Iterating the Sum-of-Divisors Function

Herman J. J. te Riele

Abstract

Let $\sigma^0(n) = n$ and $\sigma^m(n) = \sigma(\sigma^{m-1}(n))$, where $m \geq 1$ and σ is the sum-of-divisors function. We say that n is (m, k)-perfect if $\sigma^m(n) = kn$. We have tabulated all $(2, k)$-perfect numbers up to 10^9 and all $(3, k)$- and $(4, k)$-perfect numbers up to $2 \cdot 10^8$. These tables have suggested several conjectures, some of which we prove here. We ask in particular: For any fixed $m \geq 1$, are there infinitely many (m, k)-perfect numbers? Is every positive integer (m, k)-perfect, for sufficiently large $m \geq 1$? In this connection, we have obtained the smallest value of m such that n is (m, k)-perfect, for $1 \leq n \leq 1000$. We also address questions concerning the limiting behaviour of $\sigma^{m+1}(n)/\sigma^m(n)$ and $(\sigma^m(n))^{1/m}$, as $m \to \infty$.