VISUALIZING ELEMENTS IN THE SHAFAREVICH–TATE GROUP

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ABSTRACT. We review a number of ways of "visualizing" the elements of the Shafarevich–Tate group of an elliptic curve E over a number field K. We are specifically interested in cases where the elliptic curves are defined over the rationals, and are subabelian varieties of the new part of the jacobian of a modular curve (specifically, of $X_0(N)$, where N is the conductor of the elliptic curve). For a given such E with nontrivial Shafarevich–Tate group, we pose the question: *Are all the curves of genus one representing elements of the Shafarevich–Tate group of E isomorphic (over the rationals) to curves contained in a (single) abelian surface A, itself defined over the rationals, containing E as a sub-elliptic curve, and contained in turn in the new part of the jacobian of a modular curve $X_0(N)$?*

At first view, one might imagine that there are few E with nontrivial Shafarevich–Tate group for which the answer is yes. Indeed we have a small number of examples where the answer is no, and it is very likely that the answer will be no if the order of the Shafarevich–Tate group is large enough. Nonetheless, among all (modular) elliptic curves E as above, with conductors up to 5500 and with no rational point of order 2, we have found the answer to the question to be yes in the vast majority of cases. We are puzzled by this and wonder whether there is some conceptual reason for it. We present a substantial amount of data relating to the curves investigated.