right\rfloor$ and receive constructions for

$$m \in \left\{ 2n-6, 2n-5, \ldots, \left\lceil \frac{n^2-4n}{4} \right\rceil \right\}.$$

On the right hand side of Figure 8 we give a construction for $n \ge 5$ and

$$m \in \left\{ \left\lfloor \frac{n^2-4n}{4} \right\rfloor, \ldots, \left\lfloor \frac{n^2-2n-8}{4} \right\rfloor \right\}.$$

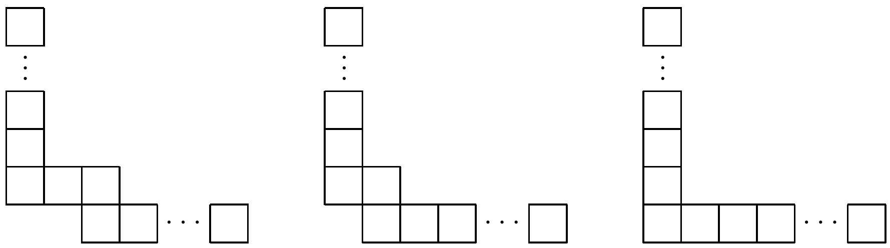

Figure 9. Construction 5: $\left\lfloor\frac{n^2-2n-6}{4}\right\rfloor \le m \le \left\lfloor\frac{n^2-2n+2}{4}\right\rfloor$

Constructions for the remaining values

$$m \in \left\{\left\lfloor\frac{n^2-2n-6}{4}\right\rfloor, \left\lfloor\frac{n^2-2n-2}{4}\right\rfloor, \left\lfloor\frac{n^2-2n+2}{4}\right\rfloor = \left\lfloor\frac{n-1}{2}\right\rfloor \left\lfloor\frac{n}{2}\right\rfloor\right\}$$

are given in Figure 9. □

3. Dimensions d ≥ 3

To prove the conjecture of Bezdek, Braß, and Harborth for dimensions $d \ge 3$ we proceed similar as in Section 2.

Definition 2.

$$f_d(l_1,\dots,l_d,v_1,\dots,v_d) = \sum_{I\subseteq\{1,\dots,d\}} \frac{1}{|I|!2^{d-|I|}} \sum_{b=0}^{2^d-1} \prod_{i\in I} q_{b,i}$$

with $d \ge 1$ *and* $b = \sum_{j=1}^{d} b_j 2^{j-1}$, $b_j \in \{0,1\}$, $q_{b,i} = \begin{cases} l_i - 1 & for \quad b_i = 0\,, \\ v_i & for \quad b_i = 1\,. \end{cases}$

Example 2.

$$\begin{aligned} f_3(l_1,l_2,l_3,v_1,v_2,v_3) &= 1 + (l_1-1) + (l_2-1) + (l_3-1) + \frac{(l_1-1)(l_2-1)}{2} + \\ &\frac{(l_1-1)(l_3-1)}{2} + \frac{(l_2-1)(l_3-1)}{2} + \frac{(l_1-1)(l_2-1)(l_3-1)}{6} + \frac{v_1(l_2-1)}{2} + \\ &\frac{v_1(l_3-1)}{2} + \frac{v_2(l_1-1)}{2} + \frac{v_2(l_3-1)}{2} + \frac{v_3(l_1-1)}{2} + \frac{v_3(l_2-1)}{2} + \\ &\frac{v_1(l_2-1)(l_3-1)}{6} + \frac{v_2(l_1-1)(l_3-1)}{6} + \frac{v_3(l_1-1)(l_2-1)}{6} + \frac{v_1v_2(l_3-1)}{6} + \\ &\frac{v_1v_3(l_2-1)}{6} + \frac{v_2v_3(l_1-1)}{6} + v_1 + v_2 + v_3 + \frac{v_1v_2}{2} + \frac{v_1v_3}{2} + \frac{v_2v_3}{2} + \frac{v_1v_2v_3}{6}\,. \end{aligned}$$

Lemma 3. *The d-dimensional volume of the convex hull of a polyomino with n unit hypercubes is at most* $f_d(l_1,\dots,l_d,v_1,\dots,v_d)$.

Proof. We prove the statement by double induction on d and n, using equation (1). Since the case $d = 2$ is already done in Theorem 3 we assume that the lemma is proven for the $\overline{d} < d$. Since for $n = 1$ only $l_i = 1$, $v_i = 0$, $i \in \{1, \ldots, d\}$ is possible and $f_d(1, \ldots, 1, 0, \ldots, 0) = 1$ the induction base for n is done. Now we assume that the lemma is proven for all possible tuples $(l_1, \ldots, l_d, v_1, \ldots, v_d)$ with $1+\sum_{i=1}^{d}(l_i - 1)+\sum_{i=1}^{d} v_i = n-1$. Due to symmetry we consider only the growth of l_1 or v_1, and the volume of the convex hull by adding the n-th hypercube.

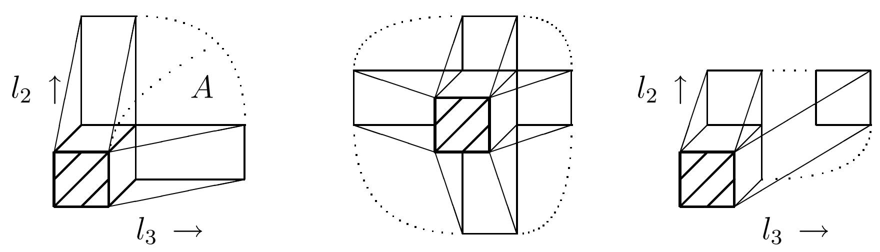

Figure 10. Increasing l_1 in the 3-dimensional case 2

(i) l_1 increases by one:
As in the proof of Lemma 1 draw lines of the convex hull of the n-th cube and its neighbor cube N, see Figure 10 for a 3-dimensional example. To be more precisely each line of the new convex hull has a corner point X of the upper face of the n-th cube as an endpoint. We will denote the second endpoint of this line by Y. In direction of axis 1 there is a corner point $\overline{X}$ of the bottom face of the n-th cube. Since $\overline{X}$ is also a corner point of N the line $\overline{X}Y$ is part of the old convex hull if Y is part of the old convex hull. In this case we draw the line $\overline{X}Y$. In the other case Y is also a corner point of the upper face of the new cube and we draw the line $\overline{XY}$ where $\overline{Y}$ is similar defined as $\overline{X}$. Additionally we draw all lines XY and $X\overline{X}$.
Doing this we have constructed a geometrical body which contains the increase of the convex hull and is subdivided into nice geometrical objects O_i with volume $\frac{\text{base} \times \text{height}}{k_i}$, for some $k_i \in \{1, \ldots, d\}$ each. For dimension $d = 3$ the cases $k_i = 1$, $k_i = 2$, or $k_i = 3$ correspond to a box, a prism, or a tetrahedron.
We project the convex hull of the whole polyomino into the hyperplane orthogonal to axis direction 1 and receive a hypervolume A. This is the convex hull of a $(d-1)$-dimensional polyomino with parameters $\overline{l}_2, \ldots, \overline{l}_d, \overline{v}_2, \ldots, \overline{v}_d$ where $\overline{l}_i \le l_i$ and $\overline{v}_i \le v_i$. From the induction hypothesis we know $A \le f_{d-1}(l_2, \ldots, l_d, v_2, \ldots, v_d)$. We apply the same projection to the O_i and objects A_i. Due to the construction the A_i are non overlapping and we have $\sum A_i \le A$. Using Cavalieri's theorem we determine the volume of O_i to be $\frac{A_i \times 1}{k_i}$. More precisely, we choose lines of the form $\overline{X}X$ as height and lift the old base up until it is orthogonal to axis direction 1. Thus we may assign a factor $\frac{1}{k}$ to each piece of A to bound the growth of the volume of the convex hull. We estimate the parts in a way that the parts with the higher factors are as big as theoretical possible.

For every $0 \le r \le d-1$ we consider the sets $\{i_1, i_2, \ldots, i_r\}$ with $1 \neq i_a \neq i_b$ for $a \neq b$. Let Z be such a set. Define $\overline{Z} = \{j_1, \ldots, j_{d-r-1}\}$ by $Z \cap \overline{Z} = \{\}$ and $Z \cup \overline{Z} = \{2, \ldots, d\}$. So the vector space spanned by the axis directions of Z and the vector space spanned by the axis directions of $\overline{Z}$ are orthogonal. If we project the convex hull in the vector space spanned by $\overline{Z}$ the resulting volume is at most $f_{d-r-1}(l_{j_1}, \ldots, l_{j_{d-r-1}}, v_{j_1}, \ldots, v_{j_{d-r-1}})$ since it is the convex hull of a $(d-r-1)$-dimensional polyomino. Since $\overline{Z}$ has cardinality $d-r-1$ the set Z yields a contribution of $\frac{1}{d-r} f_{d-r-1}(l_{j_1}, \ldots, l_{j_{d-r-1}}, v_{j_1}, \ldots, v_{j_{d-r-1}})$ to the volume of the convex hull. With the notations from Definition 2 this is

$$\frac{1}{d-r} \sum_{I \subseteq \{j_1, \ldots, j_{r-d-1}\}} \frac{1}{|I|! 2^{d-r-1-|I|}} \sum_{b=0}^{2^{d-r-1}-1} \prod_{i \in I} q_{b,i} \,.$$

Our aim is to assign the maximum possible factor to each part of A. For that reason we count for Z a maximum contribution of

$$\frac{1}{d-r} \frac{1}{|d-r-1|!} \sum_{b=0}^{2^{d-r-1}-1} \prod_{i \in \overline{Y}} q_{b,i}$$

to the volume of the convex hull.
If we do so for all possible sets Z we have assigned a factor between 1 and $\frac{1}{d}$ to every summand of $f_{d-1}(l_2, \ldots, l_d, v_2, \ldots, v_d)$. To get the induction step now we have to remark that the above described sum with its factors is exactly the difference between $f_d(l_1+1, \ldots, l_d, v_1, \ldots, v_d)$ and $f_d(l_1, \ldots, l_d, v_1, \ldots, v_d)$.

(ii) v_1 increases by one:
Due to symmetry of the l_i and v_i in Definition 2 this is similar to case (i). Additionally we remark that the maximum cannot be achieved in this case since we double count a part of the contribution of the new cube to the volume of the convex hull in our estimations. □

Theorem 4. *The d-dimensional volume of the convex hull of any facet-to-facet connected system of n unit hypercubes is*

$$\sum_{I \subseteq \{1, \ldots, d\}} \frac{1}{|I|!} \prod_{i \in I} \left\lfloor \frac{n-2+i}{d} \right\rfloor \,.$$

Proof. For given n we determine the maximum of $f_d(l_1, \ldots, f_d, v_1, \ldots, v_d)$. Due to

$$f_d(l_1+1, l_2, \ldots, l_d, v_1-1, v_2, \ldots, v_d) - f_d(l_1, l_2, \ldots, l_d, v_1, v_2, \ldots, v_d) = 0$$

and due to symmetry we assume $v_1 = \cdots = v_d = 0$ and $l_1 \le l_2 \le \cdots \le l_d$. Since

$$f_d(l_1+1, l_2, \ldots, l_{d-1}, l_d-1, 0, 0, \ldots, 0) - f_d(l_1, l_2, \ldots, l_d, 0, 0, \ldots, 0) > 0 \qquad (2)$$

we have $0 \le l_d - l_1 \le 1$. Inequality (2) due to the following consideration. If a summand of $f_d(\ldots)$ contains the term l_1 and does not contain l_d then there will be a corresponding summand with l_1 replaced by l_d, so those terms equalize each other in the above difference. Clearly the summands containing none of the terms l_1 or l_d equalize each other in the difference. So there are left only the summands with both terms l_1 and l_d. Since $(l_1+1-1)(l_d-1-1)-(l_1-1)(l_d-1) = l_d-l_1-1 > 0$ inequality (2) is valid.

Combining equation (1) with $0 \leq l_d - l_1 \leq 1$ and $l_1 \leq l_2 \leq \cdots \leq l_d$ gives $l_i = \left\lfloor \frac{n-2+i+d}{d} \right\rfloor$. Thus by inserting in Lemma 3 we receive the upper bound. The maximum is attained for example by a polyomino consisting of d pairwise orthogonal linear arms with $\left\lfloor \frac{n-2+i}{d} \right\rfloor$ cubes ($i = 1 \ldots d$) joined to a central cube. □

Conjecture 1. *Every d-dimensional polyomino P with parameters $l_1, \ldots, l_d$, $v_1, \ldots, v_d$ and maximum volume of the convex hull fulfills $v_1 = \cdots = v_d = 0$ and contains a sub-polyomino P' fulfilling:*

(i) *P' has height 1 in direction of axis i,*

(ii) *the projection of P' along i has also maximal volume of the convex hull and parameters $l_1, \ldots, l_{i-1}, l_{i+1}, \ldots, l_d$,*

(iii) *P can be decomposed into P' and up to two orthogonal linear arms.*

We remark that $v_1 = \cdots = v_d = 0$ and the maximality of the volume of the convex hull of sub-polyominoes and projections of P can be concluded from the proof of Theorem 4.

Lemma 4. *If there exists a d-dimensional polyomino with n cells and volume v of the convex hull, then $v \in V_{d,n}$ with*

$$V_{d,n} = \left\{ n + \frac{m}{d!} \middle| m \leq \sum_{I \subseteq \{1,\ldots,d\}} \frac{d!}{|I|!} \prod_{i \in I} \left\lfloor \frac{n-2+i}{d} \right\rfloor \; m \in \mathbb{N}_0 \right\}.$$

Proof. For the determination of the volume of the convex hull of a d-dimensional polyomino we only have to consider the set of S corner points of its hypercubes which lie on an integer grid. We can decompose the convex hull into d-dimensional simplices with the volume

$$\frac{1}{d!} \begin{vmatrix} x_{1,1} & \ldots & x_{1,d} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ x_{d+1,1} & \ldots & x_{d+1,d} & 1 \end{vmatrix}$$

where the coordinates of the $d+1$ points are given by $(x_{i,1}, \ldots, x_{i,d}) \in \mathbb{Z}^d$. Thus the volume of the convex hull is an integer multiple of $\frac{1}{d}$. The lower bound $n \leq v$ is obvious and the upper bound is given by Theorem 4. □

4. Remarks

We leave the description and the enumeration of the polyominoes with maximum convex hull for dimension $d \geq 3$ as a task for the interested reader. It would also be nice to see a version of Theorem 2 for higher dimensions.

The authors of [4] mention another class of problems which are related to the problems in [3] and [11]: What is the maximum area of the convex hull of all connected edge-to-edge packings of n congruent regular k-gons (also denoted as k-polyominoes, see [7]) in the plane. The methods of Section 2 might be applicable for these problems.

Conjecture 2. *The area of the convex hull of any edge-to-edge connected system of regular unit hexagons is at most* $\frac{1}{6}\left\lfloor n^2+\frac{14}{3}n+1\right\rfloor$.

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