Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 51 (2010), No. 1, 1-7.

Second Order Parallel Tensors on Generalized Sasakian Space Forms and Semi Parallel Hypersurfaces in Sasakian Space Forms

Fatiha Gherib **Mohamed Belkhelfa**

Laboratoire de Physique Quantique de la matière
et de Modélisation Mathématique (LPQ3M)
Université de Mascara, Algérie
e-mail: Fatiha_gherib@yahoo.fr
e-mail: belkhelfa@univ-mascara.dz or mohamed.belkhelfa@gmail.com

Abstract. In this paper, we show that a second order parallel symmetric tensor in a generalized Sasakian space form is proportional to the metric tensor and we deduce that there is no semi parallel hypersurface in a Sasakian space form.

MSC2000: 53D10, 53C25, 53C21
Keywords: generalized Sasakian space forms, parallel tensor, semi parallel hypersurface

1. Introduction

In 1926, Levy [4] has proved that a second order parallel symmetric non singular tensor in real space forms is proportional to the metric tensor. In 1989, Sharma [8] has proved that a second order parallel tensor in a Kähler space of constant holomorphic sectional curvature is a linear combination with constant coefficients of the Kählerian metric and the fundamental 2-form. Recently, Das [6] has established the same result for an α-Sasakian manifold ($\alpha \in \mathbb{R}_0$). In this paper we generalize this result to generalized Sasakian space form and we prove that there is no semi parallel hypersurface in a Sasakian space form.

0138-4821/93 $ 2.50 © 2010 Heldermann Verlag

2. Preliminaries

Let M denote an n-dimensional Riemannian manifold with its metric tensor g and Levi-Civita connection $\widetilde{\nabla}$. Let $\tilde{R}$ denote the Riemann curvature tensor of M. If B is a $(0,2)$ tensor which is parallel with respect to $\widetilde{\nabla}$ then we can show that

$$B\left(\tilde{R}\left(X,Y\right)Z,W\right)+B\left(Z,\tilde{R}\left(X,Y\right)W\right)=0. \tag{1}$$

Das has proved that

Theorem 2.1. [6] *On an $\alpha-K$ contact manifold ($\alpha \in R_0$) a second order symmetric parallel tensor is a constant multiple of the associated positive definite Riemannian metric tensor.*

The first purpose of this paper is to present a similar result for a generalized Sasakian space form. Let (M^{2n+1},g) be a $2n+1$ dimensional differentiable manifold and let (ϕ,ξ,η) be tensor fields of type $(1,1)$, $(1,0)$ and $(0,1)$ respectively on M, such that

$$\eta\left(\xi\right)=1 \qquad \phi^2=-I+\xi\otimes\eta$$

which implies

$$\eta\circ\phi=0 \qquad \phi\left(\xi\right)=0 \qquad rank\left(\phi\right)=2n.$$

If M admits a Riemannian metric g, such that

$$\begin{aligned} g\left(\phi X,\phi Y\right) &= g\left(X,Y\right)-\eta\left(X\right)\eta\left(Y\right) \\ g\left(X,\xi\right) &= \eta(X) \end{aligned}$$

then (ϕ,ξ,η,g) is called an almost contact metric structure on M. If moreover

$$\left(\tilde{\nabla}_X\phi\right)Y=g\left(X,Y\right)\xi-\eta\left(Y\right)X$$

where $\tilde{\nabla}$ denotes the Riemannian connection of g, then (M,ϕ,ξ,η,g) is called a Sasakian manifold [10].

The sectional curvature of the plane section spanned by the unit tangent vector field X orthogonal to ξ and ϕX is called a ϕ-sectional curvature. If M has a constant ϕ-sectional curvature c, then M is called a Sasakian space form and denoted by $M^{2n+1}\left(c\right)$. The Riemannian curvature of a Sasakian space form is given by the following formula

$$\begin{aligned} R\left(X,Y\right)Z &= \frac{c+3}{4}\left[g(Y,Z)X-g(X,Z)Y\right] \\ &+\frac{c-1}{4}\left[\eta\left(X\right)\eta\left(Z\right)Y-\eta\left(Y\right)\eta\left(Z\right)X\right] \\ &+\frac{c-1}{4}[g(X,Z)\eta\left(Y\right)\xi-g(Y,Z)\eta\left(X\right)\xi+g(Z,\phi Y)\phi X \\ &-g(Z,\phi X)\phi Y+2g(X,\phi Y)\phi Z]. \end{aligned}$$

Example 2.2. [10] We consider $\mathbb{R}^{2n+1}$ with the coordinates (x^i, y^i, z), $i=1,\ldots,n$ and its usual contact form $\eta = \frac{1}{2}\left(dz - \sum_{i=1}^{n} y^i dx^i\right)$. The characteristic field ξ is given by $\xi = 2\frac{\partial}{\partial z}$, the tensor field ϕ is given by the matrix $\begin{pmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ 0 & y^j & 0 \end{pmatrix}$ and the Riemannian metric $g = \eta \otimes \eta + \frac{1}{4}\sum_{i=1}^{n}\left(dx^i\right)^2 + \left(dy^i\right)^2$ is an associated metric for η. In this case $\mathbb{R}^{2n+1}$ is a Sasakian space form with ϕ-sectional curvature $c=-3$ denoted by $\mathbb{R}^{2n+1}(-3)$.

Given an almost contact metric (M, ϕ, ξ, η, g), M is called generalized Sasakian space form if there exist three functions f_1, f_2 and f_3 such that the Riemannian curvature tensor is given by the following formula

$$\begin{aligned} R(X,Y)Z &= f_1\left[g(Y,Z)X - g(X,Z)Y\right] + f_2[g(X,\phi Z)\phi Y \\ &\quad -g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z] + f_3[\eta(X)\eta(Z)Y \\ &\quad -\eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi]. \end{aligned} \tag{2}$$

In such a case, we will write $M(f_1, f_2, f_3)$. This kind of manifold appears as natural generalization of the Sasakian space form by taking:

$$f_1 = \frac{c+3}{4} \text{ and } f_2 = f_3 = \frac{c-1}{4}.$$

The ϕ-sectional curvature of a generalized Sasakian space form $M(f_1, f_2, f_3)$ is $f_1 + 3f_2$ [9].

Let N^{2n} be an immersed hypersurface of $M^{2n+1}(f_1, f_2, f_3)$. We denote the Levi-Civita connection of M by $\widetilde{\nabla}$ and the Levi-Civita connection of N by ∇. Then we have the formulas of Gauss and Weingarten

$$\begin{aligned} \widetilde{\nabla}_X Y &= \nabla_X Y + h(X,Y)r \\ \widetilde{\nabla}_X r &= -SX. \end{aligned}$$

X and Y are tangent vector fields, r a unit normal vector normal to N and h the second fundamental form of N and S the shape operator of N. Note that h and S are related by $h(X,Y) = g(SX,Y)$. In a hypersurface the $(0,4)$ tensor field $\tilde{R}.h$ is defined by

$$\tilde{R}.h(X,Y,Z,W) = -h\left(\tilde{R}(X,Y)Z, W\right) - h\left(Z, \tilde{R}(X,Y)W\right).$$

In [2] J. Deprez has defined semi parallel immersions which satisfy the condition $\tilde{R}.h = 0$. The authors F. Dillen, J. Fastenakels, S. Haesen, G. Van Der Veken and L. Verstraelen gave a geometrical interpretation of semi parallel submanifolds.

Proposition 2.3. [16] *A submanifold N of M is semi parallel if, $\forall p \in M$, the normal vectors $h(u,v)^{*\perp}$ and $h(u^*,v^*)$ coincide for all $u, v \in T_P M$ and for every coordinate parallelogram in M, up to second order. Where u^* and v^* are the parallel transport of u and v with respect to ∇ and $h(u,v)^{*\perp}$ is the parallel transport of $h(u,v)$ with respect to the normal connection $\nabla^{\perp}$.*

We have proved in [3] that

Theorem 2.4. *There is not a parallel connected hypersurface in a Sasakian space form $M^{2n+1}(c)$ with $n \geq 2$ and $c \neq 1$.*

The Ricci tensor is given by Kim [13]

$$S(X,Y) = (2nf_1 + 3f_2 - f_3)\, g(X,Y) - (3f_2 + (2n-1)\, f_3)\, \eta(X)\, \eta(Y).$$

3. Main results

Theorem 3.1. *In a generalized Sasakian space form $M(f_1, f_2, f_3)$ with $f_1 \neq f_3$, a second order parallel symmetric tensor B is a constant multiple of the associated positive definite metric tensor.*

Proof. If B is parallel $\left(\widetilde{\nabla} B = 0\right)$, it follows that

$$B\left(\tilde{R}(X,Y)\, Z, W\right) + B\left(Z, \tilde{R}(X,Y)\, W\right) = 0 \tag{3}$$

for X, Y, Z and W vector fields on M.

By taking $Y = \xi$ and $Z = W$ and using equation (2), we have

$$(f_1 - f_3)\,(\eta(Z)\, B(X,Z) - g(X,Z)\, B(\xi,Z) + \eta(Z)\, B(Z,X) - g(X,Z)\, B(Z,\xi)) = 0$$

since $f_1 \neq f_3$ and B is symmetric we have

$$\eta(Z)\, B(X,Z) = g(X,Z)\, B(Z,\xi)$$

so

$$B(Z,\xi) = \eta(Z)\, B(\xi,\xi)$$

which implies that

$$\eta(Z)\,(B(X,Z) - g(X,Z)\, B(\xi,\xi)) = 0.$$

If $\eta(Z) \neq 0$, we have

$$B(X,Z) = g(X,Z)\, B(\xi,\xi). \tag{4}$$

If $\eta(Z) = 0$, so $B(\xi,Z) = 0$ and by substituting $Y = W = \xi$ in equation (4) we get the above equation. □

Corollary 3.2. *If the Ricci tensor of a generalized Sasakian space form $M(f_1, f_2, f_3)$ with $f_1 \neq f_3$ is parallel, then M is Einstein.*

We also have

Theorem 3.3. *There are no semi parallel hypersurfaces in a Sasakian space form $M^{2n+1}(c)$ with $c \neq 1$ and $n \geq 2$.*

Proof. If N is a semi parallel hypersurface and h the second fundamental form of N, we have:

$$-h\left(\tilde{R}(X,Y)Z,W\right)-h\left(Z,\tilde{R}\,(X,Y)\,W\right)=0$$

by using the same argument as in Theorem 3.1 we deduce that

$$h=\lambda g$$

where λ is constant. Consequently

$$\tilde{\bigtriangledown}h=0$$

which contradicts Theorem 2.3. □

Corollary 3.4. *There are no semi parallel hypersurfaces in* $\mathbb{R}^{2n+1}(-3)$ *with* $n\geq 2$.

Remark 3.5. Let us consider the $(2n+1)$-dimensional unit sphere, i.e., $S^{2n+1}=\{z\in\mathbb{C}^{n+1}:|z|=1\}$. Any point z of S^{2n+1} can be identified to $(x^1,\ldots,x^n,y^1,\ldots,y^n)\in\mathbb{R}^{2n+2}$. We put $Jz=(-y^1,\ldots,-y^n,x^1,\ldots,x^n)$, where J is the usual complex structure on $\mathbb{C}^{n+1}$. We define the characteristic vector field ξ, the 1-form η and the $(1,1)$ tensor ϕ by

$$\xi=-Jz,\ \eta(X)=g(X,\xi)\ \text{ and }\phi=s\circ J$$

where g is the induced metric of $\mathbb{C}^{n+1}$ on S^{2n+1} and s is the orthogonal projection of $T_x\mathbb{C}^{n+1}$ on T_xS^{2n+1}. So, S^{2n+1} is a Sasakian space form with ϕ-sectional curvature equal to 1.

Now we consider the Clifford hypersurface $M_{p,q}$ defined by

$$M_{p,q}=S^{2p+1}\left(\sqrt{\frac{p}{2n}}\right)\times S^{2q+1}\left(\sqrt{\frac{q}{2n}}\right),\quad p+q=n-1.$$

Then, $M_{p,q}$ is a minimal hypersurface of S^{2n+1} tangent to the structure field ξ of S^{2n+1} and $M_{p,q}$ has a parallel second fundamental form. Therefore the assumption in Theorem 2.4 and Theorem 3.3 on the ϕ-sectional curvature $c\neq 1$ of the ambient space is essential.

Remark 3.6. The condition $n\geq 2$ in Theorem 2.4 and Theorem 3.3 is also essential, there exist parallel surfaces for $n=1$ [14] and [15].

References

[1] Dillen, F.: *Semi-parallel hypersurfaces of a real space form.* Isr. J. Math. **75** (1991), 193–202. Zbl 0765.53012

[2] Deprez, J.: *Semi-parallel hypersurfaces.* Rend. Semin. Mat. Torino **44** (2) (1986), 303–315. Zbl 0616.53018

[3] Gherib, F.; Belkhelfa, M.: *Parallel submanifolds of generalized Sasakian space form.* Bulletin of the Transilvania University of Brasov **51**(2) (2009) (to appear).

[4] Levy, H.: *Symmetric tensors of the second order whose covariant derivates vanish.* Annals of Math. **27** (1926), 91–98. JFM 51.0576.02

[5] Yano, K.; Kon, M.: *Structures on Manifolds.* Series in Pure Mathematics **3**, World Scientific, Singapore 1984. Zbl 0557.53001

[6] Das, L.: *Second order parallel tensors on α-Sasakian manifold.* Acta Math. Acad. Paedagog. Nyházi. (N.S.) **23** (2007), 65–69. Zbl 1135.53317

[7] Eisenhart, L. P.: *Symmetric tensors of the second order whose first covariant derivates are zero.* American M. S. Trans. **25** (1923), 297–306. JFM 49.0539.01

[8] Sharma, R.: *Second order parallel tensor in real and complex space forms.* Int. J. Math. Sci. **12** (1989), 787–790. Zbl 0696.53012

[9] Alegre, P.; Blair, D. E.; Carriazo, A.: *Generalized Sasakian space forms.* Isr. J. Math. **141** (2004), 157–183. Zbl 1064.53026

[10] Blair, D. E.: *Riemannian geometry of contact manifolds and symplectic manifolds.* Progress in Mathematics **203** (2002). Zbl 1011.53001

[11] Baikoussis, C.; Blair, D. E.: *Finite type integral submanifolds of the contact manifold* $\mathbb{R}^{2n+1}(-3)$. Bull. Inst. Math. Acad. Sin. **19**(4) (1991), 327–350. Zbl 0753.53030

[12] Ferus, D.: *Immersionen mit paralleler zweiter Fundamentalform: Beispiele und Nicht-Beispiele.* Manuscr. Math. **12** (1974), 153–162. Zbl 0274.53058

[13] Kim, U.: *Conformally flat generalized Sasakian space form and locally symmetric generalized Sasakian space form.* Note Mat. **26**(1) (2006), 55–67. Zbl 1118.53025

[14] Belkhelfa, M.; Dillen, F.: *Parallel surfaces in Heisenberg space.* Preprint (2000).

[15] Belkhelfa, M.; Dillen, F.; Inoguchi, J. I.: *Surfaces with parallel second fundamental form in Bianchi-Cartan-Vranceanu spaces.* In: B. Opozda (ed.) et al.: PDEs, submanifolds and affine differential geometry. Contributions of a conference, Warsaw, Poland, September 4–10, 2000. Warsaw: Polish Academy of Sciences, Institute of Mathematics, Banach Cent. Publ. **57** (2002), 67–87. Zbl 1029.53071

[16] Dillen, F.; Fastenakels, J.; Haesen, S.; Van Der Veken, J.; Verstraelen, L.: *Submanifolds theory and the parallel transport of Levi Civita*. Preprint.

Received May 28, 2008