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ON A SYMBOL CLASS OF ELLIPTIC PSEUDODIFFERENTIAL OPERATORS

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A b s t r a c t. We consider a class of symbols with prescribed smoothness and growth conditions and give examples of such symbols. The introduced class contains certain polynomial symbols and symbols with more than polynomial growth in phase space. The corresponding pseudodifferential operators defined as the Weyl transforms of the symbols are elliptic. As an application, we give a result on isomorphisms between modulation spaces. In particular, we show that the Bessel potentials establish such isomorphisms.

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1. *Introduction*

Theory of pseudodifferential operators has been established some thirty years ago, with important applications in diverse fields of theoretical and applied mathematics such as partial differential equations and quantum mechanics [17], [15], [18]. In the last decade it has been successfully applied in

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time-frequency analysis and communication theory [5], [7], [6]. In this context new classes of symbols and the corresponding operators are introduced (see [16], [8], [12]). There are different ways to define a pseudodifferential operator by the means of its symbol. In this paper we consider the so called Weyl correspondence (see (1)). As noted in [7], Feichtinger's modulation spaces introduced in [2] are the most natural framework for time-frequency analysis. Therefore, it is of particular importance to study the action of pseudodifferential operators on modulation spaces. Operators with symbols in modulation spaces are studied in [9], [8] and [12] while in [16] and [1] symbols with at most polynomial growth are considered. However, in quantum field theory it is of interest to study symbols with more than polynomial growth in momentum space in the framework of the corresponding spaces of ultradistributions [11]. A relationship between modulation spaces and ultradistributions is given in [13].

In this paper we define a class of symbols which can grow almost exponentially in phase space. It also contains a large class of polynomials, such as the symbols of the Bessel potentials. In particular, it contains the Schröedinger-type operators with appropriate almost exponentially bounded potentials.

As an application we prove that a class of partial differential operators with constant coefficients and, in particular, the Bessel potentials establish isomorphism between certain modulation spaces.

2. *Notation*

If $x = (x_1, \dots, x_d) \in \mathbf{R}^d$ $d \in \mathbf{N}$, then $|x| = \sqrt{x_1^2 + \dots + x_d^2}$, and $\langle x \rangle = (1 + |x|^2)^{1/2}$. For multi-indices $\alpha, \beta \in \mathbf{N}_0^d$, we have $|\alpha| = \alpha_1 + \dots + \alpha_d$, $\alpha! = \alpha_1! \cdots \alpha_d!$, $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ and, if $\beta \leq \alpha$, i.e., $\beta_j \leq \alpha_j$, $j = 1, 2, \dots, d$, $\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1} \cdot \dots \cdot \binom{\alpha_d}{\beta_d}$. We write $D^\alpha = D_1^{\alpha_1} \cdots D_d^{\alpha_d}$ $= \left(\frac{1}{2\pi i}\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{1}{2\pi i}\frac{\partial}{\partial x_d}\right)^{\alpha_d}$. We denote by C a positive constant, not necessarily the same at every occurrence. The symbol γ is reserved for a real number in $(0,1)$ unless otherwise is indicated. The translation and the modulation of a test function f is given by $T_x f(\cdot) = f(\cdot - x)$, $x \in \mathbf{R}^d$, and $M_\xi f(\cdot) = e^{2\pi i \xi \cdot} f(\cdot)$, $\xi \in \mathbf{R}^d$ respectively, and extended to a distribution via duality. Dual pairing is denoted by $\langle \cdot, \cdot \rangle$. For functions $\varphi, \psi \in \mathcal{S}$ ($\mathcal{S}$ is the space of rapidly decreasing functions), $\langle \varphi, \psi \rangle = \int \varphi \psi dx$. The Fourier transform of $\psi \in L^2(\mathbf{R}^d)$ is given by $\mathcal{F}\psi(\xi) = \hat{\psi}(\xi) = \int_{\mathbf{R}^d} e^{-2\pi i x \xi} \psi(x) dx$, and

$\mathcal{F}^{-1}\phi(x) = \int_{\mathbf{R}^d} e^{2\pi i x\xi}\phi(\xi)d\xi$, is the inverse Fourier transform of $\phi \in L^2(\mathbf{R}^d)$. We denote the norm in L^2 by $\|\cdot\|$, and $\|\cdot\|_\infty$ denotes L^∞ norm. Recall, Gelfand Shilov type space $\mathcal{S}^{(\gamma)}$ is defined by $\mathcal{S}^{(\gamma)} = \operatorname{proj}\lim\limits_{h\to\infty} \mathcal{S}_h^{(\gamma)}$, where $\mathcal{S}_h^{(\gamma)}$, $h \geq 0$, is the space of smooth functions f on $\mathbf{R}^d$ such that

$$\sup_{\alpha,\beta\in\mathbf{N}_0^d} \frac{h^{\alpha+\beta}}{\alpha!^{1/\gamma}\beta!^{1/\gamma}}\|x^\alpha D^\beta f(x)\|_\infty < \infty.$$

It is a Banach space and the Fourier transform is an isomorphism of $\mathcal{S}^{(\gamma)}$ into itself. For fixed $\gamma \in (0,1)$, the space $\mathcal{D}^{(\gamma)}(\Omega)$ is defined by $\mathcal{D}^{(\gamma)}(\Omega) = \operatorname{ind}\lim\limits_{K\subset\subset\Omega} \mathcal{D}^{(\gamma)}(K)$, where Ω is an open subset in $\mathbf{R}^d$ and $\mathcal{D}^{(\gamma)}(K)$ is the set of all complex valued infinitely differentiable functions $\varphi(t)$ supported by K such that for every $h > 0$ there exists a positive constant $C > 0$ such that

$$\sup_{t\in K}|D^\alpha\varphi(t)| \leq Ch^\alpha\alpha!^{1/\gamma}, \quad \alpha \in \mathbf{N}_0^d.$$

We call $\mathcal{D}'^{(\gamma)}(\Omega)$ the Beurling–Gevrey ultradistribution space and $\mathcal{S}'^{(\gamma)}$ the Beurling–Gevrey tempered ultradistribution space.

We observe pseudodifferential operators $\sigma(x,D)$ as the Weyl transforms of symbols $\sigma(x,\xi)$, i.e.

$$\sigma(x,D)f(x) = \int\limits_{\mathbf{R}^d}\int\limits_{\mathbf{R}^d} \sigma\left(\frac{x+y}{2},\xi\right)e^{2\pi i(x-y)\xi}f(y)dyd\xi, \quad f \in \mathcal{S}^{(\gamma)}(\mathbf{R}^d). \quad (1)$$

3. *A Class of Symbols*

Throughout this section $\gamma \in (0,1)$ is fixed. We consider a class of symbols $\sigma \in C^\infty(\mathbf{R}^d \times \mathbf{R}^d)$ satisfying:

(S1) $\sigma(z) \geq 1$, $z = (x,\xi) \in \mathbf{R}^d \times \mathbf{R}^d$.

(S2) $(\exists C > 0)$ $(\exists \eta \geq 0)$ such that $\sigma(z+w) \leq Ce^{\eta|z|^\gamma}\sigma(w), \quad z,w \in \mathbf{R}^{2d}$.

(S3) $(\forall h \geq 0)$ $(\exists C > 0)$ $(\exists s \geq 0)$ such that

$$\sup_{\alpha\in\mathbf{N}_0^{2d},|\alpha|\geq 1}\left|\frac{h^{|\alpha|}}{\alpha!^{1/\gamma}}D^\alpha\sigma(z)\right| \leq C\frac{\sigma(z)}{(1+|z|)^s}, \quad z \in \mathbf{R}^{2d}.$$

(S4) $\sigma(x,\xi) \leq \sigma(x,\xi')$ for all $\xi,\xi' \in \mathbf{R}^d$ such that $|\xi| \leq |\xi'|$.

Note that instead (S1) we could consider the condition $\sigma(z) \geq C$, $z \in \mathbf{R}^{2d}$, for some $C > 0$. We put $C = 1$ for the sake of simplicity. Also, (S2) implies $\sigma(z) \geq \dfrac{\sigma(0)}{C} e^{-\eta|z|^\gamma}$, $z \in \mathbf{R}^{2d}$.

If σ_1, σ_2 satisfy (S1)-(S4), then it is clear that $\sigma_1 \cdot \sigma_2$ satisfies (S1)-(S4) as well.

Theorem 1. *The following functions satisfy conditions (S1)-(S4).*

a) $\sigma(z) = \sum_{k=0}^{n} a_k \langle z\rangle^{2k}$, $z = (x,\xi) \in \mathbf{R}^d \times \mathbf{R}^d$, *where* $a_0 \geq 1$, $a_k > 0$;

b) $\sigma(x,\xi) = (1 + |x|^2 + |\xi|^2)^{s/2}$, $s \geq 0$, $x,\xi \in \mathbf{R}^d$. *In particular,* $\sigma(\xi) = (1 + |\xi|^2)^{s/2}$, $s \geq 0$, $\xi \in \mathbf{R}^d$;

c) $\sigma(x,\xi) = |\xi|^2 + V(x)$, $x,\xi \in \mathbf{R}^d$, *where*

(V1) $V \in C^\infty(\mathbf{R}^d)$, $V \geq 1$, $V(x) \to \infty$ *when* $|x| \to \infty$.

(V2) $(\exists C > 0)$ $(\exists \eta > 0)$ *such that* $V(x+y) \leq Ce^{\eta|x|^\gamma} V(y)$, $x,y \in \mathbf{R}^d$.

(V3) $(\forall h \geq 0)$ $(\exists C > 0)$ *such that*

$$\sup_{\alpha \in \mathbf{N}_0^d, |\alpha| \geq 1} \left| \frac{h^{|\alpha|}}{\alpha!^{1/\gamma}} D^\alpha V(x) \right| \leq CV(x), \quad x \in \mathbf{R}^d;$$

d) $\sigma(x,\xi) = e^{(1+|x|^2+|\xi|^2)^{\gamma/2}}$, $x,\xi \in \mathbf{R}^d$;

e) $\sigma(x,\xi) = (f * \phi)(x,\xi)$, $x,\xi \in \mathbf{R}^d$, *where*

$$f(x,\xi) = \begin{cases} e^{\mu(|x|^\gamma + |\xi|^\gamma)} & |\xi| \geq |\xi_0| > 1 \\ e^{\mu|x|^\gamma + r|\xi|} & |\xi| < |\xi_0|, \end{cases}$$

where $\mu > 0$, $r = \mu|\xi_0|^{\gamma-1}$, *and* $\phi \in \mathcal{D}^{(\gamma)}(\Omega)$, $\phi \geq 0$, $\int_{\mathbf{R}^{2d}} \phi(x,\xi)dxd\xi = 1$;

f) $\sigma(x,\xi) = C - \varphi(x,\xi)$, $x,\xi \in \mathbf{R}^d$, *where* $\varphi \in \mathcal{S}^{(\gamma)}(\Omega)$ *is an even function such that* $\varphi(x,\xi) \leq \varphi(x,\xi')$ *if* $|\xi| \leq |\xi'|$, *and* $C > \sup_{x,\xi} \varphi(x,\xi) + 1$.

Remark: The function $\sigma(z) = \sum_{k=0}^{n} a_k z^k$, $z = (x, \xi) \in \mathbf{R}^d \times \mathbf{R}^d$, $a_0 \geq 1$, $a_k > 0$ satisfies the conditions (S1), (S2), (S3).

P r o o f. It is not difficult to prove a), b) and f), so we skip these parts of the proof.

Proof of c) easily follows from the assumptions (V1), (V2) and (V3). Note that $|\xi|^2 + V(x)$ is the symbol of the Schrödinger operator $-\triangle + V$, with the increasing potential V (see also [16] in this context).

d) We show that σ satisfies (S3). To that end we use the Cauchy integral formula [10, Chapter 2] for polydisc $K \subset \mathbf{C}^n$ given by

$$K = \prod_{j=1}^{n} K_j = \Big\{ z = (z_1, \dots, z_n) \mid z_j \in K_j,\ j = 1, \dots, n \Big\},$$

where K_j are discs in $\mathbf{C}$ with the boundaries ∂K_j, $j = 1, \dots, n$. We have

$$\partial^\alpha \sigma(z) = \frac{\alpha!}{(2\pi i)^{2n}} \tag{2}$$

$$\times \int_{\partial K_1} \cdots \int_{\partial K_{2n}} \frac{\sigma(y, \eta) dy d\eta}{\prod_{j=1}^{n} (y_j - x_j)^{\alpha_j + 1} \cdot \prod_{j=1}^{n} (\eta_j - \xi_j)^{\alpha_{n+j}+1}},$$

where $z = (x, y) \in \mathbf{C}^{2n}$ and ∂K_j is the circle with center z_j and radius $r < 1/\sqrt{2n}$. Let $r = 1/(2n)$. From (2) it follows

$$\frac{|\partial^\alpha \sigma(z)|}{\alpha!} \leq \frac{1}{(2\pi i)^{2n} (1/(2n))^{|\alpha|+1}}$$

$$\times (2\pi i)^{2n} \max_{\theta_j \in [0, 2\pi], j=1,\dots,2n} e^{(1 + |x_1 + \frac{1}{2n} e^{\theta_1 i}|^2 + \cdots + |\xi_n + \frac{1}{2n} e^{\theta_{2n} i}|^2)^{\gamma/2}}$$

$$\leq \frac{1}{(1/(2n))^{|\alpha|+1}} e^{(1 + |x|^2 + |\xi|^2 + \frac{1}{n})^{\gamma/2}} \leq C \frac{1}{(1/(2n))^{|\alpha|+1}} e^{(1 + |x|^2 + |\xi|^2)^{\gamma/2}}.$$

Since $(\alpha!)^\varepsilon > a^{|\alpha|}$ for all $a, \varepsilon > 0$ and $|\alpha|$ large enough, there exists $C > 0$ such that, for every $h \geq 0$,

$$\frac{h^{|\alpha|}}{\alpha!^{1/\gamma}} \leq C \frac{1}{\alpha!}, \quad \text{for} \ \ \gamma \in (0, 1).$$

This implies (S3). The function σ obviously satisfies (S1), (S2) and (S4).

e) Conditions (S1) and (S2) are obviously satisfied. We show (S3) for $|\xi| \geq |\xi_0|$. Assume, for simplicity, that $d = 1$, and $\Omega = [-1,1] \times [-1,1]$. Then we have

$$\left| \frac{h^{|\alpha|+|\beta|}}{\alpha!^{1/\gamma}\beta!^{1/\gamma}} D_x^\alpha D_\xi^\beta \sigma(x,\xi) \right|$$

$$= \left| \frac{h^{|\alpha|+|\beta|}}{\alpha!^{1/\gamma}\beta!^{1/\gamma}} \int_{\mathbf{R}} \int_{\mathbf{R}} \left(e^{\mu(|y|^\gamma + |\eta|^\gamma)} D_x^\alpha D_\xi^\beta \phi(x-y, \xi-\eta) \right) dy d\eta \right|$$

$$\leq C \int_{|x-y|\leq 1} \int_{|\xi-\eta|\leq 1} e^{\mu(|y|^\gamma + |\eta|^\gamma)} dy d\eta \leq C \int_{|y|\leq 1} \int_{|\eta|\leq 1} e^{\mu(|x-y|^\gamma + |\xi-\eta|^\gamma)} dy d\eta$$

$$\leq C e^{\mu((|x|+1)^\gamma + (|\xi|+1)^\gamma)} \leq C \int_{|y|\leq 1} \int_{|\eta|\leq 1} e^{\mu((|x|-y+2)^\gamma + (|\xi|-\eta+2)^\gamma)} \phi(y,\eta) dy d\eta$$

$$\leq C \int_{\mathbf{R}} \int_{\mathbf{R}} e^{\mu(|x-y|^\gamma + |\xi-\eta|^\gamma)} \phi(y,\eta) dy d\eta \leq C\sigma(x,\xi),$$

that is, σ satisfies (S3). We now show (S4) for $|\xi_0| \geq |\xi|$. If $|\xi_0| \geq |\xi| \geq 1$, after an easy computation, we obtain

$$\sigma(x,\xi) = \frac{2}{r} e^{r|\xi|} \int_{\mathbf{R}} e^{\mu(|x-y|)^\gamma} \left(\int_0^1 \sinh(r\eta) \phi(x-y,\eta) d\eta \right) dy.$$

This implies (S4). Let now $|\xi| \leq |\xi'| \leq 1$. If, for example, $-1 \leq \xi' \leq -\xi \leq 0$, then

$$\sigma(x,\xi) = \int\limits_{\mathbf{R}} e^{\mu|x-y|^\gamma} \left(e^{r\xi} \int\limits_{-\xi}^{1} f(x,\eta) d\eta + e^{-r\xi} \int\limits_{\xi}^{1} f(x,\eta) d\eta \right) dy,$$

where $f(x,\eta) = e^{r|\eta|}\phi(x,\eta)$. Hence, for any fixed $x \in \mathbf{R}$, we have

$$\sigma(x,\xi') - \sigma(x,\xi)$$

$$\geq C \left(e^{r\xi'} \int\limits_{-\xi'}^{1} f(x,\eta) d\eta + e^{-r\xi'} \left(\int\limits_{\xi'}^{-\xi} f(x,\eta) d\eta + \int\limits_{-\xi}^{-\xi'} f(x,\eta) d\eta + \int\limits_{-\xi'}^{1} f(x,\eta) d\eta \right) \right.$$

$$\left. - e^{r\xi} \left(\int\limits_{-\xi}^{-\xi'} f(x,\eta) d\eta + \int\limits_{-\xi'}^{1} f(x,\eta) d\eta \right) - e^{-r\xi} \left(\int\limits_{\xi}^{-\xi'} f(x,\eta) d\eta + \int\limits_{-\xi'}^{1} f(x,\eta) d\eta \right) \right)$$

$$= 2(\cosh r\xi' - \cosh r\xi) \int_{-\xi'}^{1} f(x,\eta)d\eta + \left(e^{-r\xi'} - e^{r\xi}\right) \int_{-\xi}^{-\xi'} f(x,\eta)d\eta +$$

$$+ e^{-r\xi'} \int_{\xi'}^{-\xi} f(x,\eta)d\eta - e^{-r\xi} \int_{\xi}^{-\xi'} f(x,\eta)d\eta = I_1 + I_2 + I_3.$$

I_1 and I_2 are obviously nonnegative. After a change of variables we obtain

$$I_3 = e^{-r\xi'} \int_{\xi'}^{-\xi} f(x,\eta)d\eta - e^{-r\xi} \int_{\xi'}^{-\xi} f(x,\eta)d\eta \geq 0.$$

The other cases can be treated in a similar way.

4. *An Application*

In this section we observe the polynomial symbol $P(\xi) = \sum_{|\alpha|\leq s} a_\alpha \xi^\alpha$, $a_\alpha \in \mathbf{R}$, assuming that it satisfies (S1) and (S4). Note that the symbol $P(\xi)$ obviously satisfies (S2) and (S3). In particular, we consider the symbol $\sigma(\xi) = (1 + |\xi|^2)^{s/2} = \langle\xi\rangle^s$, $s \geq 0$, $\xi \in \mathbf{R}^d$ (see Theorem 1 b)). Then the corresponding pseudodifferential operator given by (1) is called the Bessel potential of order s. It is known that the Bessel potentials define isomorphisms between Sobolev spaces [18]. However, the question which operators establish isomorphisms between modulation spaces is still open and important in time-frequency analysis. In this section we prove that the Weyl transforms of $P(\xi)$, and $\langle\xi\rangle^s$, $s \geq 0$, establish isomorphism between certain modulation spaces.

We recall the notion of moderate weight function. A locally integrable function v is called submultiplicative weight if $v(z_1 + z_2) \leq v(z_1)v(z_2)$, $z_1, z_2 \in \mathbf{R}^{2d}$, and a locally integrable function m is moderate weight with respect to a submultiplicative weight v if

$$m(x + y, \xi + \eta) \leq Cv(x, \xi)m(y, \eta), \quad x, y, \xi, \eta \in \mathbf{R}^d.$$

Weights m_1 and m_2 are equivalent if $C_1 m_1 \leq m_2 \leq C_2 m_1$ for some positive constants C_1 and C_2. Every submultiplicative weight is equivalent to a continuous weight.

Any function w which satisfies (S1)-(S4) is moderate with respect to $e^{\eta(|x|^\gamma+|\xi|^\gamma)}$. In particular, $\sigma(\xi)=\langle\xi\rangle^s$, $s\geq 0$, $\xi\in\mathbf{R}^d$, the symbol of the Bessel potential of order s is a moderate weight.

Definition 1. *Let w be a moderate weight, $1\leq p,q<\infty$ and $0\not\equiv g\in\mathcal{S}$. Modulation space $M^{w,t}_{p,q}$ is given by*

$$M^{w,t}_{p,q}=\{f\in\mathcal{S}'\;:\;\|f\|_{M^{w,t}_{p,q}}<\infty\},$$

where

$$\|f\|_{M^{w,t}_{p,q}}=\left[\int_{\mathbf{R}^d}\left(\int_{\mathbf{R}^d}|\langle\overline{T_xM_\xi g},f\rangle|^p w(x,\xi)^p(1+|x|+|\xi|)^{tp}dx\right)^{q/p}d\xi\right]^{1/q}.$$

Modulation spaces are Banach spaces [7, Theorem 11.3.5.] independent of the choice of the analyzing function $0\not\equiv g\in\mathcal{S}$ [2]. It can be shown that $M^{1,t}_{2,2}=H^t_2\cap L^t_2$, where H^t_2 is Sobolev space and L^t_2 is weighted L^2 space with the weight $(1+|x|)^t$ [4]. Therefore $\mathcal{F}(M^{1,t}_{2,2})=M^{1,t}_{2,2}$. Obviously $M^{1,t+\mu}_{2,2}\subset M^{1,t}_{2,2}$, for any $\mu>0$.

Proof of the following theorem can be found in [14], [16].

Theorem 2. *Let $1\leq p,q<\infty$, $t\geq 0$ and let $\sigma(x,D)$ be the Weyl transform of a symbol $\sigma(x,\xi)$ satisfying (S1)-(S4). For every $f\in M^{\sigma,t}_{p,q}$ there exist positive constants C_1,C_2 and C_3 such that*

$$C_1\|f\|_{M^{\sigma,t}_{p,q}}\leq\|\sigma(x,D)f\|_{M^{1,t}_{p,q}}+C_2\|f\|_{M^{1,0}_{p,q}}\leq C_3\|f\|_{M^{\sigma,t}_{p,q}}. \tag{3}$$

If, additionally, $\sigma(z)\geq C\langle z\rangle^\mu$ for $|z|\geq K$, *for some positive constants C,μ and K, and if $\sigma(x,D)f\in M^{1,s}_{2,2}$, then f belongs to $M^{1,s+\mu}_{2,2}$.*

Immediate consequence of Theorem 2 is the continuity of the mapping $\sigma:M^{\sigma,t}_{p,q}\mapsto M^{1,t}_{p,q}$, and $\sigma(M^{\sigma,t}_{p,q})$, the image of $M^{\sigma,t}_{p,q}$ under σ, is a Banach subspace of $M^{1,t}_{p,q}$.

It is an open question to find the conditions under which the operator $\sigma(x,D)$ isomorphically maps $M^{\sigma,t}_{p,q}$ onto $M^{1,t}_{p,q}$. Here we give only a partial answer, namely we observe operators of the form $\sum_{|\alpha|\leq s}a_\alpha D^\alpha$, $a_\alpha\in\mathbf{R}$, whose symbols satisfy (S1)-(S4). More general cases, including the so called ultra-modulation spaces [13] will be considered in a separate paper.

If $\sigma=\sigma(\xi)$, $\xi\in\mathbf{R}^d$ ($\sigma=\sigma(x)$, $x\in\mathbf{R}^d$, respectively), we denote the corresponding modulation space by $M^{\sigma(\xi),t}_{p,q}$ ($M^{\sigma(x),t}_{p,q}$ respectively). If $P(\cdot)=$

$\sum_{|\alpha|\le s} a_\alpha(\cdot)^\alpha$, $a_\alpha \in \mathbf{R}$, $s \ge 0$, then $\mathcal{F}\left(M_{p,q}^{P(x),t}\right) = M_{p,q}^{P(-\xi),t}$ [3, page 365.].

Lemma 1. *Let there be given $P(\xi) = \sum_{|\alpha|\le s} a_\alpha \xi^\alpha$, $a_\alpha \in \mathbf{R}$, satisfying (S4) and $P(\xi) \ge C(1+|\xi|)^s$, $\xi \in \mathbf{R}^d$, for some $C > 0$ (Take $C = 1$ as in (S1).) Let $f \in M_{2,2}^{1,t}$. Then there exists a function $f_1 \in M_{2,2}^{P(\xi),t}$ such that $P(D)f_1 = f$. In particular, if $\sigma(\xi) = \langle\xi\rangle^s$, $s \ge 0$, and $f \in M_{2,2}^{1,t}$, then $f_1 \in M_{2,2}^{\sigma(\xi),t}$.*

P r o o f. We formally put $P(D)f_1 = f \in M_{2,2}^{1,t}$. Since $\hat{f} \in M_{2,2}^{1,t} \subset L^2$ and $\frac{\hat{f}(\xi)}{P(\xi)} \in L^2$ it follows $\hat{f}_1 \in L^2$, i.e. $f_1 \in L^2$. We than have

$$\begin{aligned}
\langle\overline{T_x M_\xi g}, f\rangle &= \langle\overline{T_x M_\xi g}, P(D)f_1\rangle \\
&= e^{2\pi i\xi x}\int_{\mathbf{R}^d} P(-D_t)\left(e^{-2\pi i\xi t}\bar{g}(t-x)\right) f_1(t)dt
\end{aligned}$$

$$\begin{aligned}
&= e^{2\pi i\xi x}\int_{\mathbf{R}^d}\sum_{|\alpha|\le s} a_\alpha(-D_t)^\alpha\left(e^{-2\pi i\xi t}\bar{g}(t-x)\right) f_1(t)dt \\
&= e^{2\pi i\xi x}\sum_{|\alpha|\le s} a_\alpha\int_{\mathbf{R}^d}\sum_{\beta\le\alpha}\binom{\alpha}{\beta}(-D_t)^{\alpha-\beta}e^{-2\pi i\xi t}(-D_t)^\beta\bar{g}(t-x)f_1(t)dt \\
&= e^{2\pi i\xi x}\sum_{|\alpha|\le s} a_\alpha\xi^\alpha\int_{\mathbf{R}^d} e^{-2\pi i\xi t}\bar{g}(t-x)f_1(t)dt \\
&+ e^{2\pi i\xi x}\sum_{|\alpha|\le s} a_\alpha\sum_{1\le|\beta|,\ \beta\le\alpha}\binom{\alpha}{\beta}\int_{\mathbf{R}^d}\xi^{\alpha-\beta}e^{-2\pi i\xi t}(-D_t)^\beta\bar{g}(t-x)f_1(t)dt \\
&= P(\xi)\langle\overline{T_x M_\xi g}, f_1\rangle + \sum_{|\alpha|\le s} a_\alpha\sum_{1\le|\beta|,\ \beta\le\alpha}\binom{\alpha}{\beta}\xi^{\alpha-\beta}\langle\overline{T_x M_\xi D_t^\beta g}, f_1\rangle \\
&= P(\xi)\langle\overline{T_x M_\xi g}, f_1\rangle + \sum_{1\le|\beta|\le s} P_\beta(\xi)\langle\overline{T_x M_\xi D_t^\beta g}, f_1\rangle,
\end{aligned}$$

where $P_\beta(\xi) = \sum_{|j|\le s-|\beta|} b_{\beta,j}\xi^j$, $1 \le |\beta| \le s$.

If $g \in \mathcal{S}$, then $D_t^\beta g \in \mathcal{S}$ wherefrom

$$\|\langle\overline{T_x M_\xi(-D_t)^\beta g}, f_1\rangle P_\beta(\xi)(1+|x|^2+|\xi|^2)^{t/2}\|$$

$$\le C_\beta\|\langle\overline{T_x M_\xi g}, f_1\rangle\langle\xi\rangle^{s-|\beta|}(1+|x|^2+|\xi|^2)^{t/2}\|.$$

This implies

$$\infty > \|\langle\overline{T_x M_\xi g}, f\rangle(1+|x|^2+|\xi|^2)^{t/2}\| \tag{4}$$

$$\geq \|\langle\overline{T_x M_\xi g}, f_1\rangle P(\xi)(1+|x|^2+|\xi|^2)^{t/2}\|$$

$$-\|\sum_{1\leq|\beta|\leq s} C_\beta\langle\overline{T_x M_\xi g}, f_1\rangle\langle\xi\rangle^{s-|\beta|}(1+|x|^2+|\xi|^2)^{t/2}\|.$$

In order to prove that $f_1 \in M_{2,2}^{P(\xi),t}$, we split $\mathbf{R}^d_\xi$ in orthants. For the sake of simplicity we show the two dimensional case and note that the case $d > 2$ can be treated in a completely analogous way.

Let $\mathbf{R}^2_\xi = \mathbf{R}^2_{(\xi_1^+,\xi_2^+)} \bigcup \mathbf{R}^2_{(\xi_1^-,\xi_2^+)} \bigcup \mathbf{R}^2_{(\xi_1^-,\xi_2^-)} \bigcup \mathbf{R}^2_{(\xi_1^+,\xi_2^-)}$, where ξ_j^+ (ξ_j^-), $j = 1,2$, are non-negative (resp. non-positive) real numbers. Consider, for example, $\mathbf{R}^2_{(\xi_1^+,\xi_2^-)}$. We take $h = (h_1, h_2)$ where $h_1 > 0$, $h_2 < 0$ are chosen such that

$$\frac{1}{2}P(\xi+h)(1+|x|^2+|\xi+h|^2)^{t/2} \geq \sum_{1\leq|\beta|\leq s} C_\beta\langle\xi+h\rangle^{s-|\beta|}(1+|x|^2+|\xi+h|^2)^{t/2}$$

holds for all $\xi \in \mathbf{R}^2_{(\xi_1^+,\xi_2^-)}$, $x \in \mathbf{R}^2$. By (4) we have

$$\infty > \|\langle\overline{T_x M_{\xi+h} g}, f\rangle(1+|x|^2+|\xi+h|^2)^{t/2}\|_{L^2(\mathbf{R}^2\times\mathbf{R}^2_{(\xi_1^+,\xi_2^-)})}$$

$$\begin{aligned}
\geq\ & \|\langle\overline{T_x M_{\xi+h} g}, f_1\rangle P(\xi+h)(1+|x|^2+|\xi+h|^2)^{t/2}\|_{L^2(\mathbf{R}^2\times\mathbf{R}^2_{(\xi_1^+,\xi_2^-)})} \\
-\ & \|\sum_{1\leq|\beta|\leq s} C_\beta\langle\overline{T_x M_{\xi+h} g}, f_1\rangle\langle\xi+h\rangle^{s-|\beta|}(1+|x|^2+|\xi+h|^2)^{t/2}\|_{L^2(\mathbf{R}^2\times\mathbf{R}^2_{(\xi_1^+,\xi_2^-)})} \\
\geq\ & \frac{1}{2}\|\langle\overline{T_x M_{\xi+h} g}, f_1\rangle P(\xi+h)(1+|x|^2+|\xi+h|^2)^{t/2}\|_{L^2(\mathbf{R}^2\times\mathbf{R}^2_{(\xi_1^+,\xi_2^-)})}.
\end{aligned}$$

After a change of variables we obtain $f_1 \in M_{2,2}^{P(\xi),t}(\mathbf{R}^2\times\mathbf{R}^2_{(\xi_1^+,\xi_2^-)})$. The same procedure could be done for all of the other orthants, so we conclude that $f_1 \in M_{2,2}^{P(\xi),t}$ if $f \in M_{2,2}^{1,t}$.

Theorem 3. *Let there be given $P(\xi) = \sum_{|\alpha|\leq s} a_\alpha\xi^\alpha$, $a_\alpha \in \mathbf{R}$, satisfying (S4) and $P(\xi) \geq C(1+|\xi|)^s$, $\xi \in \mathbf{R}^d$, for some $C > 0$. The corresponding pseudodifferenital operator defined by (1) establishes an isomorphism between $M_{2,2}^{P(\xi),t}$ and $M_{2,2}^{1,t}$, $t \geq 0$. In particular, the Bessel potential of order*

s, $s \geq 0$, *establishes an isomorphism between* $M_{2,2}^{\sigma(\xi),t}$ *and* $M_{2,2}^{1,t}$, $t \geq 0$, *where* $\sigma(\xi) = \langle \xi \rangle^s$.

P r o o f. The proof that the mapping is injective is based on the properties of the Fourier transform. Let $P(D)$ be the Weyl transform of the symbol $P(\xi)$, $f \in M_{2,2}^{P(\xi),t}$, and let $P(D)f = 0$. Then we have

$$P(D)f(x) = \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} e^{2\pi i(x-y)\xi} P(\xi) f(y) dy d\xi = \mathcal{F}^{-1}\left(P(\xi)\hat{f}(\xi)\right)(x) = 0$$

which implies $P(\xi)\hat{f}(\xi) = 0$. Since $P(\xi) \geq 1$ we have $\hat{f}(\xi) = 0$, wherefrom $f = 0$.

Let f belong to $M_{2,2}^{1,t}$. We define f_1 by $\hat{f}_1 := \frac{\hat{f}}{P}$. It remains to show that $f_1 \in M_{2,2}^{P(\xi),t}$ and $P(D)f_1 = f$. As already mentioned in the proof of Lemma 1, $f_1 \in L^2$. Since

$$\mathcal{F}(P(D)f_1)(\xi) = P(\xi)\hat{f}_1(\xi) = \hat{f}(\xi),$$

and the Fourier transforms in an isomorphism on $M_{2,2}^{1,t}$ we obtain $P(D)f_1 = f$. Finally, $f_1 \in M_{2,2}^{P(\xi),t}$ by Lemma 1, which completes the proof.

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