NEWTON'S FORMULA AND THE CONTINUED FRACTION EXPANSION OF $\sqrt{d}$

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Abstract. It is known that if the period $s(d)$ of the continued fraction expansion of $\sqrt{d}$ satisfies $s(d) \leq 2$, then all Newton's approximants

$$R_n = \frac{1}{2}\Big(\frac{p_n}{q_n} + \frac{dq_n}{p_n}\Big)$$

are convergents of $\sqrt{d}$, and moreover $R_n = p_{2n+1}/q_{2n+1}$ for all $n \geq 0$. Motivated by this fact we define $j = j(d,n)$ by $R_n = p_{2n+1+2j}/q_{2n+1+2j}$ if R_n is a convergent of $\sqrt{d}$, and define $b = b(d)$ by $b = \big|\{n : 0 \leq n \leq s-1 \text{ and } R_n \text{ is a convergent of } \sqrt{d}\}\big|$. The question is how large $|j|$ and b can be. We prove that $|j|$ is unbounded and give some examples supporting a conjecture that b is unbounded too. We also discuss the magnitude of $|j|$ and b compared with d and $s(d)$.