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# Spherical Quadrangles

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**Abstract.** We introduce the notion of quasi-well-centered spherical quadrangle, or QWCSQ for short, describing a geometrical method to construct any QWCSQ. It is shown that any spherical quadrangle is congruent to a QWCSQ. We classify such quadrangles taking into account the relative position of the spherical moons containing their sides. This allows us to conclude that the class of all QWCSQ is a differentiable manifold of dimension five.
Keywords: spherical geometry, applications of spherical trigonometry

## 1. Introduction

Let $S^2$ be the unit 2-sphere. The notion of well-centered spherical moon was introduced in [1] (a spherical moon whose vertices belong to the great circle $x = 0$, and whose bisecting semi-great circle contains the point $C = (1, 0, 0)$; in Figure 1

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a well centered spherical moon, $L_1$ is presented). Also in [1] it was established that any spherical (geodesic) quadrangle with congruent opposite angles is congruent to the intersection of two well-centered spherical moons, i.e., a well-centered spherical quadrangle.

In this paper we generalize these results to the class of all spherical quadrangles, by introducing the notion of quasi-well-centered spherical quadrangle. Some of the obtained results are based in spherical trigonometry formulas. The cosine rules state that the angles $\alpha_1$, $\alpha_2$ and $\alpha_3$ of a spherical triangle satisfy

$$\cos\alpha_1 = \frac{\cos a - \cos b\,\cos c}{\sin b\,\sin c} \quad \text{and} \quad \cos a = \frac{\cos\alpha_1 + \cos\alpha_2\,\cos\alpha_3}{\sin\alpha_2\,\sin\alpha_3}, \tag{1.1}$$

where $a$, $b$ and $c$ are the lengths of the edges opposite to $\alpha_1$, $\alpha_2$ and $\alpha_3$, respectively. For a detailed discussion on spherical trigonometry see [2].

## 2. Spherical Quadrangles

By a *quasi-well-centered spherical quadrangle* (QWCSQ) we mean a spherical quadrangle $Q$ which is the intersection of a well-centered spherical moon $L_1$ with vertices $N$ and $-N$ (where $N = (0, 0, 1)$) and a spherical moon $L_2$ with one of its vertices, say $v$, in the first octant ($x, y, z \geq 0$), see Figure 1.

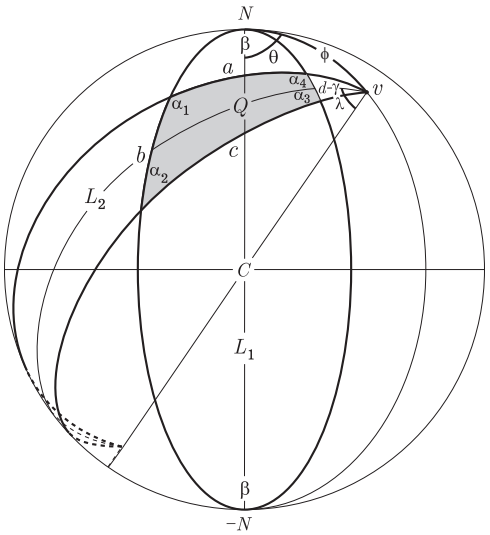


Figure 1. A quasi-well-centered spherical quadrangle $Q$

We have used the following notation:

- $\beta$ and $\gamma$ are the angles measure of the spherical moons $L_1$ and $L_2$, respectively; $\beta$, $\gamma \in (0, \pi)$;
- $\alpha_1$, $\alpha_2$, $\alpha_3$, $\alpha_4$ and $a$, $b$, $c$, $d$ are, respectively, the internal angles and the edge lengths of $Q = L_1 \cap L_2$;
- $\theta$ and $\phi$ are the spherical coordinates of the vertex $v$ of $L_2$; in other words, considering $v = (x, y, z)$, then $x = \cos\theta\sin\phi$, $y = \sin\theta\sin\phi$ and $z = \cos\phi$.

Geometrically, $\theta$ is the oriented angle between the bisector of $L_1$ and the meridian through $N$ that contains $v$; $\theta \in \left(\frac{\beta}{2}, \frac{\pi}{2}\right]$. On the other hand, $\phi$ is the oriented angle between $N$ and the vertex $v$; $\phi \in \left(0, \frac{\pi}{2}\right]$;

- $\lambda$ is the oriented angle between the line connecting $v$ and $C = (1, 0, 0)$, and the bisector of $L_2$; $\lambda \in \left(-\frac{\pi-\gamma}{2}, \frac{\pi-\gamma}{2}\right)$.

**Remark 1.** With the above notation we have the following properties.

1. If $\theta = \frac{\pi}{2}$ and $\lambda = 0$, then $\alpha_1 = \alpha_3$ and $\alpha_2 = \alpha_4$, i.e., $Q = L_1 \cap L_2$ is a spherical parallelogram. In addition, if $\phi = \frac{\pi}{2}$, then $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4$ and $Q$ is a spherical rectangle.
2. If $\theta = \frac{\pi}{2}$ and $\phi = \frac{\pi}{2}$, then $\alpha_1 = \alpha_4$ and $\alpha_2 = \alpha_3$, and so $Q = L_1 \cap L_2$ is an isosceles trapezoid.
3. If $\phi = \frac{\pi}{2}$ and $\lambda = 0$, then $\alpha_1 = \alpha_2$ and $\alpha_3 = \alpha_4$, and so $Q = L_1 \cap L_2$ is also an isosceles trapezoid.

**Proposition 2.1.** *Let $Q$ be a spherical quadrangle with internal angles $\alpha_1$, $\alpha_2$, $\alpha_3$ and $\alpha_4$, and edge lengths $a$, $b$, $c$ and $d$. Then any three of these parameters are completely determined by the remaining five.*

*Proof.* Let $Q$ be a spherical quadrangle as described above. We shall show how to determine $\alpha_4$, $c$ and $d$ as functions of $\alpha_1$, $\alpha_2$, $\alpha_3$, $a$ and $b$. Other cases are treated in a similar way.

Let $l$ be the diagonal of $Q$ through $\alpha_2$ and $\alpha_4$ as illustrated in Figure 2.

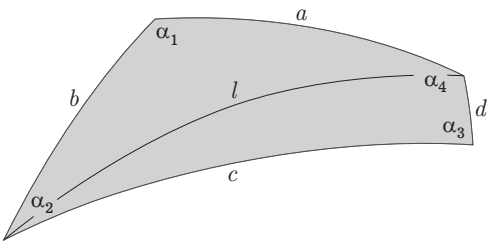


Figure 2. A spherical quadrangle

By (1.1), we have

$$\cos\alpha_1 = \frac{\cos l - \cos a \cos b}{\sin a \sin b} \qquad \text{and} \qquad \cos\alpha_3 = \frac{\cos l - \cos c \cos d}{\sin c \sin d},$$

and so

$$\cos a \cos b + \sin a \sin b \cos\alpha_1 = \cos c \cos d + \sin c \sin d \cos\alpha_3. \tag{2.1}$$

Now, extending the sides $a$ and $c$ of $Q$ one gets a spherical moon as shown in Figure 3. Let $\gamma$ be its angle measure.

Using again (1.1), one gets

$$\cos b = \frac{\cos\gamma + \cos\alpha_1 \cos\alpha_2}{\sin\alpha_1 \sin\alpha_2} \qquad \text{and} \qquad \cos d = \frac{\cos\gamma + \cos\alpha_3 \cos\alpha_4}{\sin\alpha_3 \sin\alpha_4},$$

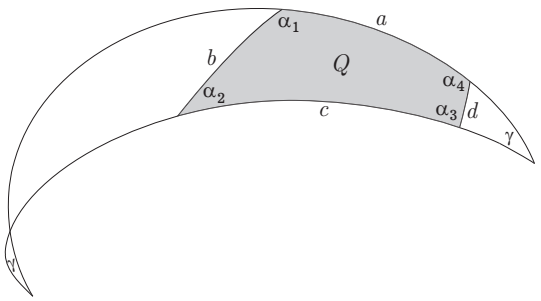


Figure 3. A spherical moon obtained by extending a pair of opposite sides of $Q$

and so

$$-\cos\alpha_1\cos\alpha_2+\sin\alpha_1\sin\alpha_2\cos b=-\cos\alpha_3\cos\alpha_4+\sin\alpha_3\sin\alpha_4\cos d. \quad (2.2)$$

Similarly, extending the other pair of opposite sides of $Q$ we get the formula

$$-\cos\alpha_1\cos\alpha_4+\sin\alpha_1\sin\alpha_4\cos a=-\cos\alpha_2\cos\alpha_3+\sin\alpha_2\sin\alpha_3\cos c. \quad (2.3)$$

From equations (2.2) and (2.3) we may obtain $d$ and $c$ as functions of $\alpha_4$, respectively. Replacing $c$ and $d$ in (2.1) by the obtained expressions, we get $\alpha_4$ as function of $\alpha_1$, $\alpha_3$, $a$ and $b$. The expressions for $c$ and $d$ follow immediately. Therefore, $\alpha_4$, $c$ and $d$ are completely determined when $\alpha_1$, $\alpha_2$, $\alpha_3$, $a$ and $b$ are fixed values. □

**Proposition 2.2.** *Any spherical quadrangle is congruent to a QWCSQ. Besides, its sides and angles are completely determined by the five parameters $\beta$, $\gamma$, $\theta$, $\phi$ and $\lambda$ defined in Figure* 1.

*Proof.* Suppose that $Q$ is a spherical quadrangle with internal angles, $\alpha_1$, $\alpha_2$, $\alpha_3$ and $\alpha_4$, and edge lengths $a$, $b$, $c$ and $d$. The extension of the edges of $Q$ give rise to two spherical moons $L_1$ and $L_2$, such that $Q=L_1\cap L_2$. Now it follows that there is a spherical isometry $\sigma$ such that $\sigma(L_1)$ is a well centered spherical moon with vertices $N$ and $-N$ ($N=(0,0,1)$) and $\sigma(L_2)$ has one of these vertices in the first octant. And so $Q$ is congruent to a QWCSQ. By Proposition 2.1 the knowledge of $\alpha_1$, $\alpha_2$, $\alpha_3$, $a$ and $b$ determines $\alpha_4$, $c$ and $d$.

Using the labelling of Figure 1 one gets the following system of equations in the five variables $\beta$, $\gamma$, $\theta$, $\phi$ and $\lambda$.

$$\begin{cases}
\cos\alpha_1 &= \cos\dfrac{\beta+2\theta}{2}\sin\dfrac{\gamma+2\lambda}{2}-\sin\dfrac{\beta+2\theta}{2}\cos\dfrac{\gamma+2\lambda}{2}\cos\phi\\
\cos\alpha_2 &= \cos\dfrac{\beta+2\theta}{2}\sin\dfrac{\gamma-2\lambda}{2}+\sin\dfrac{\beta+2\theta}{2}\cos\dfrac{\gamma-2\lambda}{2}\cos\phi\\
\cos\alpha_3 &= -\cos\dfrac{\beta-2\theta}{2}\sin\dfrac{\gamma-2\lambda}{2}+\sin\dfrac{\beta-2\theta}{2}\cos\dfrac{\gamma-2\lambda}{2}\cos\phi\\
\cos a &= \dfrac{\cos\beta+\cos\alpha_1\cos\alpha_4}{\sin\alpha_1\sin\alpha_4}\\
\cos b &= \dfrac{\cos\gamma+\cos\alpha_1\cos\alpha_2}{\sin\alpha_1\sin\alpha_2}
\end{cases}$$

We obtain the expressions of $\beta$ and $\gamma$ from the two last equations. Replacing these expressions in the first three equations and solving now the $3 \times 3$ system of equations, we also get $\theta$, $\phi$ and $\lambda$. □

Let $\mathcal{Q}$ be the set of all QWCSQ.

**Corollary 2.1.** *The degree of freedom given by the five parameters $\beta$, $\gamma$, $\theta$, $\phi$ and $\lambda$ allows us to conclude that $\mathcal{Q}$ is a differentiable manifold of dimension five; $\beta, \gamma \in (0, \pi)$, $\theta \in \left(\frac{\beta}{2}, \frac{\pi}{2}\right]$, $\phi \in \left(0, \frac{\pi}{2}\right]$, $\lambda \in \left(-\frac{\pi-\gamma}{2}, \frac{\pi-\gamma}{2}\right)$.*
*The set of all isosceles trapezoids contains a manifold of dimension three. The submanifold contained in the border of $\mathcal{Q}$ defined by the equations $\theta = \frac{\pi}{2}$ and $\phi = \frac{\pi}{2}$ has dimension three.*

## References

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