Fano Hypersurfaces in Weighted Projective 4-Spaces

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CONTENTS

We determine the full list of anticanonically embedded quasismooth Fano hypersurfaces in weighted projective 4-spaces. There are 48 infinite series and 4442 sporadic examples. In particular, the Reid–Fletcher list of 95 types of anticanonically embedded quasismooth terminal Fano threefolds in weighted projective 4-spaces is complete.

We also prove that many of these Fano hypersurfaces admit a Kähler–Einstein metric, and study the nonexistence of tigers on these Fano 3-folds.

Finally, we prove that there are only finitely many families of quasismooth Calabi–Yau hypersurfaces in weighted projective spaces of any given dimension. This implies finiteness for various families of general type hypersurfaces.

1. INTRODUCTION

A *Fano variety* is a projective variety whose anticanonical class is ample. A 2-dimensional Fano variety is called a *del Pezzo surface*. In higher dimensions, attention originally centered on smooth Fano 3-folds, but singular Fano varieties are also of considerable interest in connection with the minimal model program. The existence of Kähler–Einstein metrics on Fano varieties has also been explored; see [Bourguignon 1997] for a summary of the main results. Here again the smooth case is of primary interest, but Fano varieties with quotient singularities and their orbifold metrics have also been studied.

In a given dimension there are only finitely many families of smooth Fano varieties [Campana 1991; Nadel 1991; Kollár et al. 1992b], but very little is known about them in dimensions 4 and up. By allowing singularities, infinitely many families appear and their distribution is very poorly understood.

For a natural experimental testing ground, we turn to hypersurfaces and complete intersections in weighted projective spaces. These varieties can be written down rather explicitly, but they still provide many more examples than ordinary projective

Partial financial support for this research was provided by the NSF under grant number DMS-9970855.

1058-6458/2001 $0.50 per page

spaces. Experimental lists of certain three-dimensional complete intersections were compiled in [Iano-Fletcher 1989]. In connection with Kähler–Einstein metrics, the 2-dimensional cases were first investigated in [Demailly and Kollár 1999] and later in [Johnson and Kollár 2000].

It is also of interest to study Calabi–Yau hypersurfaces and hypersurfaces of general type in weighted projective spaces. Some lists with terminal singularities appear in [Iano-Fletcher 1989].

The aim of this paper is threefold.

First, we determine the complete list of anticanonically embedded quasismooth Fano hypersurfaces in weighted projective 4-spaces. There are 48 infinite series and 4442 sporadic examples (Theorem 2.2). As a consequence we obtain that the Reid–Fletcher list [Iano-Fletcher 1989, II.6.6] of 95 types of anticanonically embedded quasismooth terminal Fano threefolds in weighted projective 4-spaces is complete (Corollary 2.5).

Second, we prove that many of these Fano hypersurfaces admit a Kähler–Einstein metric (Corollary 3.4). We also study the nonexistence of *tigers* on these Fano 3-folds (the colorful terminology comes from [Keel and McKernan 1999]).

Third, we prove that there are only finitely many families of quasismooth Calabi–Yau hypersurfaces in weighted projective spaces of any given dimension (Theorem 4.1). This implies finiteness for various families of general type hypersurfaces (Corollary 4.3).

Definition 1.1. For positive integers a_i we denote by $\mathbb{P}(a_0,\dots,a_n)$ the *weighted projective n-space* with weights $a_0,\dots,a_n$. (See [Dolgachev 1982] or [Iano-Fletcher 1989] for basic definitions and results on weighted projective spaces.) We always assume that any n of the a_i are relatively prime. We frequently write $\mathbb{P}$ to denote a weighted projective n-space if the weights are irrelevant or clear from the context. We use $x_0,\dots,x_n$ to denote the corresponding weighted projective coordinates. We denote by

$$P_i \in \mathbb{P}(a_0,\dots,a_n)$$

the point all of whose coordinates are 0 except for the i-th one. These points are sometimes called the *vertices* of the weighted projective space. (They are uniquely determined if none of the a_i divides any other.) The affine chart where $x_i \neq 0$ can be written as

$$\mathbb{C}^n(y_0,\dots,\widehat{y_i},\dots,y_n)/\mathbb{Z}_{a_i}(a_0,\dots,\widehat{a_i},\dots,a_n). \qquad (1\text{–}1)$$

(Here and later $\widehat{}$ denotes an omitted coordinate.) This shorthand denotes the quotient of $\mathbb{C}^n$ by the action

$$(y_0,\dots,\widehat{y_i},\dots,y_n) \mapsto (\varepsilon^{a_0}y_0,\dots,\widehat{y_i},\dots,\varepsilon^{a_n}y_n),$$

where ε is a primitive a_i-th root of unity. The identification is given by $y_j^{a_i} = x_j^{a_i}/x_i^{a_j}$. (1–1) are called the *orbifold charts* on $\mathbb{P}(a_0,\dots,a_n)$.

For any i, $\mathbb{P}(a_0,\dots,a_n)$ has an index a_i quotient singularity at P_i. For any $i<j$, if

$$\gcd(a_0,\dots,\widehat{a_i},\dots,\widehat{a_j},\dots,a_n) > 1,$$

then $\mathbb{P}(a_0,\dots,a_n)$ has a quotient singularity along $(x_i = x_j = 0)$. These give all the codimension 2 singular subsets of $\mathbb{P}(a_0,\dots,a_n)$.

For every $m \in \mathbb{Z}$ there is a rank 1 sheaf $\mathcal{O}_{\mathbb{P}}(m)$ which is locally free only if $a_i | m$ for every i. A basis of the space of sections of $\mathcal{O}_{\mathbb{P}}(m)$ is given by all monomials in $x_0,\dots,x_n$ with weighted degree m. Thus $\mathcal{O}_{\mathbb{P}}(m)$ may have no sections for some $m>0$.

2. ANTICANONICALLY EMBEDDED QUASISMOOTH FANO HYPERSURFACES

Let $X \in |\mathcal{O}_{\mathbb{P}}(m)|$ be a hypersurface of degree m. The adjunction formula

$$K_X \cong \mathcal{O}_{\mathbb{P}}(K_{\mathbb{P}} + X)|_X \cong \mathcal{O}_{\mathbb{P}}(m - (a_0 + \cdots + a_n))|_X$$

holds if X does not contain any of the codimension 2 singular subsets. If this condition is satisfied then X is a Fano variety iff $m < a_0 + \cdots + a_n$. Frequently the most interesting cases are when m is as large as possible. Thus we consider the case $X_d \in |\mathcal{O}_{\mathbb{P}}(d)|$ for $d = a_0 + \cdots + a_n - 1$. Such an X is called *anticanonically embedded.*

In most cases, all hypersurfaces of a given degree d are singular and pass through some of the vertices P_i. In these cases the best one can hope is that a general hypersurface X_d is smooth in the orbifold sense, called *quasismooth.* At the vertex P_i this means that the preimage of X_d in the orbifold chart $\mathbb{C}^n(y_0,\dots,\widehat{y_i},\dots,y_n)$ is smooth. In terms of the monomials of degree d this is equivalent to saying that

$$\text{For every } i \text{ there is a } j \text{ and a monomial } x_i^{m_i} x_j \text{ of degree } d. \tag{2–1}$$

$j = i$ is allowed, corresponding to the case when the general X_d does not pass through P_i. The condition that X_d does not contain any of the singular codimension 2 subsets is equivalent to

$$\text{If } \gcd(a_0, \dots, \widehat{a_i}, \dots, \widehat{a_j}, \dots, a_n) > 1 \text{ there is a monomial of degree } d \text{ not involving } x_i, x_j. \tag{2–2}$$

For $n \geq 3$, these are the two most important special cases of the general quasismoothness condition:

$$\text{For every } I \subset \{0, \dots, n\} \text{ there is an injection } e : I \hookrightarrow \{0, \dots, n\} \text{ and monomials } x_{e(i)} \textstyle\prod_{j \in I} x_j^{m_{ij}} \text{ of degree } d \text{ for every } i \in I. \tag{2–3}$$

Remark 2.1. The quasi-smoothness condition in [Iano-Fletcher 1989, I.5.1] says that

$$\text{For every } I \subset \{0, \dots, n\} \text{ either (2–3) holds or there is a monomial } \textstyle\prod_{j \in I} x_j^{b_j} \text{ of degree } d. \tag{2–3$'$}$$

The two versions are, however, equivalent. We prove this by induction on $|I|$. Indeed, assume that there is a monomial $\prod_{j \in I} x_j^{b_j}$ of degree d and let $I' \subset I$ be all the indices which are involved in at least one such monomial. By induction (2–3) holds for $I \setminus I'$, giving monomials $x_{e(i)} \prod_{j \in I \setminus I'} x_j^{b_{ij}}$ for $i \in I \setminus I'$. By assumption these $e(i)$ are not in I, so we can choose $I' \cup e(I \setminus I')$ as the image of $e : I \to \{0, \dots, n\}$. (A suitable reordering of the values of e may be necessary.)

The computer searches carried out in connection with [Iano-Fletcher 1989; Demailly and Kollár 1999] looked at values of a_i in a certain range to find the a_i satisfying the constraints (2–1) to (2–3). This approach starts with the a_i and views (2–1) to (2–3) as linear equations in the unknown nonnegative integers m_i, m_{ij}. In the cases studied in those two works these searches seemed exhaustive. Aside from one series of examples, the computers produced solutions for low values of the a_i and then did not find any more as the range of the allowable values was extended. This of course does not ever lead to a proof that the lists were complete.

A similar search for quasismooth Fano hypersurfaces in weighted projective 4-spaces is quite time consuming. With some reasonably large bounds, say a_i a few hundred, the programs run for days and they produce a few thousand examples. We were unable to isolate the series from these lists. The finiteness of the sporadic examples was also unclear. While there were few examples with large $\min\{a_i\}$, there did not seem to be an end to the list. Indeed, it is quite unlikely that any systematic search of this kind could have discovered the example with the largest a_0:

$$(407, 547, 5311, 12528, 18792)$$

with monomials

$$x_4^2,\ x_3^3,\ x_1^{59} x_2,\ x_0 x_2^7,\ x_0^{91} x_1,$$

or the beautiful pair of sporadic examples with the largest a_4:

$$(223, 9101, 46837, 112320, 168480)$$

with monomials

$$x_4^2,\ x_3^3,\ x_1 x_2^7,\ x_0 x_1^{37},\ x_0^{1301} x_2,$$

and

$$(253, 7807, 48101, 112320, 168480)$$

with monomials

$$x_4^2,\ x_3^3,\ x_1^{37} x_2,\ x_0 x_2^7,\ x_0^{1301} x_1.$$

The biggest values of a_0 are of some interest in connection with the conjectures of [Shokurov 2000, 1.3].

Next we describe the computer programs that led to the list of anticanonically embedded quasismooth Fano hypersurfaces in weighted projective 4-spaces. The programs, written in C, are available at the address www.math.princeton.edu/~jmjohnso.

2A. Preliminary Steps

In order to find all solutions, we change the point of view. We consider (2–1) to be the main constraint with coefficients m_i and unknowns a_i. The corresponding equations can then be written as a linear system

$$(M+J+U)(a_0\ a_1\ a_2\ a_3\ a_4)^t = (-1\ \ -1\ \ -1\ \ -1\ \ -1)^t \tag{2–4}$$

where $M = \operatorname{diag}(m_0, m_1, m_2, m_3, m_4)$ is a diagonal matrix, J is a matrix with all entries -1 and U is a matrix where each row has 4 entries $= 0$ and one entry $= 1$. The main advantage is that some of the m_i can be bounded a priori. Assume for simplicity that $a_0 \leq a_1 \leq \cdots \leq a_4$.

Consider for instance m_4. The relevant equation is

$$m_4 a_4 + a_{e(4)} = a_0 + a_1 + a_2 + a_3 + a_4 - 1.$$

Since a_4 is the biggest, we get right away that $1 \leq m_3 \leq 3$. Arguing inductively with some case analysis we obtain

$$3 \leq m_2 \leq 16, \quad 2 \leq m_3 \leq 6, \quad 1 \leq m_4 \leq 3 \qquad (2\text{–}5)$$

Thus we have only finitely many possibilities for the matrix U and the numbers m_2, m_3, m_4. Fixing these values, we obtain a linear system

$$(M+J+U)(a_0\ a_1\ a_2\ a_3\ a_4)^t = (\,-1\ -1\ -1\ -1\ -1)^t,$$

where the only variable coefficients are m_0, m_1 in the upper left corner of M. Solving these formally we obtain

$$a_0 = \frac{\alpha_0 m_1 + \beta_0}{\gamma_2 m_0 m_1 + \gamma_0 m_0 + \gamma_1 m_1 + \delta},$$
$$a_1 = \frac{\alpha_1 m_0 + \beta_1}{\gamma_2 m_0 m_1 + \gamma_0 m_0 + \gamma_1 m_1 + \delta}.$$

where the $\alpha_i, \beta_i, \gamma_i, \delta$ depend only on U, m_2, m_3, m_4.

We distinguish 3 cases. The first one is the main source of examples. Cases 2 and 3 are anomalies from the point of view of our method. In both cases we ended up experimentally finding strong restrictions on the a_i. Even with hindsight we do not know how to prove these a priori.

Case 1: $\gamma_2 \neq 0$. In this case the absolute value of

$$\frac{\alpha_0 m_1 + \beta_0}{\gamma_2 m_0 m_1 + \gamma_0 m_0 + \gamma_1 m_1 + \delta}$$

goes to zero as m_0, m_1 go to infinity. It is not hard to write down the precise condition and a computer check shows that

$$\frac{\alpha_0 m_1 + \beta_0}{\gamma_2 m_0 m_1 + \gamma_0 m_0 + \gamma_1 m_1 + \delta} \geq 1$$

implies

$$\min\{m_0, m_1\} \leq 83.$$

Case 2: $\gamma_2 = 0$ and $\gamma_0\gamma_1 \neq 0$. It turns out that if this holds then $\gamma_0\gamma_1 > 0$ and a_0, a_1 are bounded by 8 for $\min\{m_0, m_1\} \geq 36$. Moreover, the 3 linear forms

$$\alpha_0 m_1 + \beta_0,\ \alpha_1 m_0 + \beta_1,\ \gamma_0 m_0 + \gamma_1 m_1 + \delta$$

are dependent. This implies that

$$\alpha_1\gamma_1 a_0 + \alpha_0\gamma_0 a_1 = \alpha_0\alpha_1.$$

A computer search shows that this is possible only if $a_0 = a_1 = 1$.

Case 3: $\gamma_2 = 0$ and $\gamma_0\gamma_1 = 0$. It turns out that one of $\alpha_0 m_1 + \beta_0, \alpha_1 m_0 + \beta_1$ equals $\gamma_0 m_0 + \gamma_1 m_1 + \delta$. Thus $a_0 = 1$ or $a_1 = 1$. Moreover, we also see by explicit computation that one of the following holds:

$$a_2 = a_3 = a_4, \quad a_2 = a_3 = \tfrac{1}{2} a_4, \quad a_2 = \tfrac{1}{2} a_3 = \tfrac{1}{3} a_4.$$

2B. Main Computer Search

Here we discuss the main case when, in addition to the inequalities (2–5) we also assume that $3 \leq m_1 \leq 83$. In this case the system (2–4) reduces to a single unknown m_0. This is very similar to the 4-variable case discussed in [Johnson and Kollár 2000].

We solve formally for a_0 to get

$$a_0 = \frac{\gamma_0}{m_0\alpha + \beta}$$

where α, β, γ_0 depend only on U and m_1, m_2, m_3, m_4. If $\alpha \neq 0$ then we get a bound on m_0 too, and we are down to finitely many possibilities all together. There are 403,455 cases of this. The resulting solutions need considerable cleaning up. Many of them occur multiply and we also have to check the other conditions, (2–2) and (2–3). Discarding repetitions, we get 15757 cases, out of which 4594 are quasismooth.

If $\alpha = 0$ then we get a series solution where the a_i are linear functions of a variable m_0. There are 550,122 cases of this. Here the main difficulty is that the program does not produce the series in a neat form. Usually one series is put together out of many pieces according to some congruence condition.

2C. Additional Cases

Assume first that we are in Case 2 of Section 2A. Since $a_0 = a_1 = 1$, the numbers a_1, a_2, a_3, a_4 and $d = a_1 + a_2 + a_3 + a_4$ satisfy the numerical conditions (2–3). This leads to a lower dimensional problem which is easy to solve.

Case 3 of Section 2A is even easier. We get solutions of the form

$$(1, a, b, b, b), \quad (1, a, b, b, 2b), \quad \text{or} \quad (1, a, b, 2b, 3b).$$

Applying (2–3) to $I = \{2,3,4\}$ gives that $b|a$. Thus b divides all but one of the weights, so $b = 1$. This implies that $a \leq 6$. At any case, all these appear also under Case 2 of Section 2A.

At the end we obtain our first main result:

Theorem 2.2. *The following is a complete list of anticanonically embedded quasismooth Fano hypersurfaces in weighted projective* 4*-spaces*:

1. 48 *infinite series of the form*
   $$X_{2k(b_1+b_2+b_3)} \subset \mathbb{P}\big(2,\, kb_1,\, kb_2,\, kb_3,\, k(b_1+b_2+b_3)-1\big)$$
   for $k = 1, 3, 5, \ldots$. *The occurring* 3*-tuples* b_1, b_2, b_3 *are described in Remark* 2.3.
2. 4442 *sporadic examples whose list is available at* www.math.princeton.edu/~jmjohnso.

2D. An Error Check

We wrote a program that looked at all 5-tuples satisfying

$$a_0 \leq 100,\ a_1 \leq 200,\ a_2 \leq 200,\ a_3 \leq 400,\ a_4 \leq 600.$$

The program ran for 4 days and produced 3610 quasismooth examples, all in complete agreement with the correspondingly truncated list of 4442 sporadic examples.

Remark 2.3. It turns out that a 3-tuple b_1, b_2, b_3 appears in Theorem 2.2(1) iff $|-2K|$ of $\mathbb{P}(b_1, b_2, b_3)$ has a quasismooth member. The list of these is implicit in Reid's list of 95 families of singular K3 surfaces in weighted projective 3-spaces. In [Iano-Fletcher 1989, II.3.3] they correspond to those quadruplets (b_1, b_2, b_3, b_4) for which $b_4 = b_1 + b_2 + b_3$. Our 48 3-tuples occured explicitly in [Yonemura 1990; Tomari 2000] in connection with the study of simple K3 singularities of multiplicity 2.

One direction of this observation is easy to establish in all dimensions.

Lemma 2.4. *Assume that* $|-2K|$ *of* $\mathbb{P}(b_1, \ldots, b_n)$ *has a quasismooth member. Then the general anticanonically embedded Fano hypersurface in*

$$\mathbb{P}(2, kb_1, \ldots, kb_n, k(b_1 + \cdots + b_n) - 1)$$

is quasismooth for $k = 1, 3, 5, \ldots$.

We conjecture that conversely, every infinite series is of this form. It is interesting that every quasismooth hypersurface in $\mathbb{P}(2, kb_1, \ldots, kb_n, k(b_1+\cdots+b_n)-1)$ has a singular set of codimension 2. Thus the preceding conjecture would imply that for every $n \geq 4$ there are only finitely many anticanonically embedded quasismooth Fano hypersurfaces with isolated singularities in weighted projective n-spaces.

It is not hard to check which of the above Fano threefolds have terminal singularities. The families in Theorem 2.2(1) always have nonisolated singularities, and for the remaining cases the conditions of [Iano-Fletcher 1989, II.4.1] work. As a consequence, we obtain the following corollary. (Reid informed us that he also has an unpublished proof of this.)

Corollary 2.5. *The Reid–Fletcher list of* 95 *families of anticanonically embedded quasismooth terminal Fano threefolds in weighted projective* 4*-spaces* [Iano-Fletcher 1989, II.6.6] *is complete.* □

3. KÄHLER–EINSTEIN METRICS AND THE NONEXISTENCE OF TIGERS

Next we study the existence of Kähler–Einstein metrics and the nonexistence of tigers on our Fano hypersurfaces. After some definitions we recall the criterion established in [Johnson and Kollár 2000]. In the case of Kähler–Einstein metrics this in turn relies on earlier work of [Nadel 1990; Demailly and Kollár 1999].

Definition 3.1. Let X be a normal variety and D a $\mathbb{Q}$-divisor on X. Assume for simplicity that K_X and D are both $\mathbb{Q}$-Cartier. Let $g : Y \to X$ be any proper birational morphism, Y smooth. Then there is a unique $\mathbb{Q}$-divisor $D_Y = \sum e_i E_i$ on Y such that

$$K_Y + D_Y \equiv g^*(K_X + D) \quad \text{and} \quad g_* D_Y = D.$$

We say that (X, D) is *klt* if $e_i > -1$ for all g and i. We call (X, D) *log canonical* if $e_i \geq -1$ for all g and i. See [Kollár and Mori 1998, Section 2.3], for instance, for a detailed introduction.

Definition 3.2 [Keel and McKernan 1999]. Let X be a normal variety. A *tiger* on X is an effective $\mathbb{Q}$-divisor D such that $D \equiv -K_X$ and (X, D) is not klt. As illustrated in [Keel and McKernan 1999], the tigers carry important information about birational transformations of log del Pezzo surfaces. They are expected to play a similar role in higher dimensions.

Proposition 3.3 [Johnson and Kollár 2000]. *Let $X_d \subset \mathbb{P}(a_0, \dots, a_n)$ be a quasismooth hypersurface of degree $d = a_0 + \dots + a_n - 1$.*

1. *X does not have a tiger if $d \le a_0 a_1$.*
2. *X admits a Kähler–Einstein metric if*

$$d < \frac{n}{n-1}\, a_0 a_1. \qquad \square$$

Corollary 3.4. *Of the sporadic series of quasismooth Fano hypersurfaces mentioned in Theorem 2.2(2), there are 1605 types where none of the members have a tiger and 1936 types where every member admits a Kähler–Einstein metric. This information is contained in the list of Theorem 2.2(2).*

4. CALABI–YAU HYPERSURFACES

Finally we study the case of Calabi–Yau hypersurfaces and hypersurfaces of general type in weighted projective spaces. For these cases there are finiteness results in all dimensions. The key part is the case of Calabi–Yau hypersurfaces.

Theorem 4.1. *For any n there are only finitely many types of quasismooth hypersurfaces with trivial canonical class in weighted projective spaces*

$$\mathbb{P}(a_0, \dots, a_n).$$

Proof. As in the Fano case, first we look at those hypersurfaces which are quasismooth at the vertices of $\mathbb{P}(a_0, \dots, a_n)$. This condition is equivalent to a linear system of equations

$$(M + J + U)(a_0, \dots, a_n)^t = (0, \dots, 0)^t \qquad (4\text{–}1)$$

where $M = \operatorname{diag}(m_0, \dots, m_n)$ is a diagonal matrix, J is a matrix with all entries -1 and U is a matrix where each row has n entries $= 0$ and one entry $= 1$. In the geometric setting the m_i and the a_i are positive integers, but it will be convenient to allow the a_i to be positive real numbers. By the homogenity of the system we may assume that $\sum a_i = 1$.

Assume now that we have an infinite sequence of solutions $(a_0(t), \dots, a_n(t))$ where a priori $M(t)$, $J(t)$, $U(t)$ also vary with t. By passing to a subsequence we may assume that $J(t)$ and $U(t)$ are constant and each $a_i(t)$ converges to a value A_i. Thus we can write $a_i(t) = A_i + c_i(t)$ where $\lim_{t\to\infty} c_i(t) = 0$, $\sum_i A_i = 1$ and $\sum_i c_i(t) = 0$. By passing to a subsequence and rearranging, we can also assume that $I := \{i : c_i(t) < 0\}$ is independent of t and that $A_0/(-c_0(t))$ is the smallest positive number among $\{A_i/(-c_i(t)) : i \in I\}$. The quasismoothness condition at the vertex P_0 translates into $m_0(t)a_0(t) + a_j(t) = 1$. We have $\lim_{t\to\infty} a_0(t) = A_0 > 0$ since $c_0(t) < 0$, hence $m_0(t)$ is bounded from above. Thus we may assume that $m_0(t) = m_0$ is constant and

$$\lim_{t\to\infty} m_0 c_0(t) + c_j(t) = 0.$$

$m_0 a_0(t) + a_j(t) = 1$ is equivalent to

$$[m_0 A_0 + A_j] + [m_0 c_0(t) + c_j(t)] = 1. \qquad (4\text{–}2)$$

By the above considerations, (4–2) splits into two equations

$$m_0 A_0 + A_j = 1 \quad \text{and} \quad m_0 c_0(t) + c_j(t) = 0. \qquad (4\text{–}3)$$

Using $\sum_i c_i(t) = 0$ and the second equation in (4–3) we obtain that

$$\sum_{i\in I} c_i(t) = -\sum_{i\notin I} c_i(t) \le -c_j(t) = m_0 c_0(t). \qquad (4\text{–}4)$$

Multiplying by $A_0/c_0(t)$ and using the special choice of $A_0/c_0(t)$ we get that

$$m_0 A_0 \le \sum_{i\in I} c_i(t) \frac{A_0}{c_0(t)} \le \sum_{i\in I} A_i. \qquad (4\text{–}5)$$

Combining with the first equation of (4–3) we get that

$$1 = m_0 A_0 + A_j \le A_j + \sum_{i\in I} A_i \le \sum_{i=0}^{n} A_i = 1. \qquad (4\text{–}6)$$

This implies that all inequalities in (4–4), (4–5) and (4–6) are equalities. Hence $A_k, c_k(t)$ are zero for $k \notin I \cup \{j\}$. By assumption the $a_k(t)$ are positive, so $I \cup \{j\} = \{0, \dots, n\}$. Moreover, the ratios $A_i/c_i(t)$ are all the same for $i \in I$.

These imply that, up to rearranging the indices, the $a_i(t)$ are of the form

$$\big(A_0(1-c(t)), \dots, A_{n-1}(1-c(t)), A_n + c(t) \textstyle\sum_{i=0}^{n-1} A_i\big).$$

Consider next the equation

$$m_n\big(A_n + c(t)\textstyle\sum_{i=0}^{n-1} A_i\big) + A_j(1-c(t)) = 1,$$

where for notational simplicity we allow $j = -1$ with $A_{-1} = 0$. For large t this implies that $\sum_{i=0}^{n-1} A_i = A_j$, which is not possible for $n \ge 2$. Thus $A_n = 0$ and the solutions become

$$\big(A_0(1 - c(t)), \dots, A_{n-1}(1 - c(t)), c(t)\big) \qquad (4\text{–}7)$$

where $\sum_{i=0}^{n-1} A_i = 1$. To get quasismoothness, we need to understand all monomials of degree $\sum a_i$, which amounts to finding all integer solutions of $\sum b_i a_i = 1$. In our case, for large t there are no solutions with $b_n = 0$ which means that every hypersurface of degree $\sum a_i$ contains the hyperplane $(x_n = 0)$, hence they are all reducible. Thus the solutions (4–7) do not correspond to quasismooth hypersurfaces. $\square$

Remark 4.2. The solutions (4–7) do correspond to interesting series of singularities. Namely, for every integer solution of $\sum_{i=0}^{n-1} 1/m_i = 1$ they give an infinite series of singularities

$$(x_0^{m_0} + \cdots + x_{n-1}^{m_{n-1}} + x_n^k)x_n = 0 \subset \mathbb{A}^{n+1}$$

for $k = 1, 2, \ldots$. These singularities are weighted homogeneous and semi log canonical (see [Kollár et al. 1992a, 16.2.1] for the definition) but not isolated. By adding a general higher degree term, we get isolated log canonical singularities.

Corollary 4.3. *For any n and $k > 0$ there are only finitely many families of quasismooth hypersurfaces $X \subset \mathbb{P}(a_0, \ldots, a_n)$ such that $\omega_X \cong \mathcal{O}_X(k)$.*

Proof. Assume that

$$X = (F(x_0, \ldots, x_n) = 0) \subset \mathbb{P}(a_0, \ldots, a_n)$$

is quasismooth of degree d and $\omega_X \cong \mathcal{O}_X(k)$. Then

$$X^* := (F(x_0, \ldots, x_n) + x_{n+1}^d + \cdots + x_{n+k}^d = 0) \subset \mathbb{P}(a_0, \ldots, a_n, \underbrace{1, \ldots, 1}_{k \text{ times}})$$

is also quasismooth of degree d and $\omega_X \cong \mathcal{O}_X$. Thus we are done by Theorem 4.1. $\square$

Remark 4.4. The finiteness result (Corollary 4.3) is in accordance with the conjectures [Kollár et al. 1992a, 18.16]. On the other hand, Theorem 4.1 seems to be a more special finiteness assertion.

ACKNOWLEDGEMENT

We thank J. McKernan for helpful comments and references.

ELECTRONIC AVAILABILITY

The computer programs that led to the list of anticanonically embedded quasismooth Fano hypersurfaces in weighted projective 4-spaces can be found at www.math.princeton.edu/~jmjohnso, together with the list itself.

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Received September 13, 2000; accepted October 10, 2000