The semilinear Feng and FMV spectra

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Abstract. In this paper we describe and compare two semilinear spectra defined for a pair (L,F), where L is a linear Fredholm operator of index zero and F is a continuous nonlinear operator. These spectra have been introduced in [8] and are useful in applications to boundary value problems for both ordinary and partial differential equations.

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1 Introduction

The last 30 years have presented an opportunity to study several spectra for nonlinear operators which are modelled on familiar spectra defined for bounded linear operators between Banach spaces.

In 1978, a nonlinear spectrum of a continuous operator F in a Banach space X was introduced by Furi, Martelli and Vignoli [10]. This spectrum is based on solvability properties of the operator equation F(x)=G(x) in X, where G is a compact operator. In a similar way, Feng [8] defined another spectrum, for G satisfying boundary conditions on spheres. The FMV-spectrum is one of the most useful nonlinear spectrum from the point of view of applications with topological character by means of the stable solvability. In addition it may be disjoint from the eigenvalues. The notion of F-regularity may be used to define another spectrum in rather the same way as the definition of FMV-spectrum by means of FMV-regularity. The FMV theory was successful until 1997 when Feng [8] introduced a new spectrum with other concepts of solvability and characteristics, based on the class of k-epi maps, which is closed, bounded, upper semicontinuous and contains all the eigenvalues, as in the linear case.

Another contribution was made in 1990 by J. Mawhin [10], using the theory of coincidence degree. The classical Leray-Schauder degree was replaced by the coincidence degree, suitable to boundary value problems. The semilinear versions of the Feng and FMV spectra take into account maps of the form L - F, where L is a linear (not necessarily invertible) operator and F is nonlinear.The semilinear Feng spectrum, denoted

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by $\sigma_F(L,F)$ has been introduced by Feng and Webb [9], for a semilinear pair (L,F), where L is a linear densely defined Fredholm operator of index zero and F is a continuous nonlinear operator with some additional requirements. This situation arises in applications to differential equations for the class of (L,k)-epi maps. When $L=I$, this class reduces to the class of k-epi maps and the semilinear spectrum reduces to the usual Feng spectrum [8]. The semilinear FMV-spectrum, denoted by $\sigma_{FMV}(L,F)$ imitates the Feng-Webb construction for the FMV- spectrum of an auxiliary map, but simpler than the map considered by Feng and Webb [9].

2 Characteristics of nonlinear operators

Let F be a continuous operator between two Banach spaces X and Y over $\Bbbk$. We recall a useful topological characteristic in the theory and applications of both linear and nonlinear analysis and some numerical characteristics for nonlinear operators to describe mapping properties, such as compactness, Lipschitz continuity or quasi-boundedness. The measure of noncompactness of a bounded subset M of X is defined by:

$$\alpha(M)=\inf\{\varepsilon:\varepsilon>0, M \text{ has a finite } \varepsilon\text{-net in } X\}, \tag{2.1}$$

where by a finite ε-net for M we understand a finite set $\{x_1,...,x_n\}\subset X$ with the property that $M\subseteq[x_1+B_\varepsilon(x)]\cup...\cup[x_n+B_\varepsilon(x)]$, for the closed ball with centre θ and radius $\varepsilon>0$ in X.

Given $F\in C(X,Y)$, the set of all continuous operators from X into Y, we will use the following notations (see [3]) :

$$[F]_{Lip}=\sup_{x\neq y}\frac{\|F(x)-F(y)\|}{\|x-y\|} \text{ and } [F]_{lip}=\inf_{x\neq y}\frac{\|F(x)-F(y)\|}{\|x-y\|} \tag{2.2}$$

$$[F]_Q=\limsup_{\|x\|\to\infty}\frac{\|F(x)\|}{\|x\|} \text{ and } [F]_q=\liminf_{\|x\|\to\infty}\frac{\|F(x)\|}{\|x\|} \tag{2.3}$$

$$[F]_B=\sup_{x\neq\theta}\frac{\|F(x)\|}{\|x\|} \text{ and } [F]_b=\inf_{x\neq\theta}\frac{\|F(x)\|}{\|x\|} \tag{2.4}$$

meaning that F is a Lipschitz continuous operator in the case of (2.2), quasi-bounded in the case of (2.3), and linear bounded in the case of (2.4).

Let X and Y be two infinite dimensional Banach spaces. Recall that a continuous operator $F:X\to Y$ is $\alpha-$Lipschitz if there exists $k>0$ such that $\alpha(F(M))\leq k\alpha(M)$, for any bounded subset $M\subset X$.

We set

$$[F]_A = \inf\{k : k > 0, \alpha(F(M)) \leq k\alpha(M)\} \tag{2.5}$$

and we say that $[F]_A$ is the *measure of noncompactness* or the α-*norm* of F. For the reverse condition, let

$$[F]_a = \sup\{k : k > 0, \alpha(F(M)) \geq k\alpha(M)\} \tag{2.6}$$

The equivalent representation, as in the linear case, are useful in infinite dimensional spaces :

$$[F]_A = \sup_{\alpha(M)>0} \frac{\alpha(F(M))}{\alpha(M)} \text{ and } [F]_a = \inf_{\alpha(M)>0} \frac{\alpha(F(M))}{\alpha(M)} \tag{2.7}$$

We define several subsets of $\Bbbk$ by means of the lower characteristics $[F]_{lip}$, $[F]_q$, $[F]_b$ and $[F]_a$:

$$\begin{cases} \sigma_{lip}(F) = \{\lambda \in \Bbbk : [\lambda I - F]_{lip} = 0\} \\ \sigma_q(F) = \{\lambda \in \Bbbk : [\lambda I - F]_q = 0\} \\ \sigma_b(F) = \{\lambda \in \Bbbk : [\lambda I - F]_b = 0\} \\ \sigma_a(F) = \{\lambda \in \Bbbk : [\lambda I - F]_a = 0\} \end{cases} \tag{2.8}$$

3 Some definitions involving nonlinear operators

Definition 3.1 ([8]). A map $F : X \to Y$ is said to be *stably solvable* if given any compact map $G : X \to Y$ with zero quasinorm, there exists at least one element X of X such that $F(x) = G(x)$.

Definition 3.2 ([10]): A map F is said to be *FMV-regular* if it is stably solvable and $[F]_q$ and $[F]_a$ are both positive.

Let be $\rho_{FMV}(F) = \{\lambda \in \mathbb{C} : \lambda I - F \text{ is } FMV - regular\}$ the *FMV-resolvent* set of F and its complement $\sigma_{FMV}(F) = \mathbb{C} \setminus \rho_{FMV}(F)$ the *FMV spectrum* of F.

Let X and Y be Banach spaces over $\Bbbk = \mathbb{R}$ or $\Bbbk = \mathbb{C}$ and let Ω be an open, bounded, connected subset of X with $\theta \in \Omega$.

Definition 3.3. A continuous operator $F : \overline{\Omega} \to Y$ is called *epi operator* on $\overline{\Omega}$ if $F(x) \neq \theta$ on $\partial\Omega$ and for any compact operator $G : \overline{\Omega} \to Y$ satisfying $G(x) \equiv \theta$ on $\partial\Omega$, the equation $F(x) = G(x)$ has a solution $x \in \Omega$.More generaly, we call F a *k-epi operator* on $\overline{\Omega}$ with $k \geq 0$ if the property mentioned before holds for all operators with $[G]_A \leq k$, (not only for compact operators).

For $F : \overline{\Omega} \to Y$ *and* $\Omega \in F(x)$, the family of all open, bounded and connected subsets Ω of X with $\theta \in \Omega$, we introduce :

$$v_{\Omega}(F) = \inf\left\{k : k > 0, F \textit{ is not } k - epi \textit{ on } \overline{\Omega}\right\} \tag{3.1}$$

$$v(F) = \inf_{\Omega\in F(x)} v_{\Omega}(F), \tag{3.2}$$

where $v(F)$ represents *the measure of solvability of F.*

The epi and k-epi operators were introduced by Furi, Martelli and Vignoli [10] and then the concept was generalized to (p, k)-epi mappings. The (p, k)-epi mappings have analogue properties with the properties of the topological degree, more precisely, with the homotopy property and the boundary dependence property, see [15]. The homotopy property gives a continuation principle for epi and k-epi operators and may be compared with its similar results in the topological degree theory.

Definition 3.4. A continuous operator $F : \overline{\Omega} \to Y$ is said to be *p-admissible* if $F(x) \neq p$ for $p \in Y$ and $x \in \partial\Omega$. A 0-admissible operator $F : \overline{\Omega} \to Y$ is said to be $(0, k) - epi$ if for each k-set contraction $G : \overline{\Omega} \to Y$ with $G(x) \equiv 0$ on $\partial\Omega$ the equation $F(x) = G(x)$ has a solution in Ω. A p-admissible mapping $F : \overline{\Omega} \to Y$ is said to be $(p, k) - epi$if the map $F - p$defined by $(F - p)(x) = F(x) - p$, $x \in \overline{\Omega}$, is $(0, k) - epi$.

Definition 3.5. (see [8]): Let be $F : X \to X$ continuous. F is said to be *regular* if $[F]_a > 0$, $[F]_b > 0$ and F is epi on $\overline{\Omega}$ for all open subsets Ω of X.

If $\lambda I - F$ is regular, for $\lambda \in \mathbb{C}$, then λ is in the resolvent set of F, denoted by $\rho_F(F)$ and the spectrum of F is defined by

$$\sigma_F(F) = \{\lambda \in \mathbb{C} : \lambda I - F \text{ is not regular}\} = \mathbb{C}\backslash\rho(F). \tag{3.3}$$

Proposition 1. 3.1. *If F is a regular map, then F is surjective.*

4 The semilinear Feng spectrum

Extending the theory of the Feng spectrum to a semilinear pair (L,F), where L is a linear Fredholm operator of index zero and F is a continuous nonlinear operator, we will obtain the Feng semilinear spectrum, denoted by $\sigma_F(L, F)$ in such a way that for $L = $ I we get the usual Feng spectrum $\sigma_F(F)$ defined by (3.3).We adopt the terminology from [13].

Let X and Y be two Banach spaces and $L : D(L) \to Y$, with $\overline{D(L)} = X$, a closed linear Fredholm operator of index zero and let $F : X \to Y$ be a continuous nonlinear operator. The spaces X and Y admit the decompositions $X = N(L) \oplus X_0$ and $Y = Y_0 \oplus R(L)$. When L is of index zero, the subspaces $N(L)$ and Y_0 have the same finite dimension. Denote by $P : X \to N(L)$ and $Q : Y \to Y_0$ the corresponding projections. The restriction L_P of L to $D(L) \cap X_0$ into $R(L)$ is invertible and denote $K_{PQ} := L_P^{-1}(I - Q) : Y \to X_0$. Let $\Pi : Y \to Y/R(L)$ be the quotient mapand

$\Lambda : Y/R\,(L) \rightarrow N\,(L)$ the natural linear isomorphism induced by L.Let $F : X \rightarrow Y$ be a continuous nonlinear operator and we are interested in the solution of the semilinear equation $Lu - Fu = 0$.

Definition 4.1. An operator $F : X \rightarrow Y$ is said to be $(L,\alpha) - \mathit{Lipschitz}$ if $[K_{PQ}]_A < \infty$,$L - \mathit{compact}$ if $[K_{PQ}]_A = 0$ and $(L,\alpha) - \mathit{contractive}$ if $[K_{PQ}]_A < 1$.

Let be $F_L(x)$ the family of all open, bounded and connected subsets Ω of X with the property that $\Omega_L = \Omega \cap D\,(L) \neq \varnothing$. An operator $F : \bar{\Omega} \rightarrow Y$ is called $(L.k) - \mathit{epi}$ on $\bar{\Omega}_L$if $F(x) \neq Lx$ on $\partial\Omega_L$ and for any operator $G : \bar{\Omega} \rightarrow Y$ satisfying $[K_{PQ}G]_A \leq k$ and $G(x) \equiv \theta$ on $\partial\Omega_L$, the equation $Lx - F(x) = G(x)$ has a solution $x \in \Omega_L$. If k=0, F is said to be $L - \mathit{epi}$ operator.

For $\lambda \in \Bbbk$ we associate with (L,F) an operator $\phi_\lambda(L,F) : X \rightarrow X$ defined by :

$$\phi_\lambda(L,F)(x) = \lambda(I-P)x - (\Lambda Q + K_{PQ})\,F(x) \tag{4.1}$$

For $\lambda, \mu \in \Bbbk$ we easily derive the identity

$$\phi_\lambda(L,F) - \phi_\mu(L,F) = (\lambda-\mu)\,(I-P) \tag{4.2}$$

Lemma 4.1 (**[12]**)**.** *Then $L + \Lambda^{-1}P : D\,(L) \rightarrow Y$ is a linear isomorphism with the inverse :*

$$\left(L + \Lambda^{-1}P\right)^{-1} = \Lambda Q + K_{PQ} \tag{4.3}$$

This lemma is illustrated by a simple example involving the theory of periodic boundary value problems for ordinary differential equations.

For fixed $\omega > 0$, denote by $C_\omega = C_\omega\,(\mathbb{R})$ the space of all continuous $\omega-$periodic functions $x : \mathbb{R} \rightarrow \mathbb{R}^n$ with the natural norm $\|x\|_{C_\omega} = \max\limits_{0\leq t\leq\omega} |x\,(t)|$ and by $C^1_\omega = C^1_\omega\,(\mathbb{R})$ the space of all continuously differentiable $\omega-$periodic functions $x : \mathbb{R} \rightarrow \mathbb{R}$ with the norm $\|x\|_{C^1_\omega} = \max\limits_{0\leq t\leq\omega} |x\,(t)| + \max\limits_{0\leq t\leq\omega} \left|x'\,(t)\right|$. Moreover, $\hat{C}_\omega = \hat{C}_\omega\,(\mathbb{R})$ and $\hat{C}{}^1_\omega = \hat{C}{}^1_\omega\,(\mathbb{R})$ are the subspaces of all $x \in C_\omega$ or C^1_ω satisfying the condition

$$Px = \tfrac{1}{\omega}\textstyle\int_0^\omega x\,(t)\;dt = 0. \tag{4.4}$$

This operator P is, in fact, a continuous projection which maps C_ω onto $\mathbb{R}^n$ and induces the decompositions $C_\omega = \hat{C}_\omega \oplus \mathbb{R}^n$ and $C^1_\omega = \hat{C}{}^1_\omega \oplus \mathbb{R}^n$ .Let $X = C^1_\omega,\ Y = C_\omega$ and $L : X \rightarrow Y$ defined by $Lx = x'$. We have $D\,(L) = X,\ N\,(L) = Y_0 = \mathbb{R}^n,\ R\,(L) = \hat{C}_\omega$and $X_0 = \hat{C}{}^1_\omega$, hence $L : X \rightarrow Y$ is a Fredholm operator with

$\dim N(L) = co\dim R(L)$, i.e., of index zero. The projections $P : C^1_\omega \to \mathbb{R}^n$ and $Q : C_\omega \to \mathbb{R}^n$ are given by (4.4). The operator $L_P x = x'$ is a bijection between $\hat{C}{}^1_\omega$ and $\hat{C}_\omega$, the projection Q associates to each $y \in C_\omega$ the class of all function in C_ω while the isomorphism Λ maps every such a class onto this common integral mean. The linear isomorphism $\Lambda\Pi + K_{PQ} : C_\omega \to C^1_\omega$ is represented by :

$$(\Lambda\Pi + K_{PQ})\, y(t) = \int_0^t y(s)\ ds - \frac{t}{\omega}\int_0^\omega y(s)\ ds + \frac{1}{\omega}\int_0^\omega \left(1 - \frac{\omega}{2} + s\right) y(s)\ ds$$

and its inverse $L + h\Lambda^{-1}P : C^1_\omega \to C_\omega$ has the form :

$$\left(L + h\Lambda^{-1}P\right) x(t) = x'(t) + \frac{1}{\omega}\int_0^\omega x(s)\ ds.$$

Usually, the nonlinear operator F is a Nemytskij operator $F(x)(t) = f(t,\ x(t))$, generated by a Carathéodory function $f : [0,\omega] \times \mathbb{R}^n \to \mathbb{R}^n$, (see [14], appendix C). For any $\lambda \in \mathbb{R}$, the operator $\phi_\lambda(L,F) : C_\omega \to C_\omega$ has the form :

$\phi_\lambda(L,F)(x)(t) = \lambda x(t) - \frac{\lambda}{\omega}\int_0^\omega x(s)\ ds - \int_0^t F(x)(s)\ ds + \frac{t}{\omega}\int_0^\omega F(x)(s)\ ds - \frac{1}{\omega}\int_0^\omega \left(1 - \frac{\omega}{2} + s\right) F(x)(s)\ ds$

The above lemma shows that the operator equation, from a space to itself

$$\phi_\lambda(L,F)(x) = y \tag{4.5}$$

is equivalent to the operator equation :

$$\lambda Lx - F(x) = z \tag{4.6}$$

which acts between X and Y, where $z = \left(L + h\Lambda^{-1}P\right) y$.We take into account of the:

Remark 4.1 ([12]). The auxiliary operator (4.1) is independent of P and Q if L is a linear bounded operator.

Let us introduce now the following subspectra :

$$\sigma_v(L,F) = \{\lambda \in \mathbb{k} : v(\phi_\lambda(L,F)) = 0\} \tag{4.7}$$

$$\sigma_a(L,F) = \{\lambda \in \mathbb{k} : [\phi_\lambda(L,F)]_a = 0\} \tag{4.8}$$

$$\sigma_b(L,F) = \{\lambda \in \mathbb{k} : [\phi_\lambda(L,F)]_b = 0\}. \tag{4.9}$$

Then, the semilinear Feng spectrum admits the decomposition :

$$\sigma_F(L,F) = \sigma_v(L,F) \cup \sigma_a(L,F) \cup \sigma_b(L,F) \tag{4.10}$$

For L=I we get $\phi_\lambda(I,F) = \lambda I - F$ so $\sigma_F(I,F) = \sigma_F(F)$ and for F=I we have :

$$\phi_\lambda(L,I) = \lambda(I-P) - \Lambda Q - K_{PQ} = (\Lambda Q + K_{PQ})(\lambda L - I),$$

so

$$\sigma_F(L,I) = \left\{ \frac{1}{\lambda} : \lambda \in \sigma(L) \setminus \{0\} \right\}$$

consists of the nonzero characteristic values of the linear operator L.

Assuming that $L \neq \Theta$ and the projection $P : X \to N(L)$ satisfies the property that $\|I-P\| \neq 0$, we have then the following theorem.

Theorem 4.1. *The spectrum $\sigma_F(L,F)$ is closed.*

Proof. Fix $\lambda \in \mathbb{k} \setminus \sigma_F(L,F)$ i.e. $[\phi_\lambda(L,F)]_a > 0$, $[\phi_\lambda(L,F)]_b > 0$ and $v(\phi_\lambda(L,F)) > 0$. We choose $\mu \in \mathbb{k}$ such that :

$$|\lambda - \mu| < \min\left\{ [\phi_\lambda(L,F)]_a,\ v(\phi_\lambda(L,F)),\ \tfrac{[\phi_\lambda(L,F)]_b}{\|I-P\|} \right\}.$$

We get from (4.2) that

$[\phi_\mu(L,F)]_a \geq [\phi_\lambda(L,F)]_a - [(\mu-\lambda)(I-P)]_A = [\phi_\lambda(L,F)]_a - |\mu-\lambda| > 0$
and

$$\|\phi_\mu(L,F)(x)\| \geq \|\phi_\lambda(L,F)(x)\| - |\mu-\lambda|\,\|(I-P)x\| \geq$$
$$\geq ([\phi_\lambda(L,F)]_b - |\mu-\lambda|\,\|I-P\|)\,\|x\|.$$

For $[\phi_\mu(L,F)]_b > 0$, we apply an homotopy argument with $H(x,t) = t(\mu-\lambda)(I-P)x$ and $F_0 = \phi_\lambda(L,F)$. Then, $[H]_A \leq |\mu-\lambda| < v(\phi_\lambda(L,F))$ and $H(x,0) \equiv \theta$ for $x \in X$. If $\phi_\lambda(L,F)(x) + t(\mu-\lambda)(I-P)x = \theta$
for some $t \in [0,1]$, then :

$[\phi_\lambda(L,F)]_b \|x\| \leq \|\phi_\lambda(L,F)(x)\| \leq |\mu-\lambda|\,\|I-P\|\,\|x\|,$

hence $x = \theta$,a contradiction. As F_0 is k-epi on $\bar{\bar{\Omega}}$ for $\Omega \in F(x)$ and $k > 0$, so using again the homotopy property we get that $F_1 = F_0 + H(\cdot,1) = \phi_\lambda(L,F) + (\mu-\lambda)(I-P) = \phi_\mu(L,F)$

is $(k - [H]_A)-epi$. Since $v(\phi_\mu(L,F)) > 0$ it follows that $\mu \notin \sigma_F(L,F)$. Therefore $\mathbb{k} \setminus \sigma_F(L,F)$ is open and so $\sigma_F(L,F)$ is closed.

Definition 4.2. A scalar $\lambda \in \mathbb{k}$ is called an*eigenvalue of the pair (L,F)* if the equation $F(x) = \lambda Lx$ has a nontrivial solution $x \in X$. The set of all eigenvalues

$$\sigma_P(L,F) = \{\lambda \in \mathbb{k} : F(x) = \lambda Lx \ for\ some\ x \neq 0\}$$

is called *the point spectrum of the pair (L,F).*

Theorem 4.2. *Let $F : X \to Y$ be L-compact, 1-homogeneous and odd. Then every $\lambda \in \sigma_F(L, F) \setminus \{0\}$ is an eigenvalue for the pair (L,F).*

Proof . It shows that every nonzero $\lambda \in \sigma_F(L, F)$ belongs to $\sigma_b(L, F)$. We suppose that $[\phi_\lambda(L, F)]_b > 0$; hence $\|\phi_\lambda(L, F)(x)\| \geq [\phi_\lambda(L, F)]_b \|x\| > 0$, for all $x \in X$ with $x \neq \theta$. Fix $\Omega \in F(X)$ and let $G : \overline{\Omega} \to X$ be compact with $G(x) \equiv \theta$ on $\partial\Omega$. We show that the equation $\phi_\lambda(L, F)(x) = G(x)$ is solvable in Ω, so $\phi_\lambda(L, F)$ is epi on $\overline{\Omega}$. Let be $H : \overline{\Omega} \to X$ defined by :

$$H(x) = Px + \tfrac{1}{\lambda} (\Lambda\Pi + K_{PQ}) F(x) + \tfrac{1}{\lambda} G(x).$$

Clearly, H is compact and $x - H(x) = \frac{1}{\lambda}\phi_\lambda(L, F)(x) \neq \theta$ on $\partial\Omega$. The restriction $H \mid_{\partial\Omega}$is odd. Applying the Borsuk theorem it follows that the degree $\deg(I - H, \Omega, \theta) \neq 0$, so there exists $\hat{x} \in \Omega$ such that :

$$\hat{x} = P\hat{x} + \tfrac{1}{\lambda} (\Lambda\Pi + K_{PQ}) F\left(\hat{x}\right) + \tfrac{1}{\lambda} G\left(\hat{x}\right)$$

and so $\phi_\lambda(L, F)$ is epi on $\overline{\Omega}$ and $\lambda \notin \sigma_F(L, F)$, contradicting the hypothesis.

The relation $[\phi_\lambda(L, F)]_b = 0$ is equivalent with the fact there exists a sequence $\{x_n\}$ in X such that

$$\left\|\lambda(I - P)x_n - \left(L + h\Lambda^{-1}P\right)^{-1} F(x_n)\right\| \leq \tfrac{1}{n} \|x_n\| .$$

We put $e_{n:=\frac{x_n}{\|x_n\|}}$. By the homogeneity of F we have

$$\left\|\lambda(I - P)e_n - \left(L + h\Lambda^{-1}P\right)^{-1} F(e_n)\right\| \to 0,\ n \to \infty.$$

The set $M = \{e_1, e_2, ...\}$ satisfies $[\phi_\lambda(L, F)]_a \alpha(M) \leq \alpha(\phi_\lambda(L, F)(M)) = 0$

which proves that the sequence $\{e_n\}$ admits a convergent subsequence $\{e_{n_k}\}$ with $e_{n_k} \to e$. By continuity, we have $\lambda(I - P)e = \left(L + h\Lambda^{-1}P\right)^{-1} F(e)$. Since $\left(L + h\Lambda^{-1}P\right)(I - P) = L$ it follows that $\lambda Le = F(e)$, i.e., $\lambda \in \sigma_P(L, F)$.

5 The semilinear Furi-Martelli-Vignoli spectrum

Let X and Y be two Banach spaces, $L : X \to Y$ a closed linear Fredholm operator of index zero and $F : X \to Y$ be a continuous nonlinear operator. We have again the decompositions $X = N(L) \oplus X_0$ and $Y = Y_0 \oplus R(L)$, where $N(L)$ and Y_0 have the same finite dimension. We denote by $P : X \to N(L)$ the projection on the nullspace of L and by $h\Lambda^{-1} : N(L) \to Y_0$ a fixed isomorphism. We have the same hypothesis of work, but in addition, we suppose that L is bounded. This assumption is not restrictive because every closed linear operator becomes bounded after a suitable renorming of X. We use the relation (4.1) and, for $\lambda \in \mathbb{k}$, we define $\phi_\lambda(L, F) : X \to X$ by :

$$\phi_\lambda(L,F)(x) = \lambda\,(I-P)\,x - \left(L + h\Lambda^{-1}P\right)^{-1} F(x).$$

We use the following subspectra :

$$\sigma_q(L,F) = \left\{\lambda \in \Bbbk : [\phi_\lambda(L,F)]_q = 0\right\} \tag{5.1}$$

$$\sigma_\delta(L,F) = \{\lambda \in \Bbbk : \phi_\lambda(L,F) \textit{ is notstably solvable}\} \tag{5.2}$$

The semilinear FMV-spectrum admits the following decomposition :

$$\sigma_{FMV}(L,F) = \sigma_q\,(L,F) \cup \sigma_a(L,F) \cup \sigma_\delta\,(L,F) \tag{5.3}$$

where $\sigma_a(L,F)$ is given by (4.8).

For L=I we get $L + h\Lambda^{-1}P = I$ and $\phi_\lambda\,(I,F) = \lambda I - F$, hence $\sigma_{FMV}\,(I,F) = \sigma_{FMV}\,(F)$. Choosing F=I we get again the set of all nonzero characteristic values of L.

Both semilinear spectra have a property in common and this fact results from the next theorem.

Theorem 5.1. *The spectrum $\sigma_{FMV}(L,F)$ is closed.*

Proof . Fix $\lambda \in \Bbbk \setminus \sigma_{FMV}\,(L,F)$ and let $0 < \delta < \frac{\min\left\{[\phi_\lambda(L,F)]_a,\ [\phi_\lambda(L,F)]_q\right\}}{\|I-P\|}$.

For any μ which satisfies $|\mu-\lambda| < \delta$ we have $\mu \in \Bbbk \setminus \sigma_{FMV}(L,F)$. From (4.2) we get :

$[\phi_\mu(L,F)]_a \geq [\phi_\lambda(L,F)]_a - |\mu-\lambda| > 0$ and

$[\phi_\mu(L,F)]_q \geq [\phi_\lambda(L,F)]_q - |\mu-\lambda| > 0$.

It follows that $\phi_\mu(L,F)$ is stably solvable for $|\mu-\lambda| < \delta$, since :

$\max\left\{[\phi_\mu(L,F) - \phi_\lambda\,(L,F)]_A\ ,\ [\phi_\mu(L,F) - \phi_\lambda(L,F)]_Q\right\} \leq$

$\leq |\mu-\lambda|\,\|I-P\| < \min\left\{[\phi_\mu(L,F)]_a\ ,\ [\phi_\lambda(L,F)]_q\right\}.$

So, λ is an interior point of $\Bbbk \setminus \sigma_{FMV}(L,F)$ and $\Bbbk \setminus \sigma_{FMV}\,(L,F)$ is open.

Theorem 5.2. *Let F be L-compact and odd. Then the following inclusion holds*: $\sigma_{FMV}(L,F) \setminus \{0\} \subseteq \sigma_q(L,F)$

This theorem shows that every nonzero spectral point is an asymptotic eigenvalue for the semilinear pair (L,F).

Proof . Suppose that $\lambda \neq 0,\ \lambda \notin \sigma_q(L,F)$ i.e. $[\phi_\lambda(L,F)]_q > 0$. Since F is compact, we have $[\phi_\mu\,(L,F)]_a = 0$ if and only if $\lambda = 0$, so $\lambda \notin \sigma_a(L,F)$. It remains to show that $\lambda \notin \sigma_\delta(L,F)$, more precisely, to prove that $\lambda L - F$ is stably solvable.

Let $G : X \to Y$ be compact with $[G]_Q = 0$. For $\lambda \neq 0$ and $[\phi_\lambda(L,F)]_q > 0$ there exists $R_1 > 0$ and $\delta > 0$ such that :

$$\left\|\lambda^{-1}\left(L+h\Lambda^{-1}P\right)^{-1}(\lambda L-F)(x)\right\|\geq\delta\|x\|$$

with$\|x\|\geq R_1$. We can find $R_2>0$ such that :

$$\left\|\lambda^{-1}\left(L+h\Lambda^{-1}P\right)^{-1}G(x)\right\|\geq\frac{\delta}{2}\|x\|$$

with $\|x\|\geq R_2$. For $\|x\|\geq R=\max\{R_1,R_2\}$ and $0\leq\mu\leq 1$ we have :

$$\left\|\lambda^{-1}\left(L+h\Lambda^{-1}P\right)^{-1}(\lambda L-F-\mu G)(x)\right\|\geq\frac{\delta}{2}\|x\|$$

and$\lambda^{-1}\left(L+h\Lambda^{-1}P\right)^{-1}(\lambda L-F-\mu G)=$

$=(I-P)-\lambda^{-1}\left(L+h\Lambda^{-1}P\right)^{-1}(F+\mu G)$

is a compact perturbation of the identity.

Using again Borsuk's theorem and the homotopy invariance of the Leray-Schauder degree, we get :

$\deg\left(\lambda^{-1}\left(L+h\Lambda^{-1}P\right)^{-1}(\lambda L-F-G),\ B_R^0(x),\ \theta\right)=$

$=\deg\left(\lambda^{-1}\left(L+h\Lambda^{-1}P\right)^{-1}(\lambda L-F),\ B_R^0(x),\ \theta\right)\equiv 1\,(\mathrm{mod}2)\,,$

so the equation $\lambda Lx=F(x)+G(x)$ has a solution in $B_R(x)$.

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