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Linear ∞-Harmonic Maps between Riemannian Manifolds

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Abstract. In this paper, we give complete classifications of linear ∞-harmonic maps between Euclidean and Heisenberg spaces, between Nil and Sol spaces. We also classify all ∞-harmonic linear endomorphisms of Sol space and show that there is a subgroup of ∞-harmonic linear automorphisms in the group of linear automorphisms of Sol space.

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1. Introduction

In this paper, all objects including manifolds, metrics, maps, and vector fields are assumed to be smooth unless it is stated otherwise.

∞-harmonic functions are solutions of the so-called ∞-Laplace equation:

$$\Delta_\infty u := \frac{1}{2}\langle \nabla\, u, \nabla\, |\nabla u|^2 \rangle = \sum_{i,j=1}^{m} u_{ij} u_i u_j = 0,$$

where $u : \Omega \subset \mathbb{R}^m \longrightarrow \mathbb{R}$, $u_i = \frac{\partial u}{\partial x^i}$ and $u_{ij} = \frac{\partial^2 u}{\partial x^i \partial x^j}$. The ∞-Laplace equation was first found by G. Aronsson ([1], [2]) in his study of "optimal" Lipschitz extension of functions in the late 1960s.

The ∞-Laplace equation can be obtained as the formal limit, as $p \to \infty$, of p-Laplace equation

$$\Delta_p u := |\nabla u|^{p-2} \left(\Delta u + \frac{p-2}{|\nabla u|^2} \Delta_\infty u \right) = 0. \tag{1}$$

In recent years, there has been a growing research work in the study of the ∞-Laplace equation. For more history and developments see e.g. [14], [3], [4], [5], [6], [7], [8], [9], [12], [13], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24]. For interesting applications of the ∞-Laplace equation in image processing see [10], [29], in mass transfer problems see e.g. [16], and in the study of shape metamorphism see e.g. [11].

Very recently, Ou, Troutman, and Wilhelm [28] introduced and studied ∞-harmonic maps between Riemannian manifolds as a natural generalization of ∞-harmonic functions and as a map between Riemannian manifolds that satisfies a system of PDE obtained as the formal limit, as $p \to \infty$, of p-harmonic map equation:

$$\frac{|\mathrm{d}\varphi|^2 \tau_2(\varphi)}{(p-2)} + \frac{1}{2} \mathrm{d}\varphi \left(\operatorname{grad} |\mathrm{d}\varphi|^2 \right) = 0.$$

According to [28], a map $\varphi : (M, g) \longrightarrow (N, h)$ between Riemannian manifolds is called an ∞-*harmonic map* if the gradient of its energy density is in the kernel of its tangent map, i.e., φ is a solution of the PDEs

$$\tau_\infty(\varphi) = \frac{1}{2} \mathrm{d}\varphi \left(\operatorname{grad} |\mathrm{d}\varphi|^2 \right) = 0, \tag{2}$$

where $|\mathrm{d}\varphi|^2 = \mathrm{Trace}_g \varphi^* h$ is the energy density of φ.

Corollary 1.1. (see [28]) *In local coordinates, a map $\varphi : (M, g) \longrightarrow (N, h)$ with $\varphi(x) = (\varphi^1(x), \varphi^2(x), \ldots, \varphi^n(x))$ is ∞-harmonic if and only if*

$$g \left(\operatorname{grad} \varphi^i, \operatorname{grad} |\mathrm{d}\varphi|^2 \right) = 0, \qquad i = 1, 2, \ldots, n. \tag{3}$$

Example 1. (see [28]) Many important and familiar families of maps between Riemannian manifolds turn out to be ∞-harmonic maps. In particular, all maps of the following classes are ∞-harmonic:

- ∞-harmonic functions,
- totally geodesic maps,
- isometric immersions,
- Riemannian submersions,
- eigenmaps between spheres,
- projections of multiply warped products (e.g., the projection of the generalized Kasner spacetimes),
- equator maps, and

- radial projections.

For more details of the above and other examples, methods of constructing ∞-harmonic maps into Euclidean spaces and into spheres, study of a subclass of ∞-harmonic maps called ∞-harmonic morphisms, study of the conformal change of ∞-Laplacian on Riemannian manifolds and other results we refer the readers to [28].

For some classifications of linear and quadratic ∞-harmonic maps from and into a sphere, quadratic ∞-harmonic maps between Euclidean spaces, linear and quadratic ∞-harmonic maps between Nil and Euclidean spaces and between Sol and Euclidean spaces see [30].

In this paper, we give complete classifications of linear ∞-harmonic maps between Euclidean and Heisenberg spaces, between Nil and Sol spaces. We also classify all ∞-harmonic linear automorphisms of Sol space and show that there is a subgroup of ∞-harmonic linear automorphisms in the group of linear automorphisms of Sol space.

2. Linear ∞-harmonic maps between Euclidean and Heisenberg spaces

2.1. Linear ∞-harmonic maps from Heisenberg space into a Euclidean space

Let $\mathbb{H}_3=(\mathbb{R}^3, g)$ denote Heisenberg space, a 3-dimensional homogeneous space with a left invariant metric whose group of isometries has dimension 4. With respect to the standard coordinates (x, y, z) in $\mathbb{R}^3$, the metric can be written as $g = \mathrm{d}x^2 + \mathrm{d}y^2 + (\mathrm{d}z + \frac{y}{2}\mathrm{d}x - \frac{x}{2}\mathrm{d}y)^2$ whose components are given by:

$$g_{11} = 1 + \frac{y^2}{4},\ g_{12} = -\frac{xy}{4},\ g_{13} = \frac{y}{2},\ g_{22} = 1 + \frac{x^2}{4},\ g_{23} = -\frac{x}{2},\ g_{33} = 1; \tag{4}$$

$$g^{11} = 1,\ g^{12} = 0,\ g^{13} = -\frac{y}{2},\ g^{22} = 1,\ g^{23} = \frac{x}{2},\ g^{33} = 1 + \frac{x^2 + y^2}{4}. \tag{5}$$

Now, let $\varphi : \mathbb{H}_3 \longrightarrow \mathbb{R}^n$ with

$$\varphi(X) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{6}$$

be a linear map from Heisenberg space into a Euclidean space. Then, we have

Theorem 2.1. *A linear map $\varphi : \mathbb{H}_3 \longrightarrow \mathbb{R}^n$ with $\varphi(X) = AX$, where A is the representation matrix with column vectors A_1, A_2, A_3, is ∞-harmonic if and only if $A_3 = 0$, or A_1, A_2, and A_3 are proportional to each other.*

Proof. A straightforward computation using (5) gives:

$$\begin{aligned}\nabla\varphi^i &= g^{\alpha\beta}\tfrac{\partial\varphi^i}{\partial x_\beta}\tfrac{\partial}{\partial x_\alpha} \\ &= (a_{i1} - \tfrac{1}{2}a_{i3}y,\ a_{i2} + \tfrac{1}{2}a_{i3}x,\ \tfrac{1}{4}a_{i3}(x^2 + y^2) + \tfrac{1}{2}a_{i2}x - \tfrac{1}{2}a_{i1}y + a_{i3}),\quad i = 1, 2, ..., n,\end{aligned}$$

$$\begin{aligned}|\mathrm{d}\varphi|^2 &= g^{\alpha\beta}{\varphi_\alpha}^i{\varphi_\beta}^j\delta_{ij}\\ &= \tfrac14\sum_{i=1}^n a_{i3}^2x^2+\sum_{i=1}^n a_{i2}a_{i3}x+\tfrac14\sum_{i=1}^n a_{i3}^2y^2-\sum_{i=1}^n a_{i1}a_{i3}y+\sum_{j=1}^3\sum_{i=1}^n a_{ij}^2,\end{aligned}$$

and

$$\begin{aligned}\frac{\partial|\mathrm{d}\varphi|^2}{\partial x_1}&=\frac{\partial|\mathrm{d}\varphi|^2}{\partial x}=\tfrac12\sum_{i=1}^n a_{i3}^2x+\sum_{i=1}^n a_{i2}a_{i3},\\ \frac{\partial|\mathrm{d}\varphi|^2}{\partial x_2}&=\frac{\partial|\mathrm{d}\varphi|^2}{\partial y}=\tfrac12\sum_{i=1}^n a_{i3}^2y-\sum_{i=1}^n a_{i1}a_{i3},\\ \frac{\partial|\mathrm{d}\varphi|^2}{\partial x_3}&=\frac{\partial|\mathrm{d}\varphi|^2}{\partial z}=0.\end{aligned}\tag{7}$$

It follows from Corollary 1.1 that φ is ∞-harmonic if and only if

$$g(\nabla\varphi^i,\nabla|\mathrm{d}\varphi|^2)=0,\qquad i=1,2,\ldots,n,\tag{8}$$

which is equivalent to

$$\begin{aligned}&\tfrac12\Big(a_{i1}\sum_{j=1}^n a_{j3}^2-a_{i3}\sum_{j=1}^n a_{j1}a_{j3}\Big)x+\tfrac12\Big(a_{i2}\sum_{j=1}^n a_{j3}^2-a_{i3}\sum_{j=1}^n a_{j2}a_{j3}\Big)y\\ &+a_{i1}\sum_{j=1}^n a_{j2}a_{j3}-a_{i2}\sum_{j=1}^n a_{j1}a_{j3}=0\end{aligned}\tag{9}$$

for $i=1,2,\ldots,n$ and for any x,y. By comparing the coefficients of the polynomial identity we have

$$a_{i1}\sum_{j=1}^n a_{j3}^2-a_{i3}\sum_{j=1}^n a_{j1}a_{j3}=0,\quad i=1,2,\ldots,n,\tag{10}$$

$$a_{i2}\sum_{j=1}^n a_{j3}^2-a_{i3}\sum_{j=1}^n a_{j2}a_{j3}=0,\quad i=1,2,\ldots,n,\tag{11}$$

$$a_{i1}\sum_{j=1}^n a_{j2}a_{j3}-a_{i2}\sum_{j=1}^n a_{j1}a_{j3}=0,\quad i=1,2,\ldots,n.\tag{12}$$

Noting that $A_i=(a_{1i},\ldots,a_{ni})^t$ for $i=1,2,3$ are the column vectors of A we conclude that the system of equations (10), (11), (12) is equivalent to $A_1//A_3$, $A_2//A_3$, and $A_1//A_2$, or, $A_3=0$, from which the theorem follows. □

Remark 1. It follows from our theorem that the maximum rank of the linear ∞-harmonic map from Heisenberg space into a Euclidean space is 2.

Example 2. Let $\varphi:\mathbb{H}_3\longrightarrow\mathbb{R}^n$, with

$$\varphi(X)=\begin{pmatrix}1&1&1\\2&2&2\\\cdots&\cdots&\cdots\\n&n&n\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}.\tag{13}$$

Then, by our theorem, φ is an ∞-harmonic map with non-constant energy density $|\mathrm{d}\varphi|^2=\frac14|A_3|^2(x^2+y^2)+|A_3|^2(x-y)+3|A_3|^2$, where $|A_3|^2=\frac{n(n+1)(2n+1)}{6}$.

2.2. Linear ∞-harmonic maps from a Euclidean space into Heisenberg space

Theorem 2.2. *Let $\varphi : \mathbb{R}^m \longrightarrow \mathbb{H}_3$ with*

$$\varphi(X) = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ a_{31} & a_{32} & \dots & a_{3m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} \tag{14}$$

be a linear map from a Euclidean space into Heisenberg space. Then, φ is ∞-harmonic if and only if the row vectors A^1, A^2 are proportional to each other.

Proof. A straightforward computation gives:

$$\nabla\varphi^i = A^i, \qquad i = 1, 2, 3, \tag{15}$$

$$\begin{aligned} |\mathrm{d}\varphi|^2 = \delta^{\alpha\beta}\varphi^i_\alpha\varphi^j_\beta g_{ij} = \frac{1}{4}|A^2|^2x^2 + \frac{1}{4}|A^1|^2y^2 - \frac{1}{2}A^1 \cdot A^2xy \\ -A^2 \cdot A^3x + A^1 \cdot A^3y + (|A^1|^2 + |A^2|^2 + |A^3|^2), \end{aligned} \tag{16}$$

$$\begin{aligned} \frac{\partial\,|\mathrm{d}\varphi|^2}{\partial x_k} = {} & \tfrac{1}{2}(a_{1k}|A^2|^2 - a_{2k}A^1 \cdot A^2)x + \tfrac{1}{2}(a_{2k}|A^1|^2 - a_{1k}A^1 \cdot A^2)y \\ & +a_{2k}A^1 \cdot A^3 - a_{1k}A^2 \cdot A^3, \qquad k = 1, 2, \dots, m. \end{aligned} \tag{17}$$

It follows from Corollary 1.1 that φ is ∞-harmonic if and only if

$$g(\nabla\,\varphi^i, \nabla\,|\mathrm{d}\varphi|^2) = 0, \qquad i = 1, 2, 3, \tag{18}$$

which is equivalent to

$$\begin{aligned} & \tfrac{1}{2}(A^i \cdot A^1|A^2|^2 - A^i \cdot A^2A^1 \cdot A^2)x + \tfrac{1}{2}(A^i \cdot A^2|A^1|^2 - A^i \cdot A^1A^1 \cdot A^2)y \\ & +A^i \cdot A^2A^1 \cdot A^3 - A^i \cdot A^1A^2 \cdot A^3 = 0, \qquad i = 1, 2, \dots, 3. \end{aligned} \tag{19}$$

Substituting $x = A^1X, y = A^2X$ into (19) we have, for any $X \in \mathbb{R}^m$,

$$(c_1A^1 + c_2A^2)X + c_3 = 0, \tag{20}$$

where

$$\begin{aligned} c_1 &= \frac{1}{2}(A^i \cdot A^1|A^2|^2 - A^i \cdot A^2A^1 \cdot A^2), \\ c_2 &= \frac{1}{2}(A^i \cdot A^2|A^1|^2 - A^i \cdot A^1A^1 \cdot A^2), \\ c_3 &= A^i \cdot A^2A^1 \cdot A^3 - A^i \cdot A^1A^2 \cdot A^3, \qquad i = 1, 2, 3. \end{aligned} \tag{21}$$

Since equation (20) holds for any $X \in \mathbb{R}^m$ it can be viewed as an identity of polynomials. It follows that φ is ∞-harmonic if and only if A^1 and A^2 are proportional to each other and $c_3 = 0$. One can check that $c_3 = 0$ is a consequence of A^1 being proportional to A^2. Therefore, we conclude that linear map φ from a Euclidean space into Heisenberg space is ∞-harmonic if and only if A^1 is proportional to A^2. $\square$

Remark 2. It follows from our theorem that the maximum possible rank of a linear ∞-harmonic map from a Euclidean space into Heisenberg space is 2 and a rank 2 linear ∞-harmonic map from a Euclidean space into Heisenberg space always has non-constant energy density. We would also like to point out that in [30] a complete classification of linear ∞-harmonic maps between Euclidean and Nil spaces is given. It is well known that Nil space is isometric to Heisenberg space. However, as the linearity of maps that we study depends on the (local) coordinates used in $\mathbb{R}^3$ and since the isometry between Nil and Heisenberg spaces is given by a quadratic polynomial map, the linear maps between Euclidean and Nil spaces and the linear maps between Euclidean and Heisenberg spaces are not isometric invariant and should be treated differently as the following examples show.

Example 3. We can check that $\sigma : (\mathbb{H}_3, g) \longrightarrow (\mathbb{R}^3, g_{Nil})$ with $\sigma(X, Y, Z) = (X, Y, Z + XY/2)$ is an isometry from Heisenberg space onto Nil space. If we identify these two spaces through this isometry, then the linear map $\varphi : \mathbb{R}^m \longrightarrow \mathbb{H}_3$ with

$$\varphi(X) = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 2 & -2 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} \tag{22}$$

becomes a quadratic map $\mathbb{R}^m \longrightarrow (\mathbb{R}^3, g_{Nil})$ with $\sigma \circ \varphi(X) = (x_1 - x_2, 2(x_1 - x_2), (x_1 - x_2)^2)$. It is interesting to note that the composition $\sigma \circ \varphi$ of φ (which is ∞-harmonic by Theorem 2.2) with an isometry σ is also ∞-harmonic. This follows from a general result in [28] that the ∞-harmonicity of a map is invariant under an isometric immersion of the target space of the map into another manifold.

Example 4. It is proved in [28] that any isometry is an ∞-harmonic morphism which preserves ∞-harmonicity in the sense that it pulls back ∞-harmonic functions to ∞-harmonic functions. One can also check that an ∞-harmonic morphism pulls back ∞-harmonic maps to ∞-harmonic maps. It follows that the isometry $\sigma : (\mathbb{H}_3, g) \longrightarrow (\mathbb{R}^3, g_{Nil})$ with $\sigma(X, Y, Z) = (X, Y, Z + XY/2)$ is an ∞-harmonic morphism. By [30], the linear map $\varphi : (\mathbb{R}^3, g_{Nil}) \longrightarrow \mathbb{R}^n$ $(n \geq 2)$

$$\varphi(X) = \begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ \dots & \dots & \dots \\ 0 & a_{n2} & a_{n3} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{23}$$

is ∞-harmonic. Therefore, the composition $\varphi \circ \sigma : (\mathbb{H}_3, g) \longrightarrow \mathbb{R}^n$ given by

$$\varphi \circ \sigma(X) = \begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ \dots & \dots & \dots \\ 0 & a_{n2} & a_{n3} \end{pmatrix} \begin{pmatrix} x \\ y \\ z + \frac{1}{2}xy \end{pmatrix} \tag{24}$$

gives an ∞-harmonic map defined by polynomials of degree 2 from Heisenberg space into a Euclidean space with constant energy density.

3. Linear ∞-harmonic maps between Nil and Sol spaces

In this section we give a complete classification of linear ∞-harmonic maps between Nil and Sol spaces. It turns out that the maximum possible rank of linear ∞-harmonic maps between Nil and Sol spaces is 2 and some of them have constant energy density while others may have non-constant energy density.

3.1. Linear ∞-harmonic maps from Nil space into Sol space

Let $(\mathbb{R}^3, g_{Nil})$ and $(\mathbb{R}^3, g_{Sol})$ denote Nil and Sol spaces, where the metrics with respect to the standard coordinates (x, y, z) in $\mathbb{R}^3$ are given by $g_{Nil} = \mathrm{d}x^2 + \mathrm{d}y^2 + (\mathrm{d}z - x\mathrm{d}y)^2$ and $g_{Sol} = e^{2z}\mathrm{d}x^2 + e^{-2z}\mathrm{d}y^2 + \mathrm{d}z^2$ respectively. In the following, we use the notations $g = g_{Nil}$, $h = g_{Sol}$, the coordinates $\{x, y, z\}$ in $(\mathbb{R}^3, g_{Nil})$ and the coordinates $\{x\acute{}, y\acute{}, z\acute{}\}$ in $(\mathbb{R}^3, g_{Sol})$, then one can easily compute the following components of Nil and Sol metrics:

$$g_{11} = 1, \quad g_{12} = g_{13} = 0, \quad g_{22} = 1 + x^2, \quad g_{23} = -x, \quad g_{33} = 1;$$

$$g^{11} = 1, \quad g^{12} = g^{13} = 0, \quad g^{22} = 1, \quad g^{23} = x, \quad g^{33} = 1 + x^2.$$

$$\begin{aligned} h_{11} &= e^{2z\acute{}}, & h_{22} &= e^{-2z\acute{}}, \quad h_{33} = 1, & &\text{all other } h_{ij} = 0; \\ h^{11} &= e^{-2z\acute{}}, & h^{22} &= e^{2z\acute{}}, \quad h^{33} = 1, & &\text{all other } h^{ij} = 0. \end{aligned}$$

Now we study the ∞-harmonicity of linear maps between Nil and Sol spaces. First, we give the following classification of linear ∞-harmonic maps from Nil space into Sol space.

Theorem 3.1. *A linear map $\varphi : (\mathbb{R}^3, g_{Nil}) \longrightarrow (\mathbb{R}^3, g_{Sol})$ from Nil space into Sol space with*

$$\varphi(X) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{25}$$

is ∞-harmonic if and only if φ takes one of the following forms:

$$\varphi(X) = \begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{26}$$

$$\varphi(X) = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{27}$$

$$\varphi(X) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \text{or} \tag{28}$$

$$\varphi(X) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_{31} & a_{32} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{29}$$

Proof. A straightforward computation gives:

$$\begin{aligned}\nabla\varphi^i &= g^{\alpha\beta}\tfrac{\partial\varphi^i}{\partial x_\beta}\tfrac{\partial}{\partial x_\alpha}\\ &= (a_{i1},\, a_{i3}x + a_{i2},\, a_{i3}x^2 + a_{i2}x + a_{i3}), \qquad i = 1, 2, 3,\end{aligned}$$

and

$$\begin{aligned}|\mathrm{d}\varphi|^2 = g^{\alpha\beta}{\varphi_\alpha}^i{\varphi_\beta}^j h_{ij}\circ\varphi &= (a_{13}^2x^2 + 2a_{12}a_{13}x + \sum_{j=1}^3 a_{1j}^2)e^{2z'}\\ &+ (a_{23}^2x^2 + 2a_{22}a_{23}x + \sum_{j=1}^3 a_{2j}^2)e^{-2z'} + (a_{33}^2x^2 + 2a_{32}a_{33}x + \sum_{j=1}^3 a_{3j}^2),\end{aligned} \tag{30}$$

where $z' = a_{31}x + a_{32}y + a_{33}z$. Also, one can check that

$$\begin{aligned}&\tfrac{\partial|\mathrm{d}\varphi|^2}{\partial x_1} = \tfrac{\partial|\mathrm{d}\varphi|^2}{\partial x} = 2\{a_{31}a_{13}^2x^2 + (a_{13}^2 + 2a_{31}a_{12}a_{13})x + a_{12}a_{13} + a_{31}\textstyle\sum_{j=1}^3 a_{1j}^2\}e^{2z'}\\ &-2\{a_{31}a_{23}^2x^2 + (2a_{31}a_{22}a_{23} - a_{23}^2)x + a_{31}\textstyle\sum_{j=1}^3 a_{2j}^2 - a_{22}a_{23}\}e^{-2z'} + 2(a_{33}^2x + a_{32}a_{33}),\\ &\tfrac{\partial|\mathrm{d}\varphi|^2}{\partial x_2} = \tfrac{\partial|\mathrm{d}\varphi|^2}{\partial y}\\ &= 2a_{32}(a_{13}^2x^2 + 2a_{12}a_{13}x + \textstyle\sum_{j=1}^3 a_{1j}^2)e^{2z'} - 2a_{32}(a_{23}^2x^2 + 2a_{22}a_{23}x + \textstyle\sum_{j=1}^3 a_{2j}^2)e^{-2z'},\\ &\tfrac{\partial|\mathrm{d}\varphi|^2}{\partial x_3}\tfrac{\partial|\mathrm{d}\varphi|^2}{\partial z}\\ &= 2a_{33}(a_{13}^2x^2 + 2a_{12}a_{13}x + \textstyle\sum_{j=1}^3 a_{1j}^2)e^{2z'} - 2a_{33}(a_{23}^2x^2 + 2a_{22}a_{23}x + \textstyle\sum_{j=1}^3 a_{2j}^2)e^{-2z'}.\end{aligned} \tag{31}$$

It follows from Corollary 1.1 that φ is an ∞-harmonic map if and only if

$$g(\nabla\,\varphi^i, \nabla\,|\mathrm{d}\varphi|^2) = 0, \quad i = 1, 2, 3, \tag{32}$$

which is equivalent to

$$\begin{aligned}&2\{a_{i3}a_{33}a_{13}^2x^4 + [(a_{i3}a_{32} + a_{i2}a_{33})a_{13}^2 + 2a_{i3}a_{33}a_{12}a_{13}]x^3\\ &+[\textstyle\sum_{k=1}^3 a_{ik}a_{3k}a_{13}^2 + 2(a_{i3}a_{32} + a_{i2}a_{33})a_{12}a_{13} + a_{i3}a_{33}\textstyle\sum_{j=1}^3 a_{1j}^2]x^2\\ &+[(a_{i3}a_{32} + a_{i2}a_{33})\textstyle\sum_{j=1}^3 a_{1j}^2 + 2\textstyle\sum_{k=1}^3 a_{ik}a_{3k}a_{12}a_{13} + a_{i1}a_{13}^2]x\\ &+[\textstyle\sum_{k=1}^3 a_{ik}a_{3k}\textstyle\sum_{j=1}^3 a_{1j}^2 + a_{i1}a_{12}a_{13}]\}e^{2z'}\\ &-2\{a_{i3}a_{33}a_{23}^2x^4 + [(a_{i3}a_{32} + a_{i2}a_{33})a_{23}^2 + 2a_{i3}a_{33}a_{22}a_{23}]x^3\\ &+[\textstyle\sum_{k=1}^3 a_{ik}a_{3k}a_{23}^2 + 2(a_{i3}a_{32} + a_{i2}a_{33})a_{22}a_{23} + a_{i3}a_{33}\textstyle\sum_{j=1}^3 a_{2j}^2]x^2\\ &+[(a_{i3}a_{32} + a_{i2}a_{33})\textstyle\sum_{j=1}^3 a_{2j}^2 + 2\textstyle\sum_{k=1}^3 a_{ik}a_{3k}a_{22}a_{23} - a_{i1}a_{23}^2]x\\ &+[\textstyle\sum_{k=1}^3 a_{ik}a_{3k}\textstyle\sum_{j=1}^3 a_{2j}^2 - a_{i1}a_{22}a_{23}]\}e^{-2z'} + 2a_{i1}(a_{33}^2x + a_{32}a_{33}) = 0,\\ &i = 1, 2, 3.\end{aligned} \tag{33}$$

Case (A): $\sum_{j=1}^{3} a_{3j}^2 = 0$. In this case, (33) becomes

$$2a_{i1}(a_{13}^2 + a_{23}^2)x + 2a_{i1}(a_{12}a_{13} + a_{22}a_{23}) = 0, \qquad i = 1, 2, 3. \tag{34}$$

Solving equation (34), we have $a_{i1} = 0$ for $i = 1, 2, 3$, or $a_{13} = a_{23} = 0$. These give the classes of linear ∞-harmonic maps corresponding to (26) and (27).

Case (B): $\sum_{j=1}^{3} a_{3j}^2 \neq 0$. In this case, we use the fact that the functions

$$1,\ x,\ xe^{2x},\ x^2e^{2x},\ x^3e^{2x},\ x^4e^{2x};\ xe^{-2x},\ x^2e^{-2x},\ x^3e^{-2x},\ x^4e^{-2x}$$

are linearly independent to conclude that (33) is equivalent to

$$\left.\begin{aligned}
a_{i1}a_{33}^2 &= 0, \quad \langle 1\rangle\\
a_{i1}a_{32}a_{33} &= 0, \quad \langle 2\rangle\\
a_{i3}a_{33}a_{13}^2 &= 0, \quad \langle 3\rangle\\
(a_{i3}a_{32} + a_{i2}a_{33})a_{13}^2 + 2a_{i3}a_{33}a_{12}a_{13} &= 0, \quad \langle 4\rangle\\
\sum_{k=1}^{3} a_{ik}a_{3k}a_{13}^2 + 2(a_{i3}a_{32} + a_{i2}a_{33})a_{12}a_{13} + a_{i3}a_{33}\sum_{j=1}^{3} a_{1j}^2 &= 0, \quad \langle 5\rangle\\
(a_{i3}a_{32} + a_{i2}a_{33})\sum_{j=1}^{3} a_{1j}^2 + 2\sum_{k=1}^{3} a_{ik}a_{3k}a_{12}a_{13} + a_{i1}a_{13}^2 &= 0, \quad \langle 6\rangle\\
\sum_{k=1}^{3} a_{ik}a_{3k}\sum_{j=1}^{3} a_{1j}^2 + a_{i1}a_{12}a_{13} &= 0, \quad \langle 7\rangle\\
a_{i3}a_{33}a_{23}^2 &= 0, \quad \langle 8\rangle\\
(a_{i3}a_{32} + a_{i2}a_{33})a_{23}^2 + 2a_{i3}a_{33}a_{22}a_{23} &= 0, \quad \langle 9\rangle\\
\sum_{k=1}^{3} a_{ik}a_{3k}a_{23}^2 + 2(a_{i3}a_{32} + a_{i2}a_{33})a_{22}a_{23} + a_{i3}a_{33}\sum_{j=1}^{3} a_{2j}^2 &= 0, \quad \langle 10\rangle\\
(a_{i3}a_{32} + a_{i2}a_{33})\sum_{j=1}^{3} a_{2j}^2 + 2\sum_{k=1}^{3} a_{ik}a_{3k}a_{22}a_{23} - a_{i1}a_{23}^2 &= 0. \quad \langle 11\rangle\\
\sum_{k=1}^{3} a_{ik}a_{3k}\sum_{j=1}^{3} a_{2j}^2 - a_{i1}a_{22}a_{23} &= 0. \quad \langle 12\rangle
\end{aligned}\right\} \tag{35}$$

It follows from ⟨1⟩ of (35) that either $a_{i1} = 0$ for $i = 1, 2, 3$, or $a_{33} = 0$.

Case (B$_1$): $\sum_{i=1}^{3} a_{i1}^2 = 0$. In this case, we have $a_{32}^2 + a_{33}^2 \neq 0$ since we are in Case (B). It follows that the equations ⟨7⟩ and ⟨12⟩ of (35) reduce to be

$$\left.\begin{aligned}
(a_{i3}a_{33} + a_{i2}a_{32})\sum_{j=1}^{3} a_{1j}^2 &= 0\\
(a_{i3}a_{33} + a_{i2}a_{32})\sum_{j=1}^{3} a_{2j}^2 &= 0, \quad i = 1, 2, 3.
\end{aligned}\right\} \tag{36}$$

Writing out the equation (36) with $i = 3$ we have that $a_{1j} = a_{2j} = 0$ for $j = 1, 2, 3$ and we can check that these, together with $a_{j1} = 0$, are solutions of the equations (35). These correspond to the class of linear ∞-harmonic maps given by (28).

Case (B_2): $a_{33}=0$ and hence $a_{31}^2+a_{32}^2\neq 0$ since we are in Case (B). In this case, equation (35) reduces to

$$\left.\begin{aligned}a_{i3}a_{32}a_{13}^2&=0,\\(a_{i2}a_{32}+a_{i1}a_{31})a_{13}^2+2a_{i3}a_{32}a_{12}a_{13}&=0,\\a_{i3}a_{32}\sum_{j=1}^{3}a_{1j}^2+2(a_{i2}a_{32}+a_{i1}a_{31})a_{12}a_{13}+a_{i1}a_{13}^2&=0,\\(a_{i2}a_{32}+a_{i1}a_{31})\sum_{j=1}^{3}a_{1j}^2+a_{i1}a_{12}a_{13}&=0,\\a_{i3}a_{32}a_{23}^2&=0,\\(a_{i2}a_{32}+a_{i1}a_{31})a_{23}^2+2a_{i3}a_{32}a_{22}a_{23}&=0,\\a_{i3}a_{32}\sum_{j=1}^{3}a_{2j}^2+2(a_{i2}a_{32}+a_{i1}a_{31})a_{22}a_{23}-a_{i1}a_{23}^2&=0,\\(a_{i2}a_{32}+a_{i1}a_{31})\sum_{j=1}^{3}a_{2j}^2-a_{i1}a_{22}a_{23}&=0,\end{aligned}\right\}\quad i=1,2,3. \tag{37}$$

It follows from the first equation of (37) that we have either $a_{13}=0$ or $a_{32}=0$. By considering following cases:

(I) $a_{13}=0, a_{32}\neq 0$,

(II) $a_{13}\neq 0, a_{32}=0$, hence, $a_{31}\neq 0$,

(III) $a_{13}=0, a_{32}=0$, hence, $a_{31}\neq 0$,

we obtain that $a_{i3}=0, a_{1i}=a_{2i}=0$, for $i=1,2,3$, are solution of the equations (35), which give the class of linear ∞-harmonic maps corresponding to (29). Thus, we obtain the theorem. □

Remark 3. It follows from our theorem that the maximum possible rank of linear ∞-harmonic maps from Nil into Sol is 2. Using the energy density formula (30) we can check that some of them have non-constant energy density while others have constant energy density.

3.2. Linear ∞-harmonic maps from Sol space into Nil space

The linear ∞-harmonic maps from Sol space into Nil space can be completely described by the following theorem.

Theorem 3.2. *A linear map $\varphi:(\mathbb{R}^3,g_{Sol})\longrightarrow(\mathbb{R}^3,g_{Nil})$ from Sol space into Nil space with*

$$\varphi(X)=\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} \tag{38}$$

is ∞-harmonic if and only if φ takes one of the following forms:

$$\varphi(X)=\begin{pmatrix}0&0&0\\a_{21}&a_{22}&0\\a_{31}&a_{32}&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}, \tag{39}$$

$$\varphi(X)=\begin{pmatrix} a_{11} & a_{12} & 0\\ 0 & 0 & 0\\ a_{31} & a_{32} & 0\end{pmatrix}\begin{pmatrix} x_1\\ x_2\\ x_3\end{pmatrix}, \tag{40}$$

$$\varphi(X)=\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & a_{23}\\ 0 & 0 & a_{33}\end{pmatrix}\begin{pmatrix} x_1\\ x_2\\ x_3\end{pmatrix}, \text{ or} \tag{41}$$

$$\varphi(X)=\begin{pmatrix} 0 & 0 & a_{13}\\ 0 & 0 & 0\\ 0 & 0 & a_{33}\end{pmatrix}\begin{pmatrix} x_1\\ x_2\\ x_3\end{pmatrix}. \tag{42}$$

Proof. Using the notations $g=g_{Sol}$, $h=g_{Nil}$, and the coordinates $\{x_1,x_2,x_3\}$ in $(\mathbb{R}^3,g_{Sol})$ and $\{y_1,y_2,y_3\}$ in $(\mathbb{R}^3,g_{Nil})$ we compute the following components of Nil and Sol metric:

$$\begin{aligned}
&g_{11}=e^{2x_3},\ g_{22}=e^{-2x_3},\ g_{33}=1,\ \text{all other } g_{ij}=0;\\
&g^{11}=e^{-2x_3},\ g^{22}=e^{2x_3},\ g^{33}=1,\ \text{all other } g^{ij}=0.\\
&h_{11}=1,\ h_{12}=g_{13}=0,\ h_{22}=1+y_1^2,\ h_{23}=-y_1,\ h_{33}=1;\\
&h^{11}=1,\ h^{12}=h^{13}=0,\ h^{22}=1,\ h^{23}=y_1,\ h^{33}=1+y_1^2.
\end{aligned} \tag{43}$$

A straightforward computation gives:

$$\begin{aligned}
\nabla\varphi^i &= g^{\alpha\beta}\tfrac{\partial\varphi^i}{\partial x_\beta}\tfrac{\partial}{\partial x_\alpha}\\
&=(a_{i1}e^{-2x_3},\,a_{i2}e^{2x_3},\,a_{i3}),\quad i=1,2,3,
\end{aligned} \tag{44}$$

and

$$\begin{aligned}
&|\mathrm{d}\varphi|^2=g^{\alpha\beta}{\varphi_\alpha}^i{\varphi_\beta}^j h_{ij}\circ\varphi\\
&=(e^{-2x_3}a_{21}^2+e^{2x_3}a_{22}^2+a_{23}^2)y_1^2-2(e^{-2x_3}a_{21}a_{31}+e^{2x_3}a_{22}a_{32}+a_{23}a_{33})y_1\\
&+\sum_{i=1}^3 a_{i1}^2e^{-2x_3}+\sum_{i=1}^3 a_{i2}^2e^{2x_3}+\sum_{i=1}^3 a_{i3}^2,
\end{aligned} \tag{45}$$

where $y_1=a_{11}x_1+a_{12}x_2+a_{13}x_3$.
Also, we can check that

$$\begin{aligned}
&\tfrac{\partial|\mathrm{d}\varphi|^2}{\partial x_1}=\\
&2(a_{11}a_{21}^2e^{-2x_3}+a_{11}a_{22}^2e^{2x_3}+a_{11}a_{23}^2)y_1\\
&-2(a_{11}a_{21}a_{31}e^{-2x_3}+a_{11}a_{22}a_{32}e^{2x_3}+a_{11}a_{23}a_{33}),
\end{aligned}$$

$$\begin{aligned}
&\tfrac{\partial|\mathrm{d}\varphi|^2}{\partial x_2}=\\
&2(a_{12}a_{21}^2e^{-2x_3}+a_{12}a_{22}^2e^{2x_3}+a_{12}a_{23}^2)y_1\\
&-2(a_{12}a_{21}a_{31}e^{-2x_3}+a_{12}a_{22}a_{32}e^{2x_3}+a_{12}a_{23}a_{33}),
\end{aligned}$$

and

$$\begin{aligned}
&\tfrac{\partial|\mathrm{d}\varphi|^2}{\partial x_3}=\\
&2(a_{22}^2e^{2x_3}-a_{21}^2e^{-2x_3})y_1^2\\
&+2\{(a_{13}a_{21}^2+2a_{21}a_{31})e^{-2x_3}+(a_{13}a_{22}^2-2a_{22}a_{32})e^{2x_3}+a_{13}a_{23}^2\}y_1\\
&-2\{(\sum_{i=1}^3 a_{i1}^2+a_{13}a_{21}a_{31})e^{-2x_3}+(a_{13}a_{22}a_{32}-\sum_{i=1}^3 a_{i2}^2)e^{2x_3}+a_{13}a_{23}a_{33})\}.
\end{aligned}$$

Using Corollary 1.1 we conclude that φ is ∞-harmonic if and only if

$$\begin{aligned}
&2(a_{i3}a_{22}^2e^{2x_3}-a_{i3}a_{21}^2e^{-2x_3})y_1^2\\
&+2\{a_{i1}a_{11}a_{21}^2e^{-4x_3}+a_{i2}a_{12}a_{22}^2e^{4x_3}\\
&+(a_{i1}a_{11}a_{23}^2+a_{i3}a_{13}a_{21}^2+2a_{i3}a_{21}a_{31}^2)e^{-2x_3}\\
&+(a_{i2}a_{12}a_{23}^2+a_{i3}a_{13}a_{22}^2-2a_{i3}a_{22}a_{32}^2)e^{2x_3}\\
&+(a_{i1}a_{11}a_{22}^2+a_{i2}a_{12}a_{21}^2+a_{i3}a_{13}a_{23}^2)y_1\\
&-2\{a_{i1}a_{11}a_{21}a_{31}e^{-4x_3}+a_{i2}a_{12}a_{22}a_{32}e^{4x_3}\\
&+(a_{i1}a_{11}a_{23}a_{33}+a_{i3}a_{13}a_{21}a_{31}+a_{i3}\sum_{j=1}^3 a_{j1}^2)e^{-2x_3}\\
&+(a_{i2}a_{12}a_{23}a_{33}+a_{i3}a_{13}a_{22}a_{32}-a_{i3}\sum_{j=1}^3 a_{j2}^2)e^{2x_3}\\
&+a_{i1}a_{11}a_{22}a_{32}+a_{i2}a_{12}a_{21}a_{31}+a_{i3}a_{13}a_{23}a_{33}\}=0,\quad i=1,2,3.
\end{aligned}\tag{46}$$

Case (A): $\sum\limits_{j=1}^3 a_{1j}^2=0$.

It follows that $y_1=a_{11}x_1+a_{12}x_2+a_{13}x_3=0$ and equation (46) reduces to

$$\begin{cases} a_{i3}\sum\limits_{j=1}^3 a_{j1}^2e^{-2x_3}=0,\\ a_{i3}\sum\limits_{j=1}^3 a_{j2}^2e^{2x_3}=0,\end{cases}\qquad i=1,2,3,\tag{47}$$

which has solutions $a_{i3}=0$, or $a_{i1}=a_{i2}=0$, for $i=1,2,3$. These give the linear ∞-harmonic maps defined by (39) and (41).

Case (B): $\sum\limits_{j=1}^3 a_{1j}^2\neq 0$.

In this case, we use equation (46) and the fact that the functions $1, te^{2t}, t^2e^{2t}; te^{-2t}, t^2e^{-2t}; te^{4t}, t^2e^{4t}; te^{-4t}, t^2e^{-4t}$ are linearly independent to conclude that φ is ∞-harmonic if and only if

$$\left.\begin{aligned} a_{i3}a_{22}^2 &=0\\ a_{i3}a_{21}^2 &=0,\end{aligned}\right\}\quad i=1,2,3,\tag{48}$$

$$\left.\begin{aligned} a_{i1}a_{11}a_{21}^2 &=0,\\ a_{i2}a_{12}a_{22}^2 &=0,\\ a_{i1}a_{11}a_{23}^2+a_{i3}a_{13}a_{21}^2+2a_{i3}a_{21}a_{31}^2 &=0,\\ a_{i2}a_{12}a_{23}^2+a_{i3}a_{13}a_{22}^2-2a_{i3}a_{22}a_{32}^2 &=0,\\ a_{i1}a_{11}a_{22}^2-a_{i2}a_{12}a_{21}^2+a_{i3}a_{13}a_{23}^2 &=0,\end{aligned}\right\}\quad i=1,2,3,\tag{49}$$

and

$$\left.\begin{aligned} a_{i1}a_{11}a_{21}a_{31} &= 0,\\ a_{i2}a_{12}a_{22}a_{32} &= 0,\\ a_{i1}a_{11}a_{23}a_{33}+a_{i3}a_{13}a_{21}a_{31}+a_{i3}\sum_{j=1}^{3}a_{j1}^2 &= 0,\\ a_{i2}a_{12}a_{23}a_{33}+a_{i3}a_{13}a_{22}a_{32}-a_{i3}\sum_{j=1}^{3}a_{j2}^2 &= 0,\\ a_{i1}a_{11}a_{22}a_{32}+a_{i2}a_{12}a_{21}a_{31}+a_{i3}a_{13}a_{23}a_{33} &= 0, \end{aligned}\right\}\quad i=1,2,3. \tag{50}$$

In this case, it is easy to check that $a_{2i} = a_{i3} = 0$ or $a_{i1} = a_{i2} = a_{2i} = 0$, for $i = 1, 2, 3$, are solutions of system (48), (49) and (50). These give the linear ∞-harmonic maps defined by (40) and (42). Thus, we obtain the theorem. □

Remark 4. Again, we remark that the maximum possible rank of linear ∞-harmonic maps from Sol into Nil is 2. Using the energy density formula (45) we can check that all rank 2 linear ∞-harmonic maps from Sol into Nil have non-constant energy density.

4. ∞-Harmonic linear endomorphisms of Sol space

In this final section, we study the ∞-harmonicity of linear endomorphisms of Sol space. We give a complete classification of ∞-harmonic linear endomorphisms of Sol space. It turns out that an ∞-harmonic linear endomorphism of Sol space can have maximum rank, i.e., there are ∞-harmonic linear diffeomorphisms from Sol space onto itself which have constant energy density and which are not isometries. We also show that there is a subgroup of ∞-harmonic linear automorphisms in the group of linear isomorphisms.

Theorem 4.1. *A linear endomorphism $\varphi : (\mathbb{R}^3, g_{Sol}) \longrightarrow (\mathbb{R}^3, g_{Sol})$ of Sol space with*

$$\varphi(X) = \begin{pmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x\\ y\\ z \end{pmatrix} \tag{51}$$

is ∞-harmonic if and only if φ takes one of the following forms:

$$\varphi(X) = \begin{pmatrix} a_{11} & 0 & 0\\ 0 & a_{22} & 0\\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x\\ y\\ z \end{pmatrix}, \tag{52}$$

$$\varphi(X) = \begin{pmatrix} 0 & a_{12} & 0\\ a_{21} & 0 & 0\\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x\\ y\\ z \end{pmatrix}, \tag{53}$$

$$\varphi(X) = \begin{pmatrix} a_{11} & a_{12} & 0\\ a_{21} & a_{22} & 0\\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x\\ y\\ z \end{pmatrix}, \tag{54}$$

$$\varphi(X) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{55}$$

$$\varphi(X) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_{31} & a_{32} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \text{ or} \tag{56}$$

$$\varphi(X) = \begin{pmatrix} 0 & 0 & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{57}$$

Proof. We use g and h to denote the metrics in the domain and the target manifolds respectively. With respect to the coordinates $\{x, y, z\}$ in the domain and $\{x', y', z'\}$ in the target manifold, we can easily write down the following components of metrics:

$$\begin{aligned} &g_{11} = e^{2z},\ g_{22} = e^{-2z},\ g_{33} = 1,\ \text{all other } g_{ij} = 0;\\ &g^{11} = e^{-2z},\ g^{22} = e^{2z},\ g^{33} = 1,\ \text{all other } g^{ij} = 0.\\ &h_{11} = e^{2z'},\ h_{22} = e^{-2z'},\ h_{33} = 1,\ \text{all other } h_{ij} = 0;\\ &h^{11} = e^{-2z'},\ h^{22} = e^{2z'},\ h^{33} = 1,\ \text{all other } h^{ij} = 0. \end{aligned}$$

A direct computation gives:

$$\begin{aligned} &\nabla\varphi^i = g^{\alpha\beta}\tfrac{\partial\varphi^i}{\partial x_\beta}\tfrac{\partial}{\partial x_\alpha}\\ &= (a_{i1}e^{-2z},\ a_{i2}e^{2z},\ a_{i3}), \qquad i = 1, 2, 3, \end{aligned}$$

and

$$\begin{aligned} &|\mathrm{d}\varphi|^2 = g^{\alpha\beta}{\varphi_\alpha}^i{\varphi_\beta}^j h_{ij}\circ\varphi = g^{\alpha\alpha}(\frac{\partial\varphi^i}{\partial x_\alpha})^2 h_{ii}\circ\varphi \\ &= (a_{12}^2e^{2z} + a_{11}^2e^{-2z} + a_{13}^2)e^{2z'}\\ &+(a_{22}^2e^{2z} + a_{21}^2e^{-2z} + a_{23}^2)e^{-2z'} + (a_{32}^2e^{2z} + a_{31}^2e^{-2z} + a_{33}^2), \end{aligned} \tag{58}$$

where $z' = a_{31}x + a_{32}y + a_{33}z$. Furthermore, we compute that

$$\begin{aligned} &\tfrac{\partial|\mathrm{d}\varphi|^2}{\partial x_1} = \tfrac{\partial|\mathrm{d}\varphi|^2}{\partial x}\\ &= 2a_{31}(a_{12}^2e^{2z} + a_{11}^2e^{-2z} + a_{13}^2)e^{2z'} - 2a_{31}(a_{22}^2e^{2z} + a_{21}^2e^{-2z} + a_{23}^2)e^{-2z'},\\ &\tfrac{\partial|\mathrm{d}\varphi|^2}{\partial x_2} = \tfrac{\partial|\mathrm{d}\varphi|^2}{\partial y}\\ &= 2a_{32}(a_{12}^2e^{2z} + a_{11}^2e^{-2z} + a_{13}^2)e^{2z'} - 2a_{32}(a_{22}^2e^{2z} + a_{21}^2e^{-2z} + a_{23}^2)e^{-2z'},\\ &\tfrac{\partial|\mathrm{d}\varphi|^2}{\partial x_3} = \tfrac{\partial|\mathrm{d}\varphi|^2}{\partial z}\\ &= 2\{(a_{12}^2 + a_{33}a_{12}^2)e^{2z} + (a_{33}a_{11}^2 - a_{11}^2)e^{-2z} + a_{33}a_{13}^2\}e^{2z'}\\ &-2\{(a_{33}a_{22}^2 - a_{22}^2)e^{2z} + (a_{33}a_{21}^2 + a_{21}^2)e^{-2z} + a_{33}a_{23}^2\}e^{-2z'}\\ &+2(a_{32}^2e^{2z} - a_{31}^2e^{-2z}). \end{aligned} \tag{59}$$

By Corollary 1.1 φ is ∞-harmonic if and only if

$$g(\nabla\,\varphi^i, \nabla\,|\mathrm{d}\varphi|^2) = 0, \quad i = 1, 2, 3, \tag{60}$$

which is equivalent to

$$\begin{aligned}
0 = \quad & \{a_{i2}a_{32}a_{12}^2e^{4z} + a_{i1}a_{31}a_{11}^2e^{-4z} \\
& +(a_{i2}a_{32}a_{13}^2 + a_{i3}a_{12}^2 + a_{i3}a_{33}a_{12}^2)e^{2z} \\
& +(a_{i1}a_{31}a_{13}^2 - a_{i3}a_{11}^2 + a_{i3}a_{33}a_{11}^2)e^{-2z} \\
& +a_{i1}a_{31}a_{12}^2 + a_{i2}a_{32}a_{11}^2 + a_{i3}a_{33}a_{13}^2\}e^{2z'} \\
& -\{a_{i2}a_{32}a_{22}^2e^{4z} + a_{i1}a_{31}a_{21}^2e^{-4z} \\
& +(a_{i2}a_{32}a_{23}^2 - a_{i3}a_{22}^2 + a_{i3}a_{33}a_{22}^2)e^{2z} \\
& +(a_{i1}a_{31}a_{23}^2 + a_{i3}a_{21}^2 + a_{i3}a_{33}a_{21}^2)e^{-2z} \\
& +a_{i1}a_{31}a_{22}^2 + a_{i2}a_{32}a_{21}^2 + a_{i3}a_{33}a_{23}^2\}e^{-2z'} \\
& +a_{i3}(a_{32}^2e^{2z} - a_{31}^2e^{-2z}), \qquad i = 1,2,3.
\end{aligned} \tag{61}$$

Case (A): $a_{31}^2 + a_{32}^2 + a_{33}^2 = 0$. It follows that $z' = a_{31}x + a_{32}y + a_{33}z = 0$, and the equation (61) becomes

$$a_{i3}\{(a_{12}^2 + a_{22}^2)e^{2z} - (a_{11}^2 + a_{21}^2)e^{-2z}\} = 0, \quad i = 1,2,3, \tag{62}$$

which gives the solutions $a_{11} = a_{12} = a_{21} = a_{22} = 0$, or, $a_{i3} = 0$, for $i = 1,2,3$. These give the linear ∞-harmonic maps of the form (54) and (57).

Case (B): $a_{31}^2 + a_{32}^2 + a_{33}^2 \neq 0$. We use equation (61) and the fact that the functions $e^{k_1t}; e^{-k_1t}; e^{k_2t}, e^{-k_2t}; e^{k_3t}, e^{-k_3t}; e^{k_4t}, e^{-k_4t}; e^{k_5t}, e^{-k_5t}$ with $k_1, \ldots, k_5$ distinctive are linearly independent to conclude that φ is ∞-harmonic if and only if

$$\left.\begin{array}{rll}
a_{i3}a_{32}^2 & = 0, & \langle 1\rangle \\
a_{i3}a_{31}^2 & = 0, & \langle 2\rangle \\
a_{i2}a_{32}a_{12}^2 & = 0, & \langle 3\rangle \\
a_{i1}a_{31}a_{11}^2 & = 0, & \langle 4\rangle \\
a_{i2}a_{32}a_{13}^2 + a_{i3}a_{12}^2 + a_{i3}a_{33}a_{12}^2 & = 0, & \langle 5\rangle \\
a_{i1}a_{31}a_{13}^2 - a_{i3}a_{11}^2 + a_{i3}a_{33}a_{11}^2 & = 0, & \langle 6\rangle \\
a_{i1}a_{31}a_{12}^2 + a_{i2}a_{32}a_{11}^2 + a_{i3}a_{33}a_{13}^2 & = 0, & \langle 7\rangle \\
a_{i2}a_{32}a_{22}^2 & = 0, & \langle 8\rangle \\
a_{i1}a_{31}a_{21}^2 & = 0, & \langle 9\rangle \\
a_{i2}a_{32}a_{23}^2 - a_{i3}a_{22}^2 + a_{i3}a_{33}a_{22}^2 & = 0, & \langle 10\rangle \\
a_{i1}a_{31}a_{23}^2 + a_{i3}a_{21}^2 + a_{i3}a_{33}a_{21}^2 & = 0, & \langle 11\rangle \\
a_{i1}a_{31}a_{22}^2 + a_{i2}a_{32}a_{21}^2 + a_{i3}a_{33}a_{23}^2 & = 0. & \langle 12\rangle
\end{array}\right\} \tag{63}$$

It follows from ⟨1⟩ and ⟨2⟩ of (63) that

$$a_{i3} = 0, \text{ or, } a_{31} = a_{32} = 0, \quad i = 1,2,3. \tag{64}$$

Case (B$_1$): $a_{i3} = 0, \quad i = 1,2,3$ and hence $a_{31}^2 + a_{32}^2 \neq 0$.

Performing ⟨3⟩ + ⟨4⟩ + ⟨7⟩, ⟨8⟩ + ⟨9⟩ + ⟨12⟩ separatively yields

$$\left.\begin{array}{rl}
(a_{11}^2 + a_{12}^2)(a_{i1}a_{31} + a_{i2}a_{32}) & = 0 \\
(a_{21}^2 + a_{22}^2)(a_{i1}a_{31} + a_{i2}a_{32}) & = 0,
\end{array}\right\} \quad i = 1,2,3, \tag{65}$$

which gives us solutions $a_{11} = a_{21} = a_{12} = a_{22} = 0$. These give ∞-harmonic linear automorphisms of the form defined in (56).

Case (B_2): $a_{31} = a_{32} = 0$, and hence $a_{33} \neq 0$.
In this case, equation (63) reduces to

$$\left.\begin{array}{rl} a_{i3}a_{12}^2(1+a_{33}) & = 0, \\ a_{i3}a_{11}^2(a_{33}-1) & = 0, \\ a_{i3}a_{33}a_{13}^2 & = 0, \\ a_{i3}a_{22}^2(a_{33}-1) & = 0, \\ a_{i3}a_{21}^2(1+a_{33}) & = 0, \\ a_{i3}a_{33}a_{23}^2 & = 0, \end{array}\right\} \quad i = 1, 2, 3. \tag{66}$$

We solve this system by considering the following three case:

(I) $a_{33} = 1$. By (66), we have

$$\left.\begin{array}{rl} a_{i3}a_{12}^2 & = 0, \\ a_{i3}a_{13}^2 & = 0, \\ a_{i3}a_{21}^2 & = 0, \\ a_{i3}a_{23}^2 & = 0, \end{array}\right\} \quad i = 1, 2, 3. \tag{67}$$

Letting $i = 3$ we conclude that $a_{12} = a_{13} = a_{21} = a_{23} = 0$, which give the solutions of the form (52).

(II) $a_{33} = -1$. In this case, (66) reduces to

$$\left.\begin{array}{rl} a_{i3}a_{11}^2 & = 0, \\ a_{i3}a_{13}^2 & = 0, \\ a_{i3}a_{22}^2 & = 0, \\ a_{i3}a_{23}^2 & = 0, \end{array}\right\} \quad i = 1, 2, 3. \tag{68}$$

Letting $i = 3$ we conclude that $a_{11} = a_{22} = a_{13} = a_{23} = 0$, which give the solutions of the form (53).

(III) $a_{33} \neq \pm 1,\ 0$. Then, (66) becomes

$$\left.\begin{array}{rl} a_{i3}a_{12}^2 & = 0, \\ a_{i3}a_{11}^2 & = 0, \\ a_{i3}a_{13}^2 & = 0, \\ a_{i3}a_{22}^2 & = 0, \\ a_{i3}a_{21}^2 & = 0, \\ a_{i3}a_{23}^2 & = 0, \end{array}\right\} \quad i = 1, 2, 3. \tag{69}$$

Letting $i = 3$ we get $a_{11} = a_{12} = a_{13} = a_{21} = a_{22} = a_{23} = 0$, which give the solutions of the form (55).

Summarizing all results in the above cases we obtain the theorem. $\square$

Corollary 4.2. *Every element of the subgroup*

$$\left\{ \varphi \in \mathrm{GL}(\mathbb{R}^3) : \ \varphi(X) = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \lambda\mu \neq 0 \right\} \tag{70}$$

of the linear automorphism group of Sol space is ∞-harmonic.

Proof. It follows from Theorem 4.1 that every element of the subgroup is an ∞-harmonic map. A straightforward checking shows that the inverse elements and the products of elements of the subgroup are also ∞-harmonic. □

Remark 5. It follows from our theorem that the maximum possible rank of linear ∞-harmonic endomorphisms of Sol space is 3, so we can have linear ∞-harmonic diffeomorphisms which have constant energy density and which are not isometries. Using the energy density formula (58) we can check that all rank 2 linear ∞-harmonic maps from Sol into itself have non-constant energy density.

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