THE TOPOLOGY OF THE RELATIVE CHARACTER VARIETIES OF A QUADRUPLY-PUNCTURED SPHERE

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Abstract. Let M be a quadruply-punctured sphere with boundary components A, B, C, D. The $\mathrm{SL}(2,\mathbb{C})$-character variety of M consists of equivalence classes of homomorphisms ρ of $\pi_1(M) \longrightarrow \mathrm{SL}(2,\mathbb{C})$ and can be identified with a quartic hypersurface in $\mathbb{C}^7$. For fixed $a, b, c, d \in \mathbb{C}$, the subset $V_{a,b,c,d}$ corresponding to representations ρ with $\mathrm{tr}(\rho(A)) = a,\ \mathrm{tr}(\rho(B)) = b,\ \mathrm{tr}(\rho(C)) = c,\ \mathrm{tr}(\rho(D)) = d$ is a cubic surface in $\mathbb{C}^3$. We determine the singular points of $V_{a,b,c,d}$ and classify its set $V_{a,b,c,d}(\mathbb{R})$ of $\mathbb{R}$-points into six topological types, at least when this set is nonsingular. $V_{a,b,c,d}(\mathbb{R})$ contains a compact component if and only if $-2 < a, b, c, d < 2$. For certain values of (a, b, c, d), this component corresponds to representations in $\mathrm{SL}(2,\mathbb{R})$.