Electronic Journal of Qualitative Theory of Differential Equations
Spec. Ed. I, 2009 No. **8**, 1–14; **http://www.math.u-szeged.hu/ejqtde/**

IMPULSIVE FRACTIONAL DIFFERENTIAL EQUATIONS IN BANACH SPACES

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Honoring the Career of John Graef on the Occasion of His Sixty-Seventh Birthday

Abstract

This paper is devoted to study the existence of solutions for a class of initial value problems for impulsive fractional differential equations involving the Caputo fractional derivative in a Banach space. The arguments are based upon Mönch's fixed point theorem and the technique of measures of noncompactness.

Key words and phrases: Initial value problem, impulses, Caputo fractional derivative, measure of noncompactness, fixed point, integral conditions, Banach space.
AMS (MOS) Subject Classifications: 26A33, 34A37, 34G20

1 Introduction

The theory of fractional differential equations is an important branch of differential equation theory, which has an extensive physical, chemical, biological, and engineering background, and hence has been emerging as an important area of investigation in the last few decades; see the monographs of Kilbas *et al.* [32], Miller and Ross [37], and the papers of Agarwal *et al.* [1, 2], Belarbi *et al.* [11], Benchohra *et al.* [12, 13, 15], Delboso and Rodino [20], Diethelm and Ford [22], El-Sayed *et al.* [23], Furati and Tatar [25, 26, 27], Momani *et al.* [38, 39], and Lakshmikantham and Devi [33] . Correspondingly, applications of the theory of fractional differential equations to different areas were considered by many authors and some basic results on fractional differential equations have been obtained see, for example, Gaul *et al.* [28], Glockle and Nonnenmacher [29], Hilfer [31], Mainardi [35], Metzler *et al.* [36] and Podlubny [42], and the references therein.

On the other hand, the theory of impulsive differential equations has undergone rapid development over the years and played a very important role in modern applied mathematical models of real processes rising in phenomena studied in physics, population dynamics, chemical technology, biotechnology and economics; see for instance the monographs by Bainov and Simeonov [10], Benchohra *et al* [14], Lakshmikantham *et al* [34], and Samoilenko and Perestyuk [43] and references therein. Moreover, impulsive differential equations present a natural framework for mathematical modeling of several real-world problems. However, the theory for fractional differential equations in Banach spaces has yet been sufficiently developed. Recently, Benchohra *et al* [16] applied the measure of noncompactness to a class of Caputo fractional differential equations of order $r \in (0,1]$ in a Banach space. Lakshmikantham and Devi [33] discussed the uniqueness and continuous dependence of the solutions of a class of fractional differential equations using the Riemann-Liouville derivative of order $r \in (0,1]$ on Banach spaces. Let E be a Banach space with norm $\|\cdot\|$. In this paper, we study the following initial value problem (IVP for short), for fractional order differential equations

$$^cD^r y(t) = f(t,y), \text{ for each } \ t \in J = [0,T], \ t \neq t_k, \ k = 1,\ldots,m, \ 0 < r \leq 1, \tag{1}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \ k = 1,\ldots,m, \tag{2}$$

$$y(0) = y_0, \tag{3}$$

where $^cD^r$ is the Caputo fractional derivative, $f : J \times E \to E$ is a given function, $I_k : E \to E$, $k = 1,\ldots,m$, and $y_0 \in E$, $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = T$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^+) = \lim_{h\to 0^+} y(t_k+h)$ and $y(t_k^-) = \lim_{h\to 0^-} y(t_k+h)$ represent the right and left limits of $y(t)$ at $t = t_k$, $k = 1,\ldots,m$. To our knowledge no paper has considered impulsive fractional differential equations in abstract spaces. This paper fills the gap in the literature. To investigate the existence of solutions of the problem above, we use Mönch's fixed point theorem combined with the technique of measures of noncompactness, which is an important method for seeking solutions of differential equations. See Akhmerov *et al.* [4], Alvàrez [5], Banaś *et al.* [6, 7, 8, 9], El-Sayed and Rzepka [24], Guo *et al.* [30], Mönch [40], Mönch and Von Harten [41] and Szufla [44].

2 Preliminaries

In what follows, we first state the following definitions, lemmas and some notation. Denote by $C(J,E)$ the Banach space of continuous functions $J \to E$, with the usual supremum norm

$$\|y\|_\infty = sup\{\|y(t)\|, \quad t \in J\}.$$

Let $L^1(J,E)$ be the Banach space of measurable functions $y : J \to E$ which are Bochner integrable, equipped with the norm

$$\|y\|_{L^1} = \int_J \|y(t)\|dt.$$

$$
\begin{aligned}
PC(J,E) \;&= \{y : J \to E : y \in C((t_k, t_{k+1}], E),\ k = 0, \ldots, m+1 \text{ and there exist } y(t_k^-) \text{ and}\\
&\quad y(t_k^+),\ k = 1, \ldots, m \text{ with } y(t_k^-) = y(t_k)\}.
\end{aligned}
$$

$PC(J,E)$ is a Banach space with norm

$$\|y\|_{PC} = \sup_{t \in J} \|y(t)\|.$$

Set $J' := [0,T] \backslash \{t_1, \ldots, t_m\}$.
Moreover, for a given set V of functions $v : J \to E$ let us denote by

$$V(t) = \{v(t), v \in V\},\ t \in J$$

and

$$V(J) = \{v(t) : v \in V,\ t \in J\}.$$

Now let us recall some fundamental facts of the notion of Kuratowski measure of noncompactness.

Definition 2.1 *([7]) Let E be a Banach space and Ω_E the bounded subsets of E. The Kuratowski measure of noncompactness is the map $\alpha : \Omega_E \to [0,\infty]$ defined by*

$$\alpha(B) = \inf\{\epsilon > 0 : B \subseteq \cup_{i=1}^{n} B_i \text{ and } diam(B_i) \le \epsilon\};\ \text{here } B \in \Omega_E.$$

Properties: The Kuratowski measure of noncompactness satisfies some properties (for more details see [7])

(a) $\alpha(B) = 0 \Leftrightarrow \overline{B}$ is compact (B is relatively compact).

(b) $\alpha(B) = \alpha(\overline{B})$.

(c) $A \subset B \Rightarrow \alpha(A) \le \alpha(B)$.

(d) $\alpha(A + B) \le \alpha(A) + \alpha(B)$

(e) $\alpha(cB) = |c|\alpha(B)$; $c \in \mathbb{R}$.

(f) $\alpha(convB) = \alpha(B)$.

For completeness we recall the definition of Caputo derivative of fractional order. Let $\varphi_r(t) = \frac{t^{r-1}}{\Gamma(r)}$ for $t > 0$ and $\varphi_r(t) = 0$ for $t \le 0$, and $\varphi_r \to \delta(t)$ as $r \to 0$, where δ is the delta function.

Definition 2.2 *([32]) The fractional order integral of the function $h \in L^1([a,b])$ of order $r \in \mathbb{R}_+$; is defined by*

$$I_a^r h(t) = \frac{1}{\Gamma(r)} \int_a^t \frac{h(s)}{(t-s)^{1-r}} dt$$

where Γ is the gamma function. When $a = 0$, we write $I^r h(t) = [h * \varphi_r](t)$.

Definition 2.3 *([32] For a function h given on the interval $[a,b]$, the Caputo fractional-order derivative of h, of order $r>0$ is defined by*

$$^{c}D^{r}_{a^+}h(t)=\frac{1}{\Gamma(n-r)}\int_a^t\frac{h^{(n)}(s)ds}{(t-s)^{1-n+r}}$$

Here $n=[r]+1$ and $[r]$ denotes the integer part of r.

For example for $0<r\leq 1$ and $h:[a,b]\to E$ an absolutely continuous function, then the fractional derivative of order r of h exists.

From the definition of Caputo derivative, we can obtain the following auxiliary results [45].

Lemma 2.1 *Let $r>0$, then the differential equation*

$$^{c}D^{r}h(t)=0$$

has solutions $h(t)=c_0+c_1t+c_2t^2+\ldots+c_{n-1}t^{n-1}$, $c_i\in E$, $i=0,1,\ldots,n-1$, $n=[r]+1$.

Lemma 2.2 *Let $r>0$, then*

$$I^{r\,c}D^{r}h(t)=h(t)+c_0+c_1t+c_2t^2+\ldots+c_{n-1}t^{n-1}$$

for some $c_i\in E$, $i=0,1,...,n-1$, $n=[r]+1$.

Definition 2.4 *A map $f:J\times E\to E$ is said to be Carathéodory if*

(i) $t\longmapsto f(t,u)$ is measurable for each $u\in E$;

(ii) $u\longmapsto F(t,u)$ is continuous for almost all $t\in J$.

For our purpose we will only need the following fixed point theorem, and the important Lemma.

Theorem 2.1 *([3, 40]) Let D be a bounded, closed and convex subset of a Banach space such that $0\in D$, and let N be a continuous mapping of D into itself. If the implication*

$$V=\overline{conv}N(V)\quad or\quad V=N(V)\cup\{0\}\Rightarrow\alpha(V)=0$$

holds for every subset V of D, then N has a fixed point.

Lemma 2.3 *([44]) Let D be a bounded, closed and convex subset of the Banach space $C(J,E)$, G a continuous function on $J\times J$ and f a function from $J\times E\to E$ which satisfies the Carathéodory conditions and assume there exists $p\in L^1(J,\mathbb{R}_+)$ such that for each $t\in J$ and each bounded set $B\subset E$ we have*

$$\lim_{k\to 0^+}\alpha(f(J_{t,k}\times B))\leq p(t)\alpha(B);\ \text{here } J_{t,k}=[t-k,t]\cap J.$$

If V is an equicontinuous subset of D, then

$$\alpha\left(\left\{\int_J G(s,t)f(s,y(s))ds:y\in V\right\}\right)\leq\int_J\|G(t,s)\|p(s)\alpha(V(s))ds.$$

3 Existence of Solutions

First of all, we define what we mean by a solution of the IVP (1)–(3).

Definition 3.1 *A function $y \in PC(J, E)$ is said to be a solution of (1)–(3) if y satisfies the equation ${}^cD^r y(t) = f(t, y(t))$ on J', and conditions*

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)),\ k = 1, \ldots, m,$$

and

$$y(0) = y_0.$$

Lemma 3.1 *Let $0 < r \leq 1$ and let $h : J \to E$ be continuous. A function y is a solution of the fractional integral equation*

$$y(t) = \begin{cases} y_0 + \dfrac{1}{\Gamma(r)} \displaystyle\int_0^t (t-s)^{r-1} h(s)ds & \text{if } t \in [0, t_1], \\ y_0 + \dfrac{1}{\Gamma(r)} \displaystyle\sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{r-1} h(s)ds & \\ + \dfrac{1}{\Gamma(r)} \displaystyle\int_{t_k}^t (t-s)^{r-1} h(s)ds & \\ + \displaystyle\sum_{i=1}^{k} I_i(y(t_i^-)), & \text{if } t \in (t_k, t_{k+1}],\ k = 1, \ldots, m, \end{cases} \tag{4}$$

if and only if y is a solution of the fractional IVP

$$ {}^cD^r y(t) = h(t),\ \textit{for each},\ t \in J', \tag{5}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)),\ k = 1, \ldots, m, \tag{6}$$

$$y(0) = y_0. \tag{7}$$

Proof: Assume y satisfies (5)-(7). If $t \in [0, t_1]$ then

$${}^cD^r y(t) = h(t).$$

Lemma 2.2 implies

$$y(t) = y_0 + \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} h(s)ds.$$

If $t \in (t_1, t_2]$ then Lemma 2.2 implies

$$\begin{aligned} y(t) &= y(t_1^+) + \frac{1}{\Gamma(r)} \int_{t_1}^t (t-s)^{r-1} h(s)ds \\ &= \Delta y|_{t=t_1} + y(t_1^-) + \frac{1}{\Gamma(r)} \int_{t_1}^t (t-s)^{r-1} h(s)ds \\ &= I_1(y(t_1^-)) + y_0 + \frac{1}{\Gamma(r)} \int_0^{t_1} (t_1 - s)^{r-1} h(s)ds \\ &+ \frac{1}{\Gamma(r)} \int_{t_1}^t (t-s)^{r-1} h(s)ds. \end{aligned}$$

If $t \in (t_2, t_3]$, then again Lemma 2.2 implies

$$\begin{aligned} y(t) &= y(t_2^+) + \frac{1}{\Gamma(r)} \int_{t_2}^{t} (t-s)^{r-1} h(s) ds \\ &= \Delta y|_{t=t_2} + y(t_2^-) + \frac{1}{\Gamma(r)} \int_{t_2}^{t} (t-s)^{r-1} h(s) ds \\ &= I_2(y(t_2^-)) + I_1(y(t_1^-)) + y_0 + \frac{1}{\Gamma(r)} \int_0^{t_1} (t_1-s)^{r-1} h(s) ds \\ &+ \frac{1}{\Gamma(r)} \int_{t_1}^{t_2} (t_2-s)^{r-1} h(s) ds + \frac{1}{\Gamma(r)} \int_{t_2}^{t} (t-s)^{r-1} h(s) ds. \end{aligned}$$

If $t \in (t_k, t_{k+1}]$, then again Lemma 2.2 implies (4).

Conversely, assume that y satisfies the impulsive fractional integral equation (4). If $t \in [0, t_1]$, then $y(0) = y_0$ and, using the fact that ${}^cD^r$ is the left inverse of I^r, we get

$${}^cD^r y(t) = h(t), \text{ for each } t \in [0, t_1].$$

If $t \in [t_k, t_{k+1}),\ k = 1, \dots, m$, and using the fact that ${}^cD^r C = 0$, where C is a constant, we get

$${}^cD^r y(t) = h(t), \text{ for each } t \in [t_k, t_{k+1}).$$

Also, we can easily show that

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)),\ k = 1, \dots, m.$$

Let us list some conditions on the functions involved in the IVP (1)–(3). Assume that

(H1) $f : J \times E \to E$ satisfies the Carathéodory conditions.

(H2) There exists $p \in L^1(J, \mathbb{R}_+) \cap C(J, \mathbb{R}_+)$, such that,

$$\|f(t, y)\| \leq p(t)\|y\|, \quad \text{for } t \in J \text{ and each } y \in E.$$

(H3) There exists $c > 0$ such that

$$\|I_k(y)\| \leq c\|y\| \text{ for each } y \in E.$$

(H4) For each bounded set $B \subset E$ we have

$$\alpha(I_k(B)) \leq c\alpha(B),\ k = 1, \dots, m.$$

(H5) For each $t \in J$ and each bounded set $B \subset E$ we have

$$\lim_{h \to 0^+} \alpha(f(J_{t,h} \times B)) \leq p(t)\alpha(B);\ \text{here } J_{t,h} = [t-h, t] \cap J.$$

Theorem 3.1 *Assume that assumptions* $(H1)-(H5)$ *hold. Let* $p^* = \sup_{t\in J} p(t)$. *If*

$$\frac{(m+1)p^*T^r}{\Gamma(r+1)} + mc < 1, \tag{8}$$

then the IVP (1)–(3) has at least one solution.

Proof. We shall reduce the existence of solutions of (1)-(3) to a fixed point problem. To this end we consider the operator $N : PC(J,E) \longrightarrow PC(J,E)$ defined by

$$\begin{aligned} N(y)(t) &= y_0 + \frac{1}{\Gamma(r)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{r-1} f(s, y(s)) ds \\ &+ \frac{1}{\Gamma(r)} \int_{t_k}^{t} (t-s)^{r-1} f(s, y(s)) ds \\ &+ \sum_{0<t_k<t} I_k(y(t_k^-)). \end{aligned}$$

Clearly, the fixed points of the operator N are solution of the problem (1)-(3). Let

$$r_0 \geq \frac{\|y_0\|}{1 - mc - \dfrac{(m+1)p^*T^r}{\Gamma(r+1)}} \tag{9}$$

and consider the set

$$D_{r_0} = \{y \in PC(J,E) : \|y\|_\infty \leq r_0\}.$$

Clearly, the subset D_{r_0} is closed, bounded and convex. We shall show that N satisfies the assumptions of Theorem 2.1. The proof will be given in three steps.

Step 1: N is continuous.

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $PC(J,E)$. Then for each $t \in J$

$$\begin{aligned} \|N(y_n)(t) - N(y)(t)\| &\leq \frac{1}{\Gamma(r)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{r-1} \|f(s, y_n(s)) - f(s, y(s))\| ds \\ &+ \frac{1}{\Gamma(r)} \int_{t_k}^{t} (t-s)^{r-1} \|f(s, y_n(s)) - f(s, y(s))\| ds \\ &+ \sum_{0<t_k<t} \|I_k(y_n(t_k^-)) - I_k(y(t_k^-))\|. \end{aligned}$$

Since I_k is continuous and f is of Carathéodory type, then by the Lebesgue dominated convergence theorem we have

$$\|N(y_n) - N(y)\|_\infty \to 0 \,\text{as}\, n \to \infty.$$

Step 2: N maps D_{r_0} into itself.

For each $y \in D_{r_0}$, by $(H2)$ and (8) we have for each $t \in J$

$$\begin{aligned}
\|N(y)(t)\| &\leq \|y_0\| + \frac{1}{\Gamma(r)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{r-1} \|f(s, y(s))\| ds \\
&+ \frac{1}{\Gamma(r)} \int_{t_k}^{t} (t - s)^{r-1} \|f(s, y(s))\| ds \\
&+ \sum_{0<t_k<t} \|I_k(y(t_k^-))\| \\
&\leq \|y_0\| + \frac{1}{\Gamma(r)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{r-1} p(t) \|y\| ds \\
&+ \frac{1}{\Gamma(r)} \int_{t_k}^{t} (t - s)^{r-1} p(t) \|y\| ds \\
&+ \sum_{0<t_k<t} c\|y\| \\
&\leq \|y_0\| + r_0 (\frac{(m+1)p^* T^r}{\Gamma(r+1)} + mc) \\
&\leq r_0.
\end{aligned}$$

Step 3: $N(D_{r_0})$ is bounded and equicontinuous.

By Step2, it is obvious that $N(D_{r_0}) \subset PC(J, E)$ is bounded.
For the equicontinuity of $N(D_{r_0})$. Let $\tau_1,\ \tau_2 \in J,\ \ \tau_1 < \tau_2,$, and let $y \in D_{r_0}$. Then

$$\begin{aligned}
\|N(y)(\tau_2) - N(y)(\tau_1)\| &= \frac{1}{\Gamma(r)} \int_0^{\tau_1} |(\tau_2 - s)^{r-1} - (\tau_1 - s)^{r-1}| \|f(s, y(s))\| ds \\
&+ \frac{1}{\Gamma(r)} \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{r-1} \|f(s, y(s))\| ds \\
&+ \sum_{0<t_k<\tau_2-\tau_1} \|I_k(y(t_k^-))\| \\
&\leq \frac{r_0 p^*}{\Gamma(r+1)} [\tau_2^r - \tau_1^r] \\
&+ \sum_{0<t_k<\tau_2-\tau_1} \|I_k(y(t_k^-))\|.
\end{aligned}$$

As $\tau_1 \longrightarrow \tau_2$, the right-hand side of the above inequality tends to zero.

Now let V be a subset of D_{r_0} such that $V \subset \overline{conv}(N(V) \cup \{0\})$.
V is bounded and equicontinuous and therefore the function $v \rightarrow v(t) = \alpha(V(t))$ is continuous on J. By (H4), (H5), Lemma 2.3 and the properties of the measure α we have for each $t \in J$

$$\begin{aligned}
v(t) &\leq \alpha(N(V)(t)\cup\{0\}) \\
&\leq \alpha(N(V)(t)) \\
&\leq \frac{1}{\Gamma(r)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{r-1}p(s)\alpha(V(s))ds \\
&+ \frac{1}{\Gamma(r)}\int_{t_k}^{t}(t-s)^{r-1}p(s)\alpha(V(s))ds \\
&+ \sum_{0<t_k<t}\alpha(V(s)) \\
&\leq \frac{1}{\Gamma(r)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{r-1}p(s)v(s)ds \\
&+ \frac{1}{\Gamma(r)}\int_{t_k}^{t}(t-s)^{r-1}p(s)v(s)ds \\
&+ \sum_{0<t_k<t}cv(s) \\
&\leq \|v\|_\infty(\frac{(m+1)p^*T^r}{\Gamma(r+1)}+mc).
\end{aligned}$$

This means that

$$\|v\|_\infty\Big(1-\Big[\frac{(m+1)p^*T^r}{\Gamma(r+1)}+mc\Big]\Big)\leq 0.$$

By (8) it follows that $\|v\|_\infty = 0$; that is, $v(t) = 0$ for each $t \in J$, and then $V(t)$ is relatively compact in $PC(J,E)$. In view of the Ascoli-Arzelà theorem, V is relatively compact in D_{r_0}. Applying now Theorem 2.1 we conclude that N has a fixed point which is a solution of the problem (1)-(3). □

4 Nonlocal impulsive differential equations

This section is concerned with a generalization of the results presented in the previous section to nonlocal impulsive fractional differential equations. More precisely we shall present some existence and uniqueness results for the following nonlocal problem

$${}^cD^ry(t) = f(t,y),\ \text{for each},\ t\in J=[0,T],\ t\neq t_k,\ k=1,\dots,m,\ 0<r\leq 1, \tag{10}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)),\ k=1,\dots,m, \tag{11}$$

$$y(0)+g(y)=y_0, \tag{12}$$

where $f,\ I_k,\ k=1,\dots,m$ are as in Section 3 and $g: PC(J,E)\to E$ is a continuous function. Nonlocal conditions were initiated by Byszewski [19] when he proved the existence and uniqueness of mild and classical solutions of nonlocal Cauchy problems. As remarked by Byszewski [17, 18], the nonlocal condition can be more useful than the standard initial condition to describe some physical phenomena. For example, in [21], the author used

$$g(y) = \sum_{i=1}^{p} c_i y(\tau_i) \tag{13}$$

where $c_i,\ i = 1, \ldots, p$, are given constants and $0 < \tau_1 < ... < \tau_p \leq T$, to describe the diffusion phenomenon of a small amount of gas in a transparent tube. In this case, (13) allows the additional measurements at $t_i,\ i = 1, \ldots, p$.

Let us introduce the following set of conditions.

(H6) There exists a constant $M^{**} > 0$ such that

$$|g(u)| \leq M^{**} \text{ for each } u \in PC(J, E).$$

(H7) For each bounded set $B \subset PC(J, E)$ we have

$$\alpha(g(B)) \leq M^{**}\alpha(B).$$

Theorem 4.1 *Assume that (H1)-(H7) hold. If*

$$\frac{(m+1)p^*T^r}{\Gamma(r+1)} + mc + M^{**} < 1, \tag{14}$$

then the nonlocal problem (10)-(12) has at least one solution on J.

Proof. Transform the problem (10)–(12) into a fixed point problem. Consider the operator $\tilde{F} : PC(J, E) \to PC(J, E)$ defined by

$$\begin{aligned} \tilde{F}(y)(t) &= y_0 - g(y) + \frac{1}{\Gamma(r)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{r-1} f(s, y(s))ds \\ &+ \frac{1}{\Gamma(r)} \int_{t_k}^{t} (t-s)^{r-1} f(s, y(s))ds \\ &+ \sum_{0<t_k<t} I_k(y(t_k^-)). \end{aligned}$$

Clearly, the fixed points of the operator $\tilde{F}$ are solution of the problem (10)-(12). We can easily show the conditions of Theorem 2.1 are satisfied by $\tilde{F}$. □

5 An Example

In this section we give an example to illustrate the usefulness of our main results. Let us consider the following impulsive fractional initial value problem,

$$^cD^r y(t) = \frac{1}{10+e^t} y(t), \quad t \in J := [0,1], \ t \neq \frac{1}{2}, \ 0 < r \leq 1, \tag{15}$$

$$\Delta y|_{t=\frac{1}{2}} = \frac{1}{5} y\left(\frac{1}{2}^{-}\right), \tag{16}$$

$$y(0) = 0. \tag{17}$$

Set

$$f(t,x) = \frac{1}{10+e^t} x, \quad (t,x) \in J \times E,$$

and

$$I_k(x) = \frac{1}{5} x, \quad x \in E.$$

Clearly conditions $(H2)$ and $(H3)$ hold with $p(t) = \frac{1}{10+e^t}$ and $c = \frac{1}{5}$.

We shall check that condition (8) is satisfied with $T = 1,\ m = 1$ and $p^* = \frac{1}{10}$. Indeed

$$\left[\frac{(m+1)p^*T^r}{\Gamma(r+1)} + \frac{1}{5}\right] < 1 \Leftrightarrow \Gamma(r+1) > \frac{1}{4}, \tag{18}$$

which is satisfied for some $r \in (0,1]$. Then by Theorem 3.1 the problem (15)-(17) has at least one solution on $[0,1]$ for values of r satisfying (18).

References

[1] R. P. Agarwal, M. Benchohra and S. Hamani, Boundary value problems for differential inclusions with fractional order, *Adv. Stud. Contemp. Math.* **12** (2008), 181-196.

[2] R. P. Agarwal, M. Benchohra and S. Hamani, Boundary value problems for fractional differential equations, *Georgian Math. J.* (to appear).

[3] R. P. Agarwal, M. Meehan and D. O'Regan, *Fixed Point Theory and Applications*, Cambridge University Press, Cambridge, 2001.

[4] R. R. Akhmerov, M. I. Kamenskii, A. S. Patapov, A. E. Rodkina and B. N. Sadovskii, *Measures of Noncompactness and Condensing Operators*, trans. from the Russian by A. Iacob, Birkhauser Verlag, Basel, 1992.

[5] J. C. Alvàrez, Measure of noncompactness and fixed points of nonexpansive condensing mappings in locally convex spaces, *Rev. Real. Acad. Cienc. Exact. Fis. Natur. Madrid* **79** (1985), 53-66.

[6] J. Banaś and B. C. Dhage, Global asymptotic stability of solutions of a fractional integral equation, *Nonlinear Anal.* **69** (2008) 1945-1952.

[7] J. Banaś and K. Goebel, *Measures of Noncompactness in Banach Spaces*, Marcel Dekker, New York, 1980.

[8] J. Banaś and B. Rzepka, An application of a measure of noncompactness in the study of asymptotique stability, *Appl. Math. Lett.* **16** (2003), 1-6.

[9] J. Banaś and K. Sadarangani, On some measures of noncompactness in the space of continuous functions, *Nonlinear Anal.* **68** (2008), 377-383.

[10] D.D. Bainov, P.S. Simeonov, *Systems with Impulsive Effect*, Horwood, Chichister, 1989.

[11] A. Belarbi, M. Benchohra and A. Ouahab, Existance results for functional differential equations of fractional order, *Appl. Anal.* **85** (2006), 1459-1470.

[12] M. Benchohra, J.R. Graef and S. Hamani, Existence results for boundary value problems with nonlinear fractional differential equations, *Appl. Anal.* **87** (7) (2008), 851-863.

[13] M. Benchohra, S. Hamani and S.K. Ntouyas, Boundary value problems for differential equations with fractional order, *Surv. Math. Appl.* **3** (2008), 1-12.

[14] M. Benchohra, J. Henderson and S. K. Ntouyas, *Impulsive Differential Equations and Inclusions*, Hindawi Publishing Corporation, Vol 2, New York, 2006.

[15] M. Benchohra, J. Henderson, S.K. Ntouyas and A. Ouahab, Existence results for fractional order functional differential equations with infinite delay, *J. Math. Anal. Appl.* **338** (2008), 1340-1350.

[16] M. Benchohra, J. Henderson and D. Seba, Measure of noncompactness and fractional differential equations in Banach spaces, *Commun. Appl. Anal.* **12** (4) (2008), 419-428.

[17] L. Byszewski, Theorems about existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem, *J. Math. Anal. Appl.* **162** (1991), 494-505.

[18] L. Byszewski, Existence and uniqueness of mild and classical solutions of semilinear functional-differential evolution nonlocal Cauchy problem. *Selected problems of mathematics*, 25–33, 50th Anniv. Cracow Univ. Technol. Anniv. Issue, 6, Cracow Univ. Technol., Krakow, 1995.

[19] L. Byszewski and V. Lakshmikantham, Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space, *Appl. Anal.* **40** (1991), 11-19.

[20] D. Delbosco and L. Rodino, Existence and uniqueness for a nonlinear fractional differential equation, *J. Math. Anal. Appl.* **204** (1996), 609-625.

[21] K. Deng, Exponential decay of solutions of semilinear parabolic equations with nonlocal initial conditions, *J. Math. Anal. Appl.* **179** (1993), 630-637.

[22] K. Diethelm and N. J. Ford, Analysis of fractional differential equations, *J. Math. Anal. Appl.* **265** (2002), 229-248.

[23] A. M. A. El-Sayed, W. G. El-Sayed and O.L. Moustafa, On some fractional functional equations, *Pure Math. Appl.* **6** (1995), 321-332.

[24] W. G. El-Sayed and B. Rzepka, Nondecreasing solutions of a quadratic integral equation of Urysohn type, *Computers Math. Appl.* **51** (2006), 1065-1074.

[25] K. M. Furati and N.-eddine Tatar, An existence result for a nonlocal fractional differential problem, *J. Fractional Calc.* **26** (2004), 43-51.

[26] K. M. Furati and N.-eddine Tatar, Behavior of solutions for a weighted Cauchy-type fractional differential problem, *J. Fractional Calc.* **28** (2005), 23-42.

[27] K. M. Furati and N.-eddine Tatar, Power type estimates for a nonlinear fractional differential equation, *Nonlinear Anal.* **62** (2005), 1025-1036.

[28] L. Gaul, P. Klein and S. Kempfle, Damping description involving fractional operators, *Mech. Systems Signal Processing* **5** (1991), 81-88.

[29] W. G. Glockle and T. F. Nonnenmacher, A fractional calculus approach of self-similar protein dynamics, *Biophys. J.* **68** (1995), 46-53.

[30] D. Guo, V. Lakshmikantham and X. Liu, *Nonlinear Integral Equations in Abstract Spaces*, Kluwer Academic Publishers Group, Dordrecht, 1996.

[31] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[32] A. A. Kilbas, H. M. Srivastava and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier Science B.V., Amsterdam, 2006.

[33] V. Lakshmikantham and J. V. Devi, Theory of fractional differential equations in a Banach space, *Eur. J. Pure Appl. Math.* **1** (2008), 38-45.

[34] V. Lakshmikantham, D.D. Bainov and P.S. Simeonov, *Theory of Impulsive Differential Equations*, Worlds Scientific, Singapore, 1989.

[35] F. Mainardi, *Fractional calculus: Some basic problems in continuum and statistical mechanics*, in: "Fractals and Fractional Calculus in Continuum Mechanics" (A. Carpinteri, F. Mainardi Eds.), Springer-Verlag, Wien, 1997, pp. 291-348.

[36] F. Metzler, W. Schick, H. G. Kilian and T.F. Nonnenmacher, Relaxation in filled polymers: a fractional calculus approach, *J. Chem. Phys.* **103** (1995), 7180-7186.

[37] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, New York, 1993.

[38] S. M. Momani and S. B. Hadid, Some comparison results for integro-fractional differential inequalities, *J. Fract. Calc.* **24** (2003), 37-44.

[39] S. M. Momani, S. B. Hadid and Z. M. Alawenh, Some analytical properties of solutions of differential equations of noninteger order, *Int. J. Math. Math. Sci.* **2004**, no. 13-16, 697-701.

[40] H. Mönch, Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces, *Nonlinear Anal.* **4** (1980), 985-999.

[41] H. Mönch and G. F. Von Harten, On the Cauchy problem for ordinary differential equations in Banach spaces, *Archiv. Math. Basel* **39** (1982), 153-160.

[42] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[43] A.M. Samoilenko, N.A. Perestyuk, *Impulsive Differential Equations*, World Scientific, Singapore, 1995.

[44] S. Szufla, On the application of measure of noncompactness to existence theorems, *Rend. Sem. Mat. Univ. Padova* **75** (1986), 1-14.

[45] S. Zhang, Positive solutions for boundary-value problems of nonlinear fractional differential equations, *Electron. J. Differential Equations* **2006**, No. 36, 12 pp. (electronic).