On the Evaluation of Euler Sums

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Abstract

Euler studied double sums of the form

$$\zeta(r,s) = \sum_{1 \le m < n} \frac{1}{n^s m^r}$$

for positive integers r and s, and inferred, for the special cases $r = 1$ or $r + s$ odd, elegant identities involving values of the Riemann zeta function. Here we establish various series expansions of $\zeta(r,s)$ for real numbers r and s. These expansions generally involve infinitely many zeta values. The series of one type terminate for integers r and s with $r + s$ odd, reducing in those cases to the Euler identities. Series of another type are rapidly convergent and therefore useful in numerical experiments.