BIASES IN THE SHANKS–RÉNYI PRIME NUMBER RACE

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ABSTRACT. Rubinstein and Sarnak investigated systems of inequalities of the form $\pi(x;q,a_1) > \cdots > \pi(x;q,a_r)$, where $\pi(x;q,b)$ denotes the number of primes up to x that are congruent to b mod q. They showed, under standard hypotheses on the zeros of Dirichlet L-functions mod q, that the set of positive real numbers x for which these inequalities hold has positive (logarithmic) density $\delta_{q;a_1,\dots,a_r} > 0$. They also discovered the surprising fact that a certain distribution associated with these densities is not symmetric under permutations of the residue classes a_j in general, even if the a_j are all squares or all nonsquares mod q (a condition necessary to avoid obvious biases of the type first observed by Chebyshev). This asymmetry suggests, contrary to prior expectations, that the densities $\delta_{q;a_1,\dots,a_r}$ themselves vary under permutations of the a_j.

Here we derive (under the hypotheses used by Rubinstein and Sarnak) a general formula for the densities $\delta_{q;a_1,\dots,a_r}$, and we use this formula to calculate many of these densities when $q \le 12$ and $r \le 4$. For the special moduli $q=8$ and $q=12$, and for $\{a_1,a_2,a_3\}$ a permutation of the nonsquares $\{3,5,7\}$ mod 8 and $\{5,7,11\}$ mod 12, respectively, we rigorously bound the error in our calculations, thus verifying that these densities are indeed asymmetric under permutation of the a_j. We also determine several situations in which the densities $\delta_{q;a_1,\dots,a_r}$ remain unchanged under certain permutations of the a_j, and some situations in which they are provably different.