The Distribution of $3x+1$ Trees

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Abstract

Backwards iteration of the $3x+1$ function starting from a fixed integer a produces a tree of preimages of a. Let $\mathcal{T}_k(a)$ denote this tree grown to depth k, and let $\mathcal{T}_k^*(a)$ denote the pruned tree resulting from the removal of all nodes $n \equiv 0 \bmod 3$. We previously computed the maximal and minimal number of leaves in $\mathcal{T}_k^*(a)$ for all $a \not\equiv 0 \bmod 3$ and all $k \leq 30$. Here we compare these data with predictions made using branching process models designed to imitate the growth of $3x+1$ trees, developed in an earlier paper by Lagarias and Weiss. We derive rigorous results for the branching process models. The range of variation exhibited by the $3x+1$ trees appears significantly narrower than that of the branching process models. We also study the variation in expected leaf-counts associated to the congruence class of a mod 3^j. This variation, when properly normalized, converges almost everywhere as $j \to \infty$ to a limit function on the invertible 3-adic integers.