Hecke Eigenforms in the Cohomology of Congruence Subgroups of $\mathrm{SL}(3,\mathbb{Z})$

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CONTENTS

We list here Hecke eigenvalues of several automorphic forms for congruence subgroups of $\mathrm{SL}(3;\mathbb{Z})$. To compute such tables, we describe an algorithm that combines techniques developed by Ash, Grayson and Green with the Lenstra–Lenstra–Lovász algorithm. With our implementation of this new algorithm we were able to handle much larger levels than those treated by Ash, Grayson and Green and by Top and van Geemen in previous work. Comparing our tables with results from computations of Galois representations, we find some new numerical evidence for the conjectured relation between modular forms and Galois representations.

1. INTRODUCTION

It is well known that one can associate Galois representations to Hecke eigenforms on congruence subgroups of $\mathrm{SL}(2,\mathbb{Z})$. It has been conjectured, as part of the Langlands program, that one can do the same for $\mathrm{SL}(3,\mathbb{Z})$ and in [van Geemen and Top 1994] we provided some evidence for this.

For any prime number p not dividing the level of the modular form/conductor of the Galois representation, one defines a local L-factor that in the $\mathrm{SL}(3,\mathbb{Z})$ case has the form

$$(1 - a_p p^{-s} + \bar{a}_p p^{1-2s} - p^{3-3s})^{-1}.$$

Here a_p is the eigenvalue of a Hecke operator E_p on the eigenform/trace of a Frobenius element at p in a 3-dimensional $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ representation and $\bar{a}_p$ is its complex conjugate.

The experimental evidence consists of an eigenform and a Galois representation with the same L-factors (that is, a_p's) for small primes.

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It is actually rather easy to find candidate Galois representations in the étale cohomology of surfaces. One family of such surfaces was discussed in [van Geemen and Top 1994] (see also our Section 3.8); two other families are constructed in [van Geemen and Top 1995]. The (computational) problem is rather to find Hecke eigenforms. (We hasten to add that none of the authors is an expert on modular forms, our interests were mainly in Galois representations and/or Algebraic Geometry and/or computational aspects).

We list in this paper some Hecke eigenvalues of several automorphic forms for congruence subgroups of $\mathrm{SL}(3,\mathbb{Z})$. By combining the methods from [Ash et al. 1984] with the Lenstra–Lenstra–Lovász algorithm, we were able to handle much larger levels than was the case in [Ash et al. 1984] and [van Geemen and Top 1994]. Comparing these tables with results from computations of Galois representations, we find further evidence for the conjectured relation between modular forms and Galois representations; see Theorem 3.9.

In the first section we recall the methods from [Ash et al. 1984] to determine the spaces of automorphic forms in terms of group cohomology and we discuss some computational aspects. Since we do not know a formula that gives the dimensions of these spaces (as function of the level of the form), we give a table with the results we found (see Table 1 in Section 3). One would also like to have a table that lists the dimension of the cuspidal part, but (with exception of the prime level case), no practical criterion that singles out the cuspidal forms is known to us.

Next we recall how to compute the action of the Hecke operators on the space of modular forms. In view of properties of cusp forms and the examples of Galois representations we know, we are mostly interested in Hecke eigenvalues that lie in CM-fields and that are small (so they satisfy the Ramanujan hypothesis). The selection criterion upon which our tables are based is given in Section 2.6.

In contrast with the $\mathrm{SL}(2,\mathbb{Z})$ case, one finds very few cusp forms of prime level for $\mathrm{SL}(3,\mathbb{Z})$. In fact the only prime levels ≤ 337 with cusp forms are the levels 53, 61, 79, 89 and 223. The CM-fields generated by the eigenvalues were imaginary quadratic with exception of the case of level 245 where we found a degree 4 extension of $\mathbb{Q}$.

2. MODULAR FORMS AND HECKE OPERATORS

2.1. We briefly recall how to compute the modular forms under consideration. The standard reference is [Ash et al. 1984].

In the case of $\mathrm{SL}(2,\mathbb{Z})$, the space $S_2(\Gamma)$ of holomorphic modular forms of weight two on a congruence subgroup Γ is a subspace of the cohomology group $H^1(\Gamma,\mathbb{C})$. This generalizes as follows.

2.2. For $N \geq 1$, define the subgroup

$$\Gamma_0(N) \subset \mathrm{SL}(3,\mathbb{Z})$$

to consist of all (a_{ij}) such that $a_{21} \equiv 0 \bmod N$ and $a_{31} \equiv 0 \bmod N$. This group has our primary interest. It is neither normal in $\mathrm{SL}(3,\mathbb{Z})$ nor torsion-free. To compute its cohomology, we introduce the finite set

$$\mathbb{P}^2(\mathbb{Z}/N) = \Pi/(\mathbb{Z}/N)^\times,$$

where $\Pi \subset (\mathbb{Z}/N)^3$ is the set consisting of all $(\bar{x},\bar{y},\bar{z})$ such that $\bar{x}\mathbb{Z}/N + \bar{y}\mathbb{Z}/N + \bar{z}\mathbb{Z}/N = \mathbb{Z}/N$. When the elements of $\mathbb{P}^2(\mathbb{Z}/N)$ are viewed as column vectors, there is a natural left action of $\mathrm{SL}(3,\mathbb{Z})$ on $\mathbb{P}^2(\mathbb{Z}/N)$. This action is transitive, and the stabilizer of $(\bar{1}:\bar{0}:\bar{0})$ equals $\Gamma_0(N)$. Therefore

$$\mathrm{SL}(3,\mathbb{Z})/\Gamma_0(N) \cong \mathbb{P}^2(\mathbb{Z}/N).$$

Under this correspondence, an element of $\mathrm{SL}(3,\mathbb{Z})$ is mapped to its first column viewed as homogeneous coordinates modulo N.

The dual of the vector space $H^3(\Gamma_0(N),\mathbb{C})$ is $H_3(\Gamma_0(N),\mathbb{C})$ and it can be computed as follows:

Theorem 2.3 [Ash et al. 1984, Thm. 3.2, Prop. 3.12]. *There is a canonical isomorphism between*

$$H_3(\Gamma_0(N),\mathbb{C})$$

and the vector space of mappings $f : \mathbb{P}^2(\mathbb{Z}/N) \to \mathbb{C}$ *that satisfy*

1. $f(\bar{x}{:}\bar{y}{:}\bar{z}) = -f(-\bar{y}{:}\bar{x}{:}\bar{z})$,
2. $f(\bar{x}{:}\bar{y}{:}\bar{z}) = f(\bar{z}{:}\bar{x}{:}\bar{y})$,
3. $f(\bar{x}{:}\bar{y}{:}\bar{z}) + f(-\bar{y}{:}\bar{x}-\bar{y}{:}\bar{z}) + f(\bar{y}-\bar{x}{:}-\bar{x}{:}\bar{y}) = 0$.

2.4. For any $\alpha \in \mathrm{GL}(3,\mathbb{Q})$ one has a ($\mathbb{C}$-linear) Hecke operator

$$T_\alpha : H^3(\Gamma_0(N),\mathbb{C}) \longrightarrow H^3(\Gamma_0(N),\mathbb{C}),$$

which defines an adjoint operator T_α^* on the dual space $H_3(\Gamma_0(N),\mathbb{C})$. We now explain how to determine this adjoint.

Let

$$\Gamma_0(N)\,\alpha\,\Gamma_0(N) = \coprod_i \beta_i\,\Gamma_0(N)$$

be the decomposition of the double coset in a (finite) disjoint union of left cosets. Such β_i's can be found in [Ash et al. 1984, p. 430].

First we need the definition of modular symbol (compare [Ash and Rudolph 1979], where however column rather than row vectors are used). These modular symbols are elements of $H_1(T_3,\mathbb{Z})$, with T_3 the Tits building for $\mathrm{SL}(3,\mathbb{Q})$, and they give rise to cohomology classes in $H^3(\Gamma_0(N),\mathbb{C})$. For the purposes of this paper it however suffices to know the following. For three nonzero row vectors $q_1, q_2, q_3 \in \mathbb{Q}^3$ we define a modular symbol

$$[Q] = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

(where we can view Q as a 3×3 matrix with rows q_i) that satisfies the following rules:

1. Permuting the rows of $\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$ changes the sign of the symbol according to the sign of the permutation.
2. $\begin{bmatrix} a_1 q_1 \\ a_2 q_2 \\ a_3 q_3 \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$.
3. $\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = 0$ when $\det \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = 0$.
4. $\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} - \begin{bmatrix} q_0 \\ q_2 \\ q_3 \end{bmatrix} + \begin{bmatrix} q_0 \\ q_1 \\ q_3 \end{bmatrix} - \begin{bmatrix} q_0 \\ q_1 \\ q_2 \end{bmatrix} = 0$.
5. $\begin{bmatrix} q_1\alpha \\ q_2\alpha \\ q_3\alpha \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \cdot \alpha$.

Here $q_0, q_1, q_2, q_3 \in \mathbb{Q}^3$ are nonzero row vectors, $a_1, a_2, a_3 \in \mathbb{Q}^\times$, $\alpha \in \mathrm{GL}(3,\mathbb{Q})$ and $\cdot$ denotes the right action of $\mathrm{GL}(3,\mathbb{Q})$ on $H_1(T_3,\mathbb{Z})$ induced by its natural right action on T_3.

A modular symbol $[Q]$ is called unimodular if $Q \in \mathrm{SL}(3,\mathbb{Z})$. Using these relations, any modular symbol is equal to the sum of unimodular symbols. An explicit algorithm we used to do this is given in 2.10. Finally we observe that if $[Q]$ is unimodular, then it defines a point of $\mathbb{P}^2(\mathbb{Z}/N) = \mathrm{SL}(3,\mathbb{Z})/\Gamma_0(N)$, denoted by the same symbol.

We continue the description of the Hecke operator. Let β_i be a coset representative as above, and let $x \in \mathbb{P}^2(\mathbb{Z}/N)$ be represented by $Q_x \in \mathrm{SL}(3,\mathbb{Z})$. Then, as modular symbols, we can write

$$[Q_x\beta_i] = \sum_j [R_{ij}],$$

with $R_{ij} \in \mathrm{SL}(3,\mathbb{Z})$. Finally we then have the formula for $T_\alpha^* : H_3(\Gamma_0(N),\mathbb{C}) \to H_3(\Gamma_0(N),\mathbb{C})$, the adjoint of the Hecke operator T_α:

$$(T_\alpha^* f)(x) = \sum_{ij} f(R_{ij}),$$

where the R_{ij} on the right-hand side are considered as elements of $\mathbb{P}^2(\mathbb{Z}/N)$.

2.5. The Hecke algebra $\mathcal{T}$ is defined to be the subalgebra of $\mathrm{End}(H^3(\Gamma_0(N),\mathbb{C}))$ generated by the T_α's with $\det(\alpha)$ relatively prime with N. The Hecke algebra is a commutative algebra and $H^3(\Gamma_0(N),\mathbb{C})$ may be decomposed as a direct sum of common eigenspaces of the operators from $\mathcal{T}$:

$$H^3(\Gamma_0(N),\mathbb{C}) = \bigoplus_\lambda V_\lambda$$

where each λ is a homomorphism of algebras $\mathcal{T} \to \mathbb{C}$, and

$$Tf = \lambda(T)f$$

for $T \in \mathcal{T}$ and $f \in V_\lambda$.

Of particular interest are the Hecke operators E_p, for p prime not dividing N, defined by $\alpha_p \in \mathrm{GL}(3,\mathbb{Q})$:

$$\alpha_p := \begin{pmatrix} p & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Given a character λ of $\mathcal{T}$, the number a_p in the local L-factor of the corresponding Hecke eigenform is

$$a_p := \lambda(E_p).$$

2.6. It is known (compare [Ash et al. 1991, Lem. 1.3, 1.4]) that the field generated by the eigenvalues a_p of a cuspidal Hecke eigenform is a number field that is either totally real or a CM-field, that is, a degree-two imaginary extension of a totally real field. Moreover, if it is totally real then the eigenclass corresponds to an essentially selfdual cuspidal automorphic representation. Conversely, if the associated automorphic representation is essentially selfdual, then a Dirichlet character χ_0 exists such that the numbers $\chi_0(p)a_p$ generate a totally real number field. One may very crudely describe results of Clozel [1991] by saying that he proves the existence of selfdual Galois representations closely related to such selfdual cuspidal automorphic ones.

Clozel [1990, Conjecture 4.5], following Langlands, predicts that the existence of Galois representations providing the same local Euler factors as the automorphic cuspidal ones, is not restricted to the selfdual case only. We are interested in testing this conjecture. For this reason, in our calculations of Hecke eigenclasses, we will restrict attention to classes whose Hecke eigenvalues generate a CM-field.

The computer determined and factorized (over $\mathbb{Q}$) the eigenvalue polynomial of the Hecke operators E_p for the first 5 primes p that do not divide N. We then considered only those V_λ for which at least one (of the five) numbers $\lambda(E_p)$ generated a CM-field.

Thus we certainly overlooked examples of nonselfdual modular forms with, say, $\lambda(E_p) \in \mathbb{Q}$ for the first 5 primes not dividing N, but with $\lambda(E_p)$ generating a CM-field for the sixth prime. Even simpler, since our selection criterion will disregard any eigenclass whose eigenvalues are all real, we will in general miss cuspidal classes corresponding to selfdual representations.

For any λ with the property that at least one of the $\lambda(E_p)$ computed was not a real number, we computed the values $a_p := \lambda(E_p)$ for the first 40 prime numbers (that is, all primes $p \leq 173$). Admittedly, the choice to use five primes in this first test looks rather arbitrary. It reflects a balance between the need to keep the time spent on the algorithms within certain bounds, versus the desire not to miss any nonselfdual classes.

Recall that we are interested in relating eigenforms to nonselfdual Galois representations, as predicted by Clozel. In this conjectured relation, the roots of the polynomial $X^3 - a_pX^2 + \bar{a}_p pX - p^3$ should be the eigenvalues of a Frobenius element (in $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$) in a 3-dimensional representation (at least if the eigenform is a cusp form). These eigenvalues of the Frobenius element should have absolute value p. Therefore we consider only eigenforms that satisfy Ramanujan's conjecture

$$|a_p| \leq 3p.$$

Examples where this is not satisfied are not listed here either, with the exception of the second column of Table 3. The first example of CM-eigenvalues (the field is $\mathbb{Q}(\sqrt{-3})$) that do not satisfy Ramanujan's conjecture occurs for $N = 49$. Note that since cuspidal classes (by Clozel's conjecture) should satisfy Ramanujan's conjecture, one expects that this second restriction on the eigenclasses will remove only noncuspidal ones.

The two properties of our eigenclasses (the first five relevant eigenvalues generate a CM-field, and all eigenvalues that we compute satisfy Ramanujan's conjecture) are the only ones we consider. In particular, in general we do not consider the question whether our eigenclasses are in fact cuspidal. In some instances practical criteria are known to determine whether a given class is cuspidal. If the level N is prime, this is worked out in detail in [Ash et al. 1984]. One can construct noncuspidal classes using for instance Eisenstein liftings; an example how one uses this to determine that certain classes are noncuspidal is given in Example 3.4. However, for general level we do not know a practical conclusive test to determine whether a given class is cuspidal or not. This implies that at present we are unable to adapt our programs in such a way that they might be used to verify Ramanujan's conjecture for certain levels.

2.7. Dimensions. The computer first determined the space $H_3(\Gamma_0(N), \mathbb{C})$ using Theorem 2.3. Table 1 on page 169 lists the dimension of that space. Representing a map $f : \mathbb{P}^2(\mathbb{Z}/N) \to \mathbb{C}$ by the vector of its values, the equations listed in 2.3 give a system of linear equations. The number of variables is first reduced using the first two equations and there remains a sparse linear system with small integer coefficients. This system is reduced further, roughly by eliminating equations with fewer than three terms. For example, in case $N = 223$ (a prime number) we are left with a system of 7005 equations in 1963 variables. We will use this example to explain how we proceed.

2.8. Lattice reduction. In smaller cases we solved the sparse linear system by Gauss elimination, plus a Euclidean algorithm to keep the entries small. In these smaller cases we observed that the solution space is always spanned by vectors with remarkably small coordinates. But for larger systems like in our example case $N = 223$ our Pascal program crashes because of integer overflow during the Gauss elimination. Therefore we solve the system only modulo the prime 32503. (Since $2 \times 32503 \times 32503 < \texttt{Maxint}$ in our Pascal implementation, overflow is now easily avoided without much change to the program.)

We find that over the field $\mathbb{Z}/32503\mathbb{Z}$ the solution space is spanned by a basis of 38 vectors. Now the trick is to apply the LLL algorithm [Lenstra et al. 1982; Pohst 1987] to the lattice L of integral vectors of length 1963 whose reduction modulo 32503 is spanned by these 38 vectors. The LLL algorithm finds 38 independent vectors with their 1963 integer coordinates all between -42 and 64, and so that their residues mod 32503 still form a basis of the solution space of the modular system. (The program aims for coordinates between -150 and 150. This works in all examples, with some room to spare.) One now plugs these new vectors in the original system, to see that we are in luck and that they satisfy it over $\mathbb{Z}$. (In all cases we had such luck.) It follows that they span the solution space over $\mathbb{Q}$, so by this trick we succeeded in solving the 7005 by 1963 system over $\mathbb{Q}$. Here the LLL algorithm that we use is `lllint` in GP/PARI Calculator Version 1.37.

Actually we do not really apply the LLL algorithm to the lattice $L \subset \mathbb{Z}^{1963}$. This $\mathbb{Z}^{1963}$ is too big. But note that, to describe a new basis of the solution space of the modular system, all one needs is a 38 by 38 transformation matrix. One can start looking for a useful matrix using just a small sample of the 1963 coordinates. We increase the sample until success is achieved. This finishes the explanation of how we solve our large sparse linear systems.

2.9. Finding a subspace. Next we compute the 38 by 38 matrix describing the Hecke operator for some prime p, compute its minimal polynomial and factorize it. There is just one factor that has CM-eigenvalues and it has degree two. Next we plug the matrix into this factor of degree two. This results in a corank-two matrix whose kernel we compute. From this we get two vectors of length 1963, spanning our interesting subspace. Applying LLL once more, now with the prime 224737, we can get a new pair, spanning the same subspace over $\mathbb{Q}$ (this we check), and with coordinates between -72 and 90. (At this step we aimed for coordinates between -4500 and 4500, as in practice the coordinates of the generators of the subspace are not as small as those for the full solution space.)

2.10. Reducing symbols. We now describe the algorithm we used to reduce a modular symbol to a sum of unimodular symbols. Large parts of it are borrowed from the algorithm given by Ash and Rudolph [1979]. We shall constantly refer to the properties enjoyed by the modular symbol, listed in Section 2.4.

By property 2, we may restrict our attention to modular symbols whose underlying matrices have integer entries. Let Q be a 3×3 matrix (with integer entries), all whose rows are nonzero. By properties 2 and 3, we may assume that $|\det Q| > 1$. For any nonzero row vector $v \in \mathbb{Z}^3$ and $1 \le i \le 3$, let $Q_i\{v\}$ denote the matrix Q with its i-th row replaced with v. It follows from properties 1 and 4 that

$$[Q] = [Q_1\{v\}] + [Q_2\{v\}] + [Q_3\{v\}].$$

A vector v will be constructed such that each matrix $Q_i\{v\}$ has smaller $|\det|$ than Q. Let q_1, q_2 and q_3 denote the rows of Q, and write

$$v = t_1 q_1 + t_2 q_2 + t_3 q_3 \tag{2.1}$$

with $t_1, t_2, t_3 \in \mathbb{Q}$. Since

$$\det Q_i\{v\} = \sum_{j=1}^{3} t_j \det Q_i\{q_j\} = t_i \det Q,$$

we need to find t_i with $|t_i| < 1$ such that the vector given in (2.1) has integer coefficients.

In order to do this, we shall find a row vector $x \in \mathbb{Z}^3$ and an integer m such that

$$xQ \equiv 0 \bmod m \tag{2.2}$$

and

$$x \not\equiv 0 \bmod m. \tag{2.3}$$

From such a congruence, a suitable vector v can be constructed as follows. Write $x = (x_1, x_2, x_3)$. We may assume that $|x_i| \le \frac{1}{2}|m|$ for $1 \le i \le 3$. It then follows easily from (2.2) and (2.3) that we may take $t_i = x_i/m$.

It remains to find x and m satisfying (2.2) and (2.3). A Gauss-like elimination procedure is applied to (2.2), without specifying the value of m yet. The trick is to choose the modulus m only after enough elimination steps have been performed. We begin working on the first row of Q. By means of elementary column operations, (2.2) is transformed into an equivalent congruence relation

$$(x_1, x_2, x_3) \begin{pmatrix} m_1 & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix} \equiv (0,0,0) \bmod m. \tag{2.4}$$

Since $|\det Q| > 1$ and the column operations do not change $|\det|$ of the matrix, m_1 cannot vanish. Now if $|m_1| > 1$, we take $x = (1,0,0)$ and $m = m_1$, and we have found a solution to (2.2) and (2.3). If $|m_1| = 1$, we turn to the second row of the matrix in (2.4). By elementary column operations we get

$$(x_1, x_2, x_3) \begin{pmatrix} \pm 1 & 0 & 0 \\ * & m_2 & 0 \\ * & * & * \end{pmatrix} \equiv (0,0,0) \bmod m. \tag{2.5}$$

Again m_2 cannot vanish. If $|m_2| > 1$, we take $m = m_2$ and find a solution of the form $x = (*, 1, 0)$. If $|m_2| = 1$, (2.5) takes the form

$$(x_1, x_2, x_3) \begin{pmatrix} \pm 1 & 0 & 0 \\ * & \pm 1 & 0 \\ * & * & m_3 \end{pmatrix} \equiv (0,0,0) \bmod m.$$

Since $m_3 = \pm \det Q$, we have $|m_3| > 1$, so we can take $m = m_3$ and find a solution of the form $x = (*, *, 1)$.

A close look at the algorithm reveals that

$$|\det Q_i\{v\}| \le \tfrac{1}{2} |\det Q|$$

for $i = 1, 2$ and $|\det Q_3\{v\}| \le 1$. Also we would like to point out that our algorithm, like that of Ash and Rudolph, works over any Euclidean domain and for any dimension.

3. NUMERICAL RESULTS

Remark 3.1. For prime level p one knows [Ash et al. 1984, Thm. 3.19] that

$$\dim H^3(\Gamma_0(p), \mathbb{C}) = \dim H^3_{\text{cusp}}(\Gamma_0(p), \mathbb{C}) \ + \ 2 \dim S_2(p),$$

$x =$	0	1	2	3	4	5	6	7	8	9
N	dim									
$1x$	2	2	7	0	4	4	6	2	7	2
$2x$	9	4	8	4	17	4	6	6	13	4
$3x$	20	4	12	10	10	8	21	4	12	8
$4x$	23	6	26	6	21	15	16	8	34	9
$5x$	20	14	21	10	25	14	31	14	20	10
$6x$	55	10	20	19	26	12	42	10	29	20
$7x$	38	12	51	10	22	28	33	18	44	14
$8x$	48	23	26	14	71	18	28	24	49	16
$9x$	67	16	24	41	32	22	68	14	43	33
$10x$	59	16	60	16	51	48	42	18	69	16
$11x$	58	28	64	18	66	28	57	35	40	26
$12x$	125	29	44	40	53	28	89	20	58	34
$13x$	60	22	107	26	44	51	67	22	82	22
$14x$	101	40	50	30	111	32	46	55	61	24
$15x$	122	24	75	51	76	36	119	24	62	50
$16x$	100	36	101	26	69	74	56	28	161	40
$17x$	80	53	73	28	106	56	102	50	64	30
$18x$	177	28	82	54	93	40	106	40	81	67
$19x$	94	32	146	30	62	80	121	32	139	32
$20x$	141	54	66	44	155	48	68	67	108	44
$21x$	?	34	109	60	72	50	163	44	70	58
$22x$	?	44	?	38	130	107	74	38	?	36
$23x$	?	94	129	38	?	56	?	70	?	40
$24x$	38	38	?	79	117	83	?	46	119	70
$25x$	42	42	?	54	84	?	?	42	?	?

TABLE 1. Dimension of $H_3(\Gamma_0(N), \mathbb{C})$, for all N between 10 and 209 and for a few more cases. A question mark indicates that we did not pursue this level, because time or memory requirements exceeded some reasonable limit. These requirements do not necessarily increase with N. Indeed, as mentioned in 3.1, for prime N it was practical to go all the way to $N = 337$.

where $S_2(p)$ is the dimension of the space of weight two cusp forms for the congruence subgroup

$$\Gamma_0(p) \subset \mathrm{SL}(2, \mathbb{Z}).$$

Recall that $\dim S_2(p) = k-1,\ k,\ k,\ k+1$ when $p = 12k + r$ and $r = 1, 5, 7, 11$. Thus in this case it is easy to determine if H^3_{cusp} is nonzero. We checked that the only prime levels below 338 for which $\dim H^3(\Gamma_0(p), \mathbb{C})$ differs from $2 \dim S_2(p)$ are the levels 53, 61, 79, 89 and 223.

Remark 3.2. In case there is a newform of level N, then in level pN we find 3 copies of it (for example, the form of level 53 appears 3 times in level 106 and 3 times in level 159). It appears 6 times in level $212 = 2^2 \cdot 53$. Such old forms, especially for levels $N = p^k$, were studied in [Reeder 1991].

3.3. We now exhibit the data about eigenclasses satisfying our criteria, in cases where the field generated by the eigenvalues is *not* $\mathbb{Q}(i)$. The case $\mathbb{Q}(i)$ is treated in Section 3.6. Table 2 lists the Hecke eigenvalues a_p for Hecke operators E_p, with $2 \le p \le 173$, of eigenforms of certain levels. The eigenvalues for small p and level 53, 61 and 79 were already given in [Ash et al. 1984]. Moreover, it seems that the form of level 223 we found has already been predicted by P. Green in 1986; his unpublished computations are mentioned in [Ash and McConnell 1992].

Example 3.4. At level 245 we found two four-dimensional spaces, V_a and V_b, invariant under the Hecke action, and the eigenvalue polynomial of the E_p's, for $p \in \{2, 3, 11, 13, 17\}$ on each space is an irreducible polynomial of degree 4, listed in Table 3 on page 171. The field K generated by the roots of these polynomials is the same for both spaces:

$$\begin{aligned} K &= \mathbb{Q}[X]/(x^4 + 2x^2 + 4) \\ &\cong \mathbb{Q}\big(\sqrt{-1 + \sqrt{-3}}\big) = \mathbb{Q}(\sqrt{2},\ \sqrt{-3}). \end{aligned}$$

The four roots of each of these polynomials $X^4 - c_p X^3 + \cdots$ are the eigenvalues of E_p, and by the Ramanujan conjecture for cusp forms their absolute value should be at most $3p$, so $|c_p| \le 12p$. The c_p we found on V_b do not satisfy this condition, those listed for V_a do. Of course, this condition on c_p is weaker than the one on a_p given in 2.6. For instance, taking $p = 11$, two of the eigenvalues of E_p on V_a have absolute value $23 + 22\sqrt{2} > 3p = 33$.

In fact, following a suggestion made to us by Ash, it is easy to give a precise description of V_a and V_b in terms of classical modular forms. This description shows that neither space contains any cusp forms. Namely, using (unpublished) tables of

$N =$	53	58	61	79	88	153	223
p	eigenvalue						
2	$-2-\delta$	$**$	$-\beta$	-1	$**$	1	1
3	$-1+\delta$	$-1+\gamma$	$-3+2\beta$	$-1+\varepsilon$	$-1+\gamma$	$**$	$-3+\iota$
5	1	$-4-2\gamma$	2β	$-4-2\varepsilon$	$-4-2\gamma$	1	1
7	-3	$1+2\gamma$	$-3-3\beta$	$-3-\varepsilon$	$1-2\gamma$	$-3+6\alpha$	1
11	1	$7-\gamma$	$-1+\beta$	$1+2\varepsilon$	$**$	$-5+6\alpha$	$1-\iota$
13	$-8-6\delta$	$-6-2\gamma$	$-4-2\beta$	$-6-2\varepsilon$	$1+4\gamma$	$-9-12\alpha$	-1
17	22	13	$-15+4\beta$	-1	-1	$**$	$-2-4\iota$
19	$11+3\delta$	$-11-4\gamma$	$17+4\beta$	$5+4\varepsilon$	$-11-4\gamma$	9	-3
23	$-11+\delta$	$-7+8\gamma$	$5-9\beta$	$17+2\varepsilon$	$21-5\gamma$	$-11+6\alpha$	$-11-\iota$
29	$16+2\delta$	$**$	$7+4\beta$	-9	$-11-4\gamma$	13	22
31	-7	$-15-11\gamma$	$17-4\beta$	$1+2\varepsilon$	$-15-\gamma$	$-15-6\alpha$	$-3+6\iota$
37	$-24+6\delta$	$21+4\gamma$	$1-16\beta$	-1	$-14+18\gamma$	-15	-2
41	-17	15	$-22-36\beta$	43	$1-8\gamma$	31	$-32-4\iota$
43	$29+6\delta$	$-25+7\gamma$	$-27+16\beta$	$-11-8\varepsilon$	$17-6\gamma$	$33+12\alpha$	$-11+6\iota$
47	$1-14\delta$	$-39+13\gamma$	$33+4\beta$	$-39-5\varepsilon$	$17+16\gamma$	$-11-12\alpha$	$-11-6\iota$
53	$-38+14\delta$	$56-2\gamma$	-25	$-15-4\varepsilon$	$-21+8\gamma$	$19-12\alpha$	$-44-12\iota$
59	$1-14\delta$	69	$19-\beta$	$15-\varepsilon$	$-1+7\gamma$	$49+12\alpha$	$25-11\iota$
61	-7	$17+4\gamma$	$30+30\beta$	$9+4\varepsilon$	$-39-28\gamma$	9	17
67	$-11-12\delta$	$-35+8\gamma$	$71+3\beta$	$-43+4\varepsilon$	$-21+23\gamma$	$-27-36\alpha$	$25-9\iota$
71	$13-5\delta$	$17-14\gamma$	$-15+4\beta$	$-67+31\varepsilon$	$101+\gamma$	$-35-30\alpha$	$25-4\iota$
73	$-39-12\delta$	$13-24\gamma$	$-42-4\beta$	27	$13+8\gamma$	$-33+72\alpha$	$20+12\iota$
79	$-39+9\delta$	$-7-17\gamma$	$-7+31\beta$	$41-17\varepsilon$	$-63-10\gamma$	$33-18\alpha$	25
83	$67-\delta$	$-27-36\gamma$	$13+32\beta$	$33+10\varepsilon$	$1-6\gamma$	$-47+12\alpha$	$-23+22\iota$
89	$-29+16\delta$	$-53-16\gamma$	$-19+8\beta$	$-18-12\varepsilon$	$-60-4\gamma$	$-89+96\alpha$	$16-4\iota$
97	-58	$-69+48\gamma$	$3+32\beta$	$-58+16\varepsilon$	$106+16\gamma$	$27-24\alpha$	$-81+24\iota$
101	$43-20\delta$	$-43+4\gamma$	$-15-48\beta$	$46-6\varepsilon$	$27+8\gamma$	55	$-53+16\iota$
103	$-99+33\delta$	$129+6\gamma$	$-67-72\beta$	$-51+15\varepsilon$	-39	$69-36\alpha$	$-79+15\iota$
107	$85-18\delta$	$-63-38\gamma$	$81+38\beta$	$-89-41\varepsilon$	$-63+18\gamma$	$-89+114\alpha$	$-11-24\iota$
109	$101+12\delta$	$84+18\gamma$	$14-14\beta$	$-61-8\varepsilon$	$-77-40\gamma$	$-63+72\alpha$	63
113	$-68+24\delta$	3	$-94+80\beta$	$69-16\varepsilon$	$122+8\gamma$	$115-24\alpha$	$-41+24\iota$
127	$-7-21\delta$	129	$5-46\beta$	$-15+9\varepsilon$	$-95-8\gamma$	$-99-144\alpha$	$-79+6\iota$
131	$-107-50\delta$	$45+16\gamma$	$-127-64\beta$	$25+22\varepsilon$	$-39+6\gamma$	$-53-102\alpha$	$25+10\iota$
137	$25+12\delta$	$21+8\gamma$	$90-36\beta$	$117+8\varepsilon$	$70-4\gamma$	43	$-149+44\iota$
139	$-19-12\delta$	$-83+4\gamma$	$-21-13\beta$	$115-23\varepsilon$	$113+26\gamma$	$39-6\alpha$	$5+6\iota$
149	$46-38\delta$	$14-30\gamma$	$-10-58\beta$	$-1-32\varepsilon$	$231-16\gamma$	$-137+12\alpha$	$175+8\iota$
151	$-35-45\delta$	$49-26\gamma$	$-75-57\beta$	$-79+58\varepsilon$	$49+34\gamma$	$-27-72\alpha$	$-11-15\iota$
157	$-51+48\delta$	-113	221	$-85+8\varepsilon$	$104+18\gamma$	$-57-96\alpha$	-45
163	$277-6\delta$	$91-25\gamma$	$85-66\beta$	-19	$189-24\gamma$	$39+54\alpha$	$125-12\iota$
167	$157+15\delta$	$1+22\gamma$	$-147-136\beta$	$-31+6\varepsilon$	$-55+12\gamma$	$-107-150\alpha$	$-155-59\iota$
173	$-53-56\delta$	$-109+56\gamma$	$19+56\beta$	$-135-12\varepsilon$	$3-8\gamma$	$13+24\alpha$	$181-8\iota$

TABLE 2. Hecke eigenvalues a_p for Hecke operators E_p, with $2 \le p \le 173$, of eigenforms of certain levels. For each column of the table we fix an algebraic integer with the following property:

$$\alpha^2 = -2, \quad \beta^2 = -3, \quad \gamma^2 = -7, \quad \delta^2 = -11, \quad \varepsilon^2 = -15, \quad \iota^2 = -23.$$

p	V_a	V_b
2	$x^4+6x^3+35x^2+6x+1$	$x^4+10x^3+77x^2+230x+529$
3	$x^4+8x^3+66x^2-16x+4$	$x^4+20x^3+302x^2+1960x+9604$
11	$x^4+46x^3+2555x^2-20194x+192721$	$x^4+246x^3+45395x^2+3719766x+228644641$
13	$x^4+100x^3+1046x^2-72700x+528529$	$x^4-668x^3+167318x^2-18624508x+777350161$
17	$x^4+70x^3+5987x^2-76090x+1181569$	$x^4+582x^3+254051x^2+49279686x+7169516929$

TABLE 3. Eigenvalue polynomial of the E_p's on each of the two four-dimensional spaces V_a, V_b invariant under the Hecke action.

Cohen, Skoruppa and Zagier it turns out that there exists a unique newform $f = q + b_2q^2 + b_3q^3 + \cdots$ of weight 2, level 245 and with trivial character, having $b_2 = 1+\sqrt{2}$, $b_3 = 1-\sqrt{2}$, $b_{11} = 2-2\sqrt{2}$ and $b_{13} = -2-2\sqrt{2}$. Such a newform lifts in two different ways to eigenclasses in our H^3. One has eigenvalue pb_p+1 at E_p; the other b_p+p^2. (On the Galois side of the Langlands correspondence, if f corresponds to a 2-dimensional representation V, the two lifts are $V(-1)\oplus\mathbb{Q}_\ell$ and $V\oplus\mathbb{Q}_\ell(-2)$; the (-1) and (-2) denote Tate twists and $\mathbb{Q}_\ell$ is the trivial representation.)

Now take χ_0 to be a nontrivial cubic Dirichlet character modulo 7. Twisting the lifted eigenclasses by χ_0 and by the complex conjugate character, one again finds eigenclasses that in our case are still of level 245. In terms of Galois representations, this means one takes χ to be the character of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ corresponding to χ_0 and one considers $(V(-1)\oplus\mathbb{Q}_\ell)\otimes\chi$ and $(V\oplus\mathbb{Q}_\ell(-2))\otimes\chi$, respectively.

Running over f and its conjugate, and χ_0 and its complex conjugate one finds in this way 4 lifted eigenclasses of the first type, exactly generating V_a, and similarly 4 lifts of the other type generating V_b. It is well known that the space of such "Eisenstein lifts" contains no nonzero cuspidal classes.

3.5. Within the range of our search, we found at precisely one level eigenclasses satisfying our selection criteria, with eigenvalues generating a *quartic* totally complex field. This occurs at level 244. The CM-field in this case is generated by a root of X^4+9X^2+12.

The following short table presents eigenvalues for Hecke operators E_p acting on one of these eigenclasses. The eigenvalues are given in terms of τ, which satisfies $4\tau^4+9\tau^2+3=0$. Although τ is not integral, the entries in the table of course are.

$p=$	eigenvalue
3	1
5	$-8+14\tau-8\tau^2+8\tau^3$
7	$-3-3\tau-4\tau^3$
11	$-5-15\tau-4\tau^3$
13	$-6\tau-8\tau^3$
17	$-5+28\tau+16\tau^3$
19	$45+6\tau+32\tau^2+8\tau^3$
23	$13+27\tau+20\tau^3$
31	$-15-16\tau^3$
37	$-45+84\tau-32\tau^2+48\tau^3$
41	$-14+60\tau+16\tau^3$
43	$-51-156\tau-16\tau^2-96\tau^3$
47	$-47-140\tau-32\tau^2-80\tau^3$
53	$79+48\tau^2$

3.6. Finally, Table 4 lists the Hecke eigenvalues for Hecke operators E_p, with $2\le p\le 173$, of eigenforms with eigenvalues $a_p\in\mathbb{Z}[i]$. The form for level $N=89$ here already appeared in [Ash et al. 1984].

Remark 3.7. The numbers a_p listed are conjectured to be the traces of the automorphisms through which a Frobenius element at p acts on three-dimensional $\mathbb{Q}_l$ vector spaces. Note that the trace of the identity map on such a vector space is equal to three.

$N =$	89	$106 = 2 \cdot 53$	$116 = 2^2 \cdot 29$	$128 = 2^7$	$160 = 2^5 \cdot 5$	$205 = 5 \cdot 41$	$212 = 2^2 \cdot 53$	$221 = 13 \cdot 17$
p	eigenvalue							
2	$-1-2i$	$**$	$**$	$**$	$**$	-1	$**$	$-1+2i$
3	$-1+i$	$-1+i$	$-1+i$	$1+2i$	$1+2i$	$1+2i$	$-1+i$	$-1+4i$
5	$2-2i$	$-4-5i$	$2-2i$	$-1-4i$	$**$	$**$	$-1+4i$	$-1-4i$
7	$-7+14i$	$2+5i$	$1-4i$	$1+4i$	$1-2i$	$1+2i$	$5+2i$	$3-4i$
11	$-3-10i$	$6+5i$	$-5+5i$	$-7-10i$	$-3-12i$	$-7-10i$	$-3-10i$	5
13	$-1-4i$	$-8+4i$	$-6-14i$	$-1+4i$	$-5-8i$	$3-8i$	$16-2i$	$**$
17	$-6+8i$	$-8-10i$	$-17+24i$	7	-5	-5	$-2-16i$	$**$
19	$11-i$	$-9+13i$	$1+2i$	$1-14i$	$13+8i$	$-15-14i$	$-9+i$	$21+8i$
23	$-11-19i$	$-1-9i$	$-7-10i$	$17-4i$	$-15+26i$	$-7-20i$	$-19+3i$	$37-4i$
29	$-19+32i$	$6-28i$	$**$	$-9-12i$	$15-16i$	$-13+24i$	$6+26i$	$-19-32i$
31	$17-5i$	-7	$-15+25i$	1	$33+4i$	1	$-7-30i$	$-1-20i$
37	$15+32i$	$26-24i$	$33+4i$	$-25+28i$	$11+24i$	$-13+8i$	$-10-18i$	$3+36i$
41	$25-20i$	$-7+50i$	-27	-5	$47-16i$	$**$	$-37-40i$	$-35-40i$
43	$19+i$	$-26-19i$	$35-23i$	$-7+30i$	$-31-22i$	$53-8i$	$-23-16i$	$25+8i$
47	$13-16i$	$1+16i$	$57-5i$	$17+40i$	$1+54i$	$17+14i$	$-23+10i$	$9+32i$
53	$-22-10i$	$**$	$20-38i$	$23-20i$	$-45-24i$	$83-8i$	$**$	$3+40i$
59	$41+30i$	$-49-34i$	$-39-30i$	$-39+22i$	$-11-16i$	$-43+16i$	$41+14i$	$41-32i$
61	$15+20i$	$18-25i$	$-49+40i$	$63+20i$	$-21+24i$	$31-16i$	$-9+20i$	-7
67	$-7-76i$	$-11-62i$	$37+20i$	$65-22i$	$-23-58i$	$-23+22i$	$-23+70i$	$-55+48i$
71	$-55-10i$	$-67+125i$	$-31-50i$	$-31+20i$	$-23-28i$	$-31+38i$	$77+35i$	$11+20i$
73	$60-28i$	$86-7i$	$-35+72i$	$-57-80i$	$-45-32i$	$-33+80i$	$-85-148i$	$-35-72i$
79	$41-46i$	$41+19i$	$41+37i$	$81-24i$	$-15-88i$	$-63-74i$	$41-35i$	$-59-52i$
83	$-47+130i$	$7+49i$	$57-6i$	$-63+106i$	$17+58i$	$-43+28i$	$103+25i$	$-11-56i$
89	$**$	$51+6i$	$-59-64i$	$-9+16i$	107	-21	-69	$11+24i$
97	$-12-16i$	$72-40i$	$45+72i$	7	$-77+64i$	$-77-128i$	$-24-64i$	$13-64i$
101	45	$58+25i$	$77-20i$	$-105-100i$	$-33+64i$	$115-40i$	$61+40i$	$-25+40i$
103	$-27+85i$	$-69-137i$	$-63-126i$	$-127-220i$	$113+50i$	$-39-40i$	$117+19i$	$-59+152i$
107	$33-26i$	$40+17i$	$-27+124i$	$-7+86i$	$-39-130i$	$109-36i$	$-95+32i$	$35+68i$
109	$-74-94i$	$-39+92i$	$6+90i$	$-9+68i$	$-21-40i$	$59+40i$	$21+20i$	$-69-36i$
113	$87-76i$	$222-16i$	-57	$-61+64i$	$11-64i$	$-1+64i$	$-78+104i$	$91+32i$
127	$-111+183i$	$3-i$	$33-126i$	$161-16i$	$1-34i$	$161-44i$	$-87+119i$	$-19+64i$
131	$-31-20i$	$-82-125i$	$-87-50i$	$-63-70i$	$69+12i$	$-91-52i$	$-79-80i$	$-25-60i$
137	$-125+72i$	$-30+77i$	$-99-184i$	$235-32i$	$-13+160i$	$-45+96i$	$-57+44i$	$-37+176i$
139	$-59-8i$	$81+28i$	$1+46i$	$121-50i$	$37-16i$	$-155-224i$	$-39+166i$	$-149+180i$
149	$101+36i$	$-124+2i$	$26-138i$	$-49+76i$	$259+8i$	$99+56i$	$146+86i$	$11+64i$
151	$-47-50i$	$-145-175i$	$-95-20i$	$17+60i$	$-71+148i$	$-63+126i$	$101-115i$	$5-80i$
157	$-141+48i$	$-146-197i$	$-179-324i$	$-113-140i$	$19+136i$	$155+8i$	$-77+124i$	$31-56i$
163	$-141+31i$	$-138+149i$	$19+131i$	$1+2i$	$-143-70i$	$-139+164i$	$-63+104i$	$-79+152i$
167	$-175-188i$	$77+5i$	$-47-26i$	$-95-172i$	$1-34i$	$65+50i$	$-163-205i$	$7+100i$
173	$54-54i$	$87+14i$	$11+8i$	$-49-188i$	$99+104i$	$-153-288i$	$-189+248i$	$49-136i$

TABLE 4. Hecke eigenvalues a_p for Hecke operators E_p, with $2 \leq p \leq 173$, of eigenforms with eigenvalues $a_p \in \mathbb{Z}[i]$.

For $p \leq 173$ we verified that the a_p's for the modular form of level 128 are such traces. R. Schoof observed that as far as the table goes we have

$$a_p \equiv \begin{cases} 3 \bmod 4 & \text{for} \quad p \equiv 1 \bmod 4 \\ 1+2i \bmod 8 & \text{for} \quad p \equiv 3 \bmod 8 \\ 1 \bmod 8 & \text{for} \quad p \equiv 7 \bmod 8 \end{cases}$$

and that moreover $a_p \equiv 3 \bmod 8$ when $p = a^2 + 32b^2$. (Note that $41 = 3^2 + 32 \cdot 1^2$, $113 = 9^2 + 32 \cdot 1^2$, and $137 = 3^2 + 32 \cdot 2^2$.)

3.8. The background for this paragraph can be found in [van Geemen and Top 1994]. There a 3-dimensional (compatible system of l-adic) Galois representation V_l was constructed in $H^2(S_a, \mathbb{Q}_l)$ (étale cohomology) of the (smooth, minimal, projective) surface S_a defined by the (affine) equation:

$$t^2 = xy(x^2-1)(y^2-1)(x^2-y^2+axy)$$

After a twist by the nontrivial character

$$\chi : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gal}(\mathbb{Q}(\sqrt{-2})/\mathbb{Q}) \cong \pm 1,$$

the L-factors of the Galois representation on V_l for $a = 2$ coincide with the L-factors of a modular form of level 128 (the one also listed in the table here) for all primes ≤ 173. With similar computations we found two more examples:

Theorem 3.9. *For all odd primes $p \leq 173$ the L-factors of the modular form of level 160 listed here coincide with the twist by the nontrivial character $\varepsilon : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gal}(\mathbb{Q}(\sqrt{-1})/\mathbb{Q}) \cong \pm 1$ of the L-factors of the Galois representation V_l from the surface S_a with $a = 1$. A similar statement holds for the modular form of level 205, with $a = 1$ replaced by $a = 1/16$.*

3.10. It may be expected that more examples of the kind given in Theorem 3.9 can be found. There is no particular reason why the family of surfaces S_a given above will provide such examples. In fact, in [van Geemen and Top 1995] different families of surfaces were used to compute tables of traces of Frobenius for the corresponding 3-dimensional Galois representations V_l. Table 5 shows similar data, giving for various values $a \in \mathbb{Z}$ the traces of Frobenius on a V_l in the cohomology of S_a, for good primes $p \leq 29$. "Good primes" here means primes p that do not divide $2a(a^2+4)$; our table displays the symbol $(*)$ for primes that do divide this quantity. The method by which traces are computed is explained in [van Geemen and Top 1994, (3.6–9)]. For amusement, and to stress the point that it is indeed easy to do such calculations for many primes, the prime $p = 173$ is included as well.

To illustrate Theorem 3.9, note that the numbers in the column $a = 1$, for primes $\equiv 3 \bmod 4$ multiplied by -1, exactly equal the complex conjugates

$a =$	1	2	3	4	5	6	7	8	9
p	trace								
3	$-1+2i$	$1+2i$	$(*)$	$-1+2i$	$1+2i$	$(*)$	$-1+2i$	$1+2i$	$(*)$
5	$(*)$	$1+4i$	$1-4i$	$(*)$	$(*)$	$(*)$	$1+4i$	$1-4i$	$(*)$
7	$-1-2i$	$-1-4i$	$1+2i$	$-1+2i$	$1-4i$	$1-2i$	$(*)$	$-1-2i$	$-1-4i$
11	$3-12i$	$-7-10i$	$-9+6i$	-13	$7+14i$	$-7+14i$	13	$9+6i$	$7-10i$
13	$-5+8i$	$1-4i$	$(*)$	$3-8i$	$9-8i$	$-3-8i$	$-3+8i$	$9+8i$	$3+8i$
17	-5	7	$5-8i$	$3+16i$	$-15+4i$	$9+20i$	$1+8i$	$(*)$	$(*)$
19	$-13+8i$	$1-14i$	$-7-18i$	$3+20i$	$-9-2i$	$15-14i$	$15+18i$	5	$-21-4i$
23	$15+26i$	$-17+4i$	-17	$15-26i$	$-7+12i$	$-1-10i$	$33+2i$	$15-2i$	$15+24i$
29	$15+16i$	$9+12i$	$-7+16i$	$23+16i$	$(*)$	$-13-8i$	$-21+24i$	$1-4i$	$-13+24i$
173	$99-104i$	$49+188i$	$-43-96i$	$-93-56i$	$99+56i$	$27-72i$	$-135+68i$	$-79-68i$	$295+48i$

TABLE 5. Traces of Frobenius on the Galois representation V_ℓ for various a.

of the numbers listed for $N = 160$ in Table 3.6. So far we did not find eigenclasses corresponding to other columns in the above table. The case $a = 1/16$ may also be partially verified using the above table, by noting $a \bmod 3 \equiv 1$, $a \bmod 7 \equiv 4$, etc.

It would be very interesting to compute a "conductor" of these spaces V_l, and to predict a relation with the level of a hypothetical corresponding eigenclass.

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