THE SIZE OF THE FUNDAMENTAL SOLUTIONS OF CONSECUTIVE PELL EQUATIONS

MICHAEL J. JACOBSON, JR. AND HUGH C. WILLIAMS

ABSTRACT. Let D be a positive integer such that D and $D-1$ are not perfect squares; denote by X_0, Y_0, X_1, Y_1 the least positive integers such that $X_0^2 - (D-1)Y_0^2 = 1$ and $X_1^2 - DY_1^2 = 1$; and put $\rho(D) = \log X_1 / \log X_0$. We prove here that $\rho(D)$ can be arbitrarily large. Indeed, we exhibit an infinite family of values of D for which $\rho(D) \gg D^{1/6}/\log D$. We also provide some heuristic reasoning which suggests that there exists an infinitude of values of D for which $\rho(D) \gg \sqrt{D} \log\log D / \log D$, and that this is the best possible result under the Extended Riemann Hypothesis. Finally, we present some numerical evidence in support of this heuristic.