A Class of 1-Additive Sequences and Quadratic Recurrences

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Abstract

For odd $v \geq 5$, Schmerl and Spiegel have proved that the 1-additive sequence $(2, v)$ has precisely two even terms and, consequently, is regular. For $5 \leq v \equiv 1 \bmod 4$, we prove, using a different approach, that the 1-additive sequence $(4, v)$ has precisely three even terms. The proof draws upon the periodicity properties of a certain ternary quadratic recurrence.

Unlike the case of $(2, v)$, the regularity of $(4, v)$ can be captured by expressions in closed form. For example, its period can be written as an exponential sum of binary digit sums. Therefore the asymptotic density $\Delta(v)$ of $(4, v)$ tends to 0 as $v \to \infty$, but is misbehaved in the sense that

$$\liminf_{\substack{v \to \infty \\ v \equiv 1 \bmod 4}} \left(\frac{v}{2}\right)^{2-\log_2 3} \Delta(v) = \frac{1}{4},$$

$$\limsup_{\substack{v \to \infty \\ v \equiv 1 \bmod 4}} \left(\frac{v}{2}\right)^{2-\log_2 3} \Delta(v) > 0.27164.$$

This is proved using techniques adapted from Harborth and Stolarsky.