Algorithms for Finding Good Examples for the abc and Szpiro Conjectures

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Abstract

The radical $\mathrm{rad} n$ of an integer $n \neq 0$ is the product of the primes dividing n. The abc-conjecture and the Szpiro conjecture imply that, for any positive relatively prime integers a, b, and c such that $a+b=c$, the expressions

$$\frac{\log c}{\log \mathrm{rad}(abc)} \quad \text{and} \quad \frac{\log abc}{\log \mathrm{rad}(abc)}$$

are bounded. We give an algorithm for finding triples (a, b, c) for which these ratios are high with respect to their conjectured asymptotic values. The algorithm is based on approximation methods for solving the equation $Ax^n - By^n = Cz$ in integers x, y, and z with small $|z|$.

Additionally, we employ these triples to obtain semistable elliptic curves over $\mathbb{Q}$ with high Szpiro ratio

$$\sigma = \frac{\log |\Delta|}{\log N},$$

where Δ is the discriminant and N is the conductor.