Abel's Equation and Regular Growth: Variations on a Theme by Abel

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CONTENTS

Following a 70-year old suggestion of Paul Lévy, a condition is formulated for the regularity of growth of real functions. The condition, which is quite explicit, makes use of the iterates of the function and solutions of Abel's functional equation, and is well adapted to a computer testing.

Numerous computer experiments reveal interesting properties of the proposed regularity criterion.

1. INTRODUCTION

Abel's functional equation,

$$A(F(x)) = A(x) + 1, \tag{1–1}$$

plays a central role in the theory of continuous or fractional interaction. It appeared first in a brief note of Abel [1881], which was among his left-over papers rescued by Holmboe for posterity. He observed that if A is a strictly increasing continuous solution of (1–1), which for brevity we shall call an Abel function of F, then the functions

$$F^{\sigma}(x) = A^{-1}(A(x) + \sigma), \quad \text{for } \sigma \in \mathbb{R}, \tag{1–2}$$

form a family of fractional iterates of F with the property that $F^0 = \mathrm{id}$, $F^1 = F$, and

$$F^{\sigma} \circ F^{\tau} = F^{\tau} \circ F^{\sigma} = F^{\sigma+\tau} \quad \text{for all } \sigma, \tau \in \mathbb{R}.$$

Clearly, if A is an Abel function of F then so is $A + c$ for any constant c and, from the point of view of iteration, not essentially different from A, as it supplies the same family of iterates. There are of course many other solutions of (1–1) and Abel wrote down their general form:

1058-6458/1998 $0.50 per page

Lemma 1.1 [Abel 1881]. *If A, A^* are strictly increasing C^1 solutions of* (1–1) *then*

$$A^*(x) = A(x) + \varphi(A(x)) \tag{1–3}$$

for some periodic φ with period 1 *and such that $\varphi'(x) > -1$ for all x. Conversely, any A^* of the form* (1–3) *is an invertible solution of* (1–1).

Abel never mentions the condition $\varphi'(x) > -1$ (for that matter he mentions no conditions whatever about his functions) but it must be satisfied if ${A^*}'(x) = A'(x)\big(1 + \varphi'(A(x))\big)$ is to be positive. It is now natural to ask whether it is possible to single out one particular solution of (1–1) (of course modulo an additive constant), and hence one particular family of fractional iterates, that in some sense can be regarded as the "best" solution of the iteration problem. A typical well-known instance is the problem of the half-iterate of the exponential function, that is, an F satisfying $F(F(x)) = e^x$ for any $x \geq 0$, say. Such an F is certainly not an elementary function, since its growth cannot be fitted into Hardy's logarithmico-exponential scale [Hardy 1910]. Our problem is just a particular instance of a far more general question, namely whether it is possible to give an objective meaning to the concept of "regular growth" of real functions. What we propose here is to formulate an intrinsic property of real functions that can be interpreted as a criterion of regular growth, and has at least the potentiality to become the source of a universal comparison scale. Unfortunately at present there is no way of deciding whether the suggested, quite explicit, criterion can in fact fulfil such an ambitious program (there is no a priori guarantee that such a property exists at all) and the evidence so far is largely computational.

Following the frustrations of last century there was one serious attempt in the present century to formulate a criterion of "croissance régulière": that of Paul Lévy [1928], based on Abel's functional equation. The present attempt grew out of Lévy's work and is essentially a recasting of his proposal in a more workable and computationally more accessible form.

The program is briefly this: with every convex analytic $F : \mathbb{R}^{>0} \to \mathbb{R}^{>0}$ satisfying $F(0) = 0$, $F'(0) \geq 1$, and $F''(x) > 0$ for $x > 0$ we shall associate, through a construction which involves both the local (at $x = 0$) and global behaviour of the function, a continuous $\varphi : \mathbb{R}^{>0} \to \mathbb{R}$ with the property that $\varphi(F(x)) = \varphi(x)$ for all $x > 0$. Once we have such a function it can be converted, again with the help of Abel's equation, into a periodic ψ with period 1. By taking the Fourier coefficients $\hat{\psi}_m$, we transform the problem of regular growth of F into one concerning the "regularity" of real sequences. Details of the construction and the ensuing regularity criterion will be discussed in Section 2. The rest of the paper is taken up by the presentation of some of the computational evidence in support of the criterion and the description of the link between our criterion and Lévy's original suggestion.

The precursor of the present work was a similar but rather perfunctory attempt that I made some years ago [Szekeres 1984] without coming to a definite formulation of a criterion. The main stumbling block was the very limited computational accuracy (standard 16 decimal figures) available to me at the time—not nearly sufficient to extract any useful information from the numerical data obtained. The experimental evidence presented in Section 3 relies on 180 decimal figures, using Richard Brent's multiple precision package. This was quite adequate for all examples discussed here.

2. THE CRITERION OF REGULAR GROWTH

For $c \geq 1$, let $\mathcal{C}_c$ denote the set of strictly convex analytic functions $F : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ satisfying $F(0) = 0$, $F'(0) = c$, and

$$F''(x) > 0 \quad \text{for } x \geq 0.$$

The regularity criterion will refer to members of $\mathcal{C} = \bigcup_{c \geq 1} \mathcal{C}_c$.

For $F \in \mathcal{C}$ there is always a uniquely determined Abel function that has a best local behaviour at $x = 0$; we call it the *principal Abel function* of F. It is distinguished by the property that, if

$$F(x) = \sum_{j=1}^{\infty} b_j x^j$$

is the Taylor series of F, with $b_1 = c \geq 1$, then the principal Abel function has an expansion

$$A(x) = \frac{\log x}{\log c} + \sum_{j \geq 1} a_j x^j \tag{2–1}$$

at $x = 0$ if $F \in \mathcal{C}_c$ with $c > 1$, and

$$A(x) \simeq -\frac{1}{b_2 x} + a_0 \log x + \sum_{j \geq 1} a_j x^j \quad \text{as } x \to 0^+ \tag{2–2}$$

if $F \in \mathcal{C}_1$, the coefficients of expansion being obtained by formal substitution into (1–1). The series (2–1) converges, by a century-old theorem of Königs [1884]; see [Kuczma 1968, Chapters VI and VII] for results of this kind. The series (2–2) is not necessarily convergent; nevertheless there is a unique analytic solution of (1–1) having (2–2) as an asymptotic expansion at $x = 0^+$ [Szekeres 1958]. Although the principal Abel function is the one that behaves best near 0, there is no reason to assume that it is also the solution that has the most "regular" manner of growth at $x \to \infty$.

Let $D(x) = 1/A'(x)$ denote the reciprocal of the derivative of the principal Abel function. If $c > 1$ it has an expansion

$$D(x) = \sum_{j \geq 1} d_j x^j, \tag{2–3}$$

with

$$d_1 = \log c,\ d_2 = \frac{b_2 \log c}{c(c-1)},\ \dots;$$

if $F \in \mathcal{C}_1$ the expansion is

$$D(x) \simeq \sum_{j \geq 2} d_j x^j \tag{2–4}$$

with

$$d_2 = b_2,\ d_3 = b_3 - b_2^2,\ d_4 = b_4 + \tfrac{1}{2} b_2 (3b_2^2 - 5b_3),\ \dots.$$

In a sense, $D(x)$ is more basic than $A(x)$, being the infinitesimal generator of the iteration group (1–2). Unlike the Abel function, which has the character of an indefinite integral (it admits an arbitrary additive constant), $D(x)$ is a proper function. It satisfies

$$\begin{aligned} D(F(x)) &= F'(x) D(x), \\ D'(F(x)) &= D'(x) + D(x) \frac{F''(x)}{F'(x)}. \end{aligned} \tag{2–5}$$

We shall refer to $D(x)$ as the *principal iteration generator* of F.

Another function that we shall need is $E(x)$, satisfying

$$E(F(x)) = F'(x)(E(x) + D'(x)) \tag{2–6}$$

when $c > 1$ and

$$E(F(x)) = F'(x) E(x) + D'(F(x)) \tag{2–7}$$

when $F \in \mathcal{C}_1$, and having an expansion

$$E(x) = \sum_{j \geq 0} e_j x^j, \tag{2–8}$$

where

$$e_0 = -\frac{c \log c}{c-1}$$

if $c > 1$, and $e_0 = -1$, $e_1 = b_2$ if $F \in \mathcal{C}_1$. It is a kind of inhomogeneous iteration generator of F. The coefficients e_j are obtained by formal substitution into (2–6) and (2–7) respectively, except for e_1 in the first case and e_2 in the second case, for which the defining equation for the coefficients becomes an identity, given the values d_1, d_2 and d_2, d_3, d_u respectively in (2–3) and (2–4). The reason of course is that, if $E(x)$ is a solution of (2–6) or (2–7), then $E(x) + \mu D(x)$ is also a solution for any constant μ. The missing coefficients will be fixed later.

The last auxiliary function we will need is

$$t(x) = \sum_{k=0}^{\infty} \frac{1}{F^{k\,\prime}(x)}, \quad \text{for } x > 0, \tag{2–9}$$

where $F^k = F \circ F^{k-1}$, $F^0 = \mathrm{id}$, and

$$\begin{aligned} F^{k\,\prime}(x) &= F'(x)F'(F(x)) \cdots F'(F^{k-1}(x)) \\ &= \prod_{j=1}^{k} F'(F^{j-1}(x)). \end{aligned}$$

It satisfies

$$\begin{aligned} t(F(x)) &= F'(x)(t(x)-1), \\ t(x) &= 1 + \frac{t(F(x))}{F'(x)}. \end{aligned} \tag{2–10}$$

The convergence of (2–9) is obvious because of the strict convexity of F. We also note that

$$t'(F(x)) = t'(x) + \frac{F''(x)}{F'(x)}(t(x)-1). \tag{2–11}$$

We now give the definition of φ envisaged in the introduction.

Theorem 2.1. *For $F \in \mathcal{C}_c$, with $c > 1$, let D, E, t be defined by* (2–5), (2–6), *and* (2–9). *Then*

$$\varphi(x) = \frac{1}{D(x)} \big(t(x)D'(x) - t'(x)D(x) + E(x) \big) \tag{2–12}$$

satisfies

$$\varphi(F(x)) = \varphi(x) \quad \textit{for all } x > 0.$$

Theorem 2.2. *For $F \in \mathcal{C}_1$, let D, E, t be defined by* (2–5), (2–7) *and* (2–9) *respectively. Then*

$$\varphi(x) = \frac{1}{D(x)} \big((t(x)-1)D'(x) - t'(x)D(x) + E(x) \big) \tag{2–13}$$

satisfies

$$\varphi(F(x)) = \varphi(x) \quad \textit{for all } x > 0.$$

Proof of Theorems 2.1 and 2.2. Let $F \in \mathcal{C}_c$, with $c > 1$. From (2–5), (2–6), (2–10), (2–11), and (2–12) we have

$$\begin{aligned} \varphi(F(x)) &= \frac{1}{D(F(x))} \big(t(F(x))D'(F(x)) \\ &\qquad - t'(F(x))D(F(x)) + E(F(x)) \big) \\ &= \frac{1}{D(x)} \Big((t(x)-1)\Big(D'(x) + D(x)\frac{F''(x)}{F'(x)}\Big) \\ &\qquad - D(x)\Big(t'(x) + \frac{F''(x)}{F'(x)}(t(x)-1)\Big) \\ &\qquad + E(x) + D'(x) \Big) \\ &= \frac{1}{D(x)} \big(t(x)D'(x) - D(x)t'(x) + E(x) \big) \\ &= \varphi(x). \end{aligned}$$

This proves Theorem 2.1. The proof of Theorem 2.2 is similar, involving (2–7) and (2–13) instead of (2–6) and (2–12). □

Note that in the definition of $\varphi(x)$ the functions $D(x)$ and $E(x)$ are determined by the local expansion coefficients of $F(x)$ at $x = 0$, whereas $t(x)$ uses the full global behaviour of $F(x)$ throughout $(0, \infty)$. Note also that the arbitrary multiple $\mu D(x)$ of $D(x)$ that can be added to $E(x)$ appears here as an arbitrary constant μ that can be added to $\varphi(x)$.

We can transform $\varphi(x)$ into a periodic function by making yet another use of Abel's functional equation. Consider $\psi(\sigma) = \varphi(A^{-1}(\sigma))$, where, as before, A is the principal Abel function of F. Then ψ is periodic with period 1, by (1–1). Since A is only determined up to an arbitrary additive constant, ψ is only determined up to an arbitrary phase constant (plus a free additive constant). We can take care of both these constants by considering the Fourier coefficients

$$\hat{\psi}_n = \int_0^1 e^{2\pi i n \sigma} \psi(\sigma)\, d\sigma$$

of ψ, which we write as

$$\hat{\psi}_n = e^{\alpha_n + 2\pi i \beta_n} \tag{2–14}$$

with α_n, β_n real and $0 \le \beta_n < 1$, for $n \ge 1$. In particular, the sequence $\alpha = \{\alpha_n\}_1^\infty$ is independent of the phase constant of ψ. The free additive constant of ψ is fixed by the normalizing condition

$$2\pi\hat{\psi}_0 = \int_0^1 \psi(\sigma)\, d\sigma = 0. \tag{2–15}$$

This is achieved by setting in (2–8) the values

$$e_1 = \frac{2b_2}{c(c-1)}\left(1 - \frac{c\log c}{c-1}\right) \tag{2–16}$$

if $c > 1$, and

$$e_2 = \tfrac{3}{2}b_3 - \tfrac{5}{3}b_2^2 \tag{2–17}$$

if $c = 1$; an indirect proof will be supplied later.

In writing $\hat{\psi}_n$ in the form (2–14) we have assumed of course that no Fourier coefficient apart from $\hat{\psi}_0$ is 0, an assumption that will not cause us any problem. We mention here that in the trivial linear case $F(x) = cx$ for $c > 1$, which is not strictly convex and hence is not in $\mathcal{C}$, the function ψ is identically 0. Indeed, $D(x) = (\log c)x$ and $t(x) = c/(c-1)$ by (2–9); hence $t'(x) = 0$,

$$E(x) = -\frac{c\log c}{c-1}$$

by (2–8), and so $\psi(x) = 0$ for all $x > 0$, by (2–12).

With the normalization (2–16), (2–17) we have now a perfectly well defined mapping Γ from $\mathcal{C}$ to the set of real sequences $\Gamma F = \alpha = \{\alpha_n\}$. We shall call α the sequence of Lévy–Fourier (LF) coefficients associated with F. As we have noted earlier, the construction of ψ and hence of the LF coefficients involves the entire global behaviour of $F(x)$ at large values of x. It is therefore reasonable to expect that regularity of growth properties of F will translate somehow in a suitable regularity property of the LF coefficients. We shall find in complete monotonity the principal ingredient of such a property, but before formulating precisely our regularity condition, we examine in some detail the main invariance properties of the iteration wave ψ.

Theorem 2.3. *For $F \in \mathcal{C}$ and $b > 0$, let $F^*(x) = (1/b)F(bx)$. Let α and α^* be the LF coefficients of F and F^*. Then*

$$\alpha_n^* = \alpha_n + \log b \quad \textit{for } n \ge 1.$$

Proof. Suppose $F \in \mathcal{C}_c$ for $c > 1$. Denote by D^*, E^*, t^* the various iteration functions pertaining to F^* in Theorem 2.1. Then ${F^*}'(x) = F'(bx)$, ${F^{*k}}' = {F^k}'(bx)$, and $D^*(x) = (1/b)D(bx)$ by (2–3) and (2–5); ${D^*}'(x) = D'(bx)$, $E^*(x) = E(bx)$ as seen from (2–6) and (2–8); $t^*(x) = t(bx)$ as seen from (2–9); and ${t^*}'(x) = bt'(bx)$. Hence

$$\begin{aligned}\varphi^*(x) &= \frac{1}{D^*(x)}\big(t^*(x){D^*}'(x) - {t^*}'(x)D^*(x) + E^*(x)\big)\\ &= b\varphi(bx).\end{aligned}$$

Also $A^*(x) = A(bx)$ for the principal Abel function of F^*,

$$\psi^*(\sigma) = \varphi^*(A^{*-1}(\sigma)) = b\varphi(A^{-1}(\sigma)) = b\psi(\sigma),$$

apart from a phase constant. Hence $\alpha_n^* = \alpha_n + \log b$ for each n.

With minor modifications the proof is the same if $F \in \mathcal{C}_1$. □

Theorem 2.4. *Let $F \in \mathcal{C}$ and set $F^* = F^p$ for some positive integer p. Then*

$$\alpha_n = \alpha_{np}^* + \log p.$$

The theorem will follow from an auxiliary result:

Lemma 2.5. *Let $F^* = F^p$ for some positive integer p. Then, apart from an additive constant, which does not matter,*

$$\varphi(x) = \sum_{j=0}^{p-1} \varphi^*(F^j(x)). \tag{2–18}$$

Proof. First assume that $F \in \mathcal{C}_c$, with $c > 1$. We show that

$$\sum_{j=0}^{p-1} \frac{1}{{F^j}'x} t^*(F^j x) = tx. \tag{2–19}$$

(We omit the parentheses around x for economy of notation.) To verify (2–19), write

$$\begin{aligned} \frac{1}{{F^j}'x} t^*(F^j x) &= \frac{1}{{F^j}'x} \sum_{k=0}^{\infty} \frac{1}{{F^{kp}}'(F^j x)} \\ &= \sum_{k=0}^{\infty} \frac{1}{{F^{kp+j}}'x} \end{aligned}$$

and

$$\sum_{k=0}^{\infty} \sum_{j=0}^{p-1} \frac{1}{{F^{kp+j}}'x} = \sum_{k=0}^{\infty} \frac{1}{{F^k}'x} = tx.$$

Now (1–1) implies $A(F^* x) = A(F^p x) = Ax + p$; hence $A^* x = (1/p) Ax$ and $D^* x = pDx$. Furthermore, $D^*(F^j x) = {F^j}'x \, D^* x = p{F^j}'x \, Dx$; hence

$$\begin{aligned} &\sum_{j=0}^{p-1} t^*(F^j x) \frac{{D^*}'(F^j x)}{D^*(F^j x)} \\ &\qquad = \sum_{j=0}^{p-1} \frac{t^*(F^j x)}{{F^j}'x \, Dx} \Bigl(D'x + Dx \frac{{F^j}''x}{{F^j}'x} \Bigr) \\ &\qquad = tx \frac{D'x}{Dx} + \sum_{j=0}^{p-1} \frac{{F^j}''x}{({F^j}'x)^2} t^*(F^j x), \end{aligned}$$

by (2–19). But, from (2–19),

$$t'x = \sum_{j=0}^{p-1} {t^*}'(F^j x) - \sum_{j=0}^{p-1} t^*(F^j x) \frac{{F^j}''x}{({F^j}'x)^2},$$

hence

$$\sum_{j=0}^{p-1} \Bigl(t^*(F^j x) \frac{{D^*}'(F^j x)}{D^*(F^j x)} - {t^*}'(F^j x) \Bigr) = tx \frac{D'x}{Dx} - t'x. \tag{2–20}$$

Therefore in order to prove (2–18) we only have to show, by Theorem 2.1, that

$$\sum_{j=0}^{p-1} \frac{E^*(F^j x)}{D^*(F^j x)} = \frac{Ex}{Dx}$$

or

$$\sum_{j=0}^{p-1} \frac{E^*(F^j x)}{{F^j}'x} = pEx. \tag{2–21}$$

But, by definition, $E^*(F^p x) = {F^p}'x(E^* x + pD'x)$. Hence, setting

$$\sum_{j=0}^{p-1} \frac{E^*(F^j x)}{{F^j}'x} = pE^{**}x, \tag{2–22}$$

we get

$$\begin{aligned} &pE^{**}(Fx) \\ &\quad = F'x \sum_{j=0}^{p-1} \frac{E^*(F^{j+1} x)}{{F^{j+1}}'x} \\ &\quad = F'x \Bigl(\sum_{j=1}^{p-1} \frac{E^*(F^j x)}{{F^j}'x} + \frac{E^*(F^p x)}{{F^p}'x} \Bigr) \\ &\quad = F'x \Bigl(\sum_{j=0}^{p-1} \frac{E^*(F^j x)}{{F^j}'x} - E^* x + E^* x + pD'x \Bigr) \\ &\quad = F'x (pE^{**}x + pD'x). \end{aligned}$$

This shows that $E^{**}x$ satisfies the same equation as Ex. Thus the two functions are the same, modulo a constant multiple of Dx which does not matter, proving (2–21).

Now suppose instead that $F \in \mathcal{C}_1$. The proof is similar but details are more involved. Instead of (2–20) we have

$$\begin{aligned} &\sum_{j=0}^{p-1} \Bigl((t^*(F^j x) - 1) \frac{{D^*}'(F^j x)}{D^*(F^j x)} - {t^*}'(F^j x) \Bigr) \\ &= (tx - 1) \frac{D'x}{Dx} - t'x - \frac{D'x}{Dx} \sum_{j=1}^{p-1} \frac{1}{{F^j}'x} - \sum_{j=1}^{p-1} \frac{{F^j}''x}{({F^j}'x)^2}, \end{aligned}$$

and we have to show that

$$\sum_{j=0}^{p-1} \frac{E^*(F^j x)}{F^{j\,\prime} x} = p\Big(Ex + D'x \sum_{j=1}^{p-1} \frac{1}{F^{j\,\prime} x} + Dx \sum_{j=1}^{p-1} \frac{F^{j\,\prime\prime} x}{(F^{j\,\prime} x)^2}\Big).$$

Set

$$E^{**}x = \frac{1}{p}\sum_{j=0}^{p-1} \frac{E^*(F^j x)}{F^{j\,\prime} x} - D'x \sum_{j=1}^{p-1} \frac{1}{F^{j\,\prime} x} - Dx \sum_{j=1}^{p-1} \frac{F^{j\,\prime\prime} x}{(F^{j\,\prime} x)^2}.$$

Using the equalities

$$F^{j\,\prime}(Fx) = \frac{1}{F'x} F^{j+1\,\prime} x,$$

$$F^{j\,\prime\prime}(Fx) = \frac{F^{j+1\,\prime\prime} x}{(F^{j+1\,\prime} x)^2} - F^{j+1\,\prime} x \frac{F''x}{(F'x)^2},$$

and

$$E^*(F^p x) = F^{p\,\prime} x\, E^* x + pD'x + pDx \frac{F^{p\,\prime\prime} x}{F^{p\,\prime} x}$$

we get, after some cancellations,

$$E^{**}(Fx) = Ex\, F'x + D'x + Dx \frac{F''x}{F'x} = E(Fx)$$

according to (2–7) and (2–5). Hence $E^{**} = E$ and the lemma is proved. □

Proof of Theorem 2.4. Write

$$\sigma = \frac{1}{p} A(x).$$

Now $A^*(F^j x) = (1/p)A(F^j x) = (1/p)(Ax + j)$, so $F^j x = A^{*-1}(\sigma + j/p)$ and $\varphi^*(F^j x) = \psi^*(\sigma + j/p)$, and Lemma 2.5 gives

$$\psi(p\sigma) = \sum_{j=0}^{p-1} \psi^*\Big(\sigma + \frac{j}{p}\Big).$$

The theorem is proved by observing that

$$\begin{aligned} \hat{\psi}_n &= \int_0^1 \psi(\tau) e^{2\pi i n\tau}\, d\tau \\ &= \sum_{j=0}^{p-1} \int_0^1 \psi^*\Big(\frac{\tau + j}{p}\Big) e^{2\pi i n(\tau+j)}\, d\tau \\ &= \sum_{j=0}^{p-1} p \int_0^1 \psi^*\Big(\sigma + \frac{j}{p}\Big) e^{2\pi i n(p\sigma + j)}\, d\sigma \\ &= p\hat{\psi}^*_{np}. \end{aligned}$$

□

The importance of Theorem 2.4 is that it allows an extension of the LF coefficients α_λ from positive integral λ to any real $\lambda > 0$. Define α_λ for rational $\lambda = n/p$ by

$$\alpha_\lambda = \alpha_{n/p} = \alpha^*_n + \log p \tag{2–23}$$

where α^*_n is the n-th LF coefficient of $F^* = F^p$; the definition is independent of the representation n/p of λ. For if $\lambda = (kn)/(kp)$ then $\alpha_\lambda = \alpha^{**}_{kn} + \log p + \log k$, where α^{**}_n is the n-th LF coefficient of $F^{**} = F^{kp}$, hence $\alpha^*_n = \alpha^{**}_{kn} + \log k$.

We can express (2–23) in a slightly different form. Still with $F^* = F^p$, and setting $F^{**} = F^{*1/n} - F^{p/n}$, we have

$$\begin{aligned} \alpha_{1/p} &= \alpha^*_1 + \log p, \\ \alpha_{n/p} &= \alpha^*_n + \log p = \alpha^{**}_1 + \log p - \log n. \end{aligned}$$

Redefining $F^* = F^{1/\lambda}$, this gives

$$\alpha_\lambda = \alpha^*_1 - \log \lambda.$$

This was only shown to be true for rational λ, but we can now define the function

$$L(\lambda) = L_F(\lambda) = \alpha^*_1 - \log \lambda$$

for all $\lambda > 0$, where F^* is the principal $1/\lambda$-iterate of F. We call $L_F(\lambda)$ the LF function of F; it is the continuous generalization of the LF coefficients.

Clearly,

$$L_{F^{1/r}}(\lambda) = L_F(r\lambda) + \log r \tag{2–24}$$

for any $r > 0$. Thus the LF function has the pleasant property that it is the same (apart from scaling of its variable λ and an irrelevant additive constant) for all principal iterates of F.

The ground is now prepared for the formulation of our criterion of regular growth. The property that presents itself most naturally (partly from numerical evidence) is complete monotonity, that is, the condition

$$(-1)^{n-1} L_F^{(n)}(\lambda) > 0 \quad \text{for all } n \geq 1 \text{ and } \lambda > 0.$$

It is a property that is not affected by an additive constant or a change of scale of the variable in (2–24) hence is preserved for all principal iterates of F. Numerical evidence suggests that complete monotonity is somewhat too strong; a more reasonable condition is that $L'(\lambda)$ should be the difference of two completely monotonic functions. This is the same as saying that $L'(\lambda)$ is a Laplace–Stieltjes transform,

$$L'(\lambda) = \int_0^\infty e^{-\lambda t}\, d\gamma(t),$$

where γ is of bounded variation [Widder 1941, p. 160]. Let us call a function with this property L-regular. Our criterion of regular growth can then be stated as follows:

Criterion 2.6 (regular growth). $F \in \mathcal{C}$ *is said to be regularly growing if the LF function* $L_F(\lambda)$ *of* F *is* L*-regular*.

Clear-cut as it is, the criterion in this form is not very suitable for numerical experimentation. Ultimately one is compelled to examine the complete monotonity and L-regularity of the LF coefficients themselves. A nonnegative sequence $\lambda = \{\lambda_n\}_{n=0}^\infty$ is called *completely monotonic* if all higher-order backward differences

$$\begin{aligned}(\Delta^0\lambda)_n &= \lambda_n,\\ (\Delta^k\lambda)_n &= (\Delta^{k-1}\lambda)_n - (\Delta^{k-1}\lambda)_{n+1}\\ &= \sum_{j=0}^{k} (-1)^j \binom{k}{j} \lambda_{n+j}, \quad \text{for } k \geq 1,\ n \geq 0,\end{aligned}$$

are nonnegative. The sequence $\alpha = \{\alpha_n\}_{n=1}^\infty$ will be called *L-regular* if its first difference sequence

$$\mu_n = (\Delta^1\alpha)_{n+1} = \alpha_{n+1} - \alpha_{n+2}, \quad \text{for } n \geq 0,$$

can be expressed as the difference of two completely monotonic bounded sequences. By a theorem of Hausdorff, this is equivalent to saying that $\mu = \{\mu_n\}$ is a moment sequence, that is,

$$\mu_n = \int_0^1 t^n\, d\chi(t),$$

where $\chi(t)$ is bounded and of bounded variation [Widder 1941, Chapter III]. It is now natural to postulate:

Criterion 2.7 (regular growth, discrete form). F *is said to be regularly growing if the LF sequence of* F *is* L*-regular*.

Strictly speaking, the condition ought to be satisfied for all natural iterates F^p of F according to Theorem 2.4 if we want the two forms of the criterion to be equivalent, but we shall be content with this seemingly weaker form.

From the identity

$$\sum_{j=0}^{k} \binom{k}{j} (\Delta^{k-j}\mu)_j = \mu_0$$

[Hardy 1949, p. 252] we see that

$$\sum_{j=0}^{k} \binom{k}{j} \bigl|(\Delta^{k-j}\mu)_j\bigr| - \mu_0$$

vanishes for all $k \geq 0$ if and only if μ is completely monotonic, and is bounded if and only if μ is a moment sequence. Consequently α is L-regular if and only if the sequence $\Lambda = \{\Lambda_k\}$ defined by

$$\Lambda_k = \sum_{j=1}^{k} \binom{k-1}{j-1} \bigl|(\Delta^{k-j}\alpha)_j\bigr| - \alpha_1, \quad \text{for } k \geq 1, \tag{2–25}$$

is bounded. Here we have a sensitive test for the L-regularity of α; if the LF sequence of F is not L-regular Λ tends to blow up. The Λ-test will be used extensively in the experiments described in the next section.

3. NUMERICAL EVIDENCE

Two things have to be remembered when we calculate the LF coefficients of various functions. One is that, however many coefficients we evaluate, they can never *prove* L-regularity. The other is that being in unexplored territory we cannot really predict (or even make reliable guesses of) what functions will turn out to be regularly growing. If we have preconceived ideas (as people did in early last century about the great inland sea of Australia) we must be prepared for surprises.

Our examples will mostly, but not exclusively, be taken from $\mathcal{C}^+ = \bigcup_{c>1} \mathcal{C}_c$, largely for practical reasons; the computational demands are considerably higher when $F \in \mathcal{C}_1$. For instance, if $c = 2$ and x is taken in the neighbourhood of 10^{-6}, then 30 expansion coefficients will be amply sufficient to calculate $E(x)$, $D(x)$ and $D'(x)$ from (2–3) and (2–8) with an error less than 10^{-170} say, and at most 30 terms will be needed in the series (2–9)—or rather the recursion's (2–10) and (2–11)—for $t(x)$ and $t'(x)$. In contrast, if $F \in \mathcal{C}_1$ then x cannot be taken less than say 10^{-3} if we want to keep the number of terms in (2–9) under 500 and then at least 80 expansion terms may be needed for $E(x)$ and $D(x)$. For some functions such an increase in computational effort by a factor of 10 or more could be prohibitive, given that up to 2000 function values of $\psi(x)$ need to be computed if we want to determine 200 to 300 Fourier coefficients. Examples will show that no essentially new information comes from the study of functions in $\mathcal{C}_1$, and we may just as well stick to $F \in \mathcal{C}_c$ for $c \geq 2$, say.

We begin with some simple elementary functions in $\mathcal{C}_2$ such as

$$2x + x^2 = (1+x)^2 - 1.$$

Intuitively one would like this function (together with its iterates) to be regularly growing, particularly since its principal iterates $F^\sigma(x) = (1 + x)^{2^\sigma} - 1$ have such a simple explicit form. Actually the functions that we have examined were $F(x) = (1+x)^{e^b} - 1$ for various values of $b > 0$. Taking $b = 1$, the Fourier coefficients of $\psi(x)$ went down very rapidly from $|\hat{\psi}_1| = \exp(-9.88211180139)$ to

$$|\hat{\psi}_{25}| = \exp(-248.3695682035),$$

the limit of reliability with 1000 $\psi(x)$ values and 180 decimal figures accuracy. The test quantity Λ_k given by (2–25) was found to be 0 for $k = 1, \cdots, 25$, showing that the LF coefficients were not only L-regular but completely monotonic, at least for the first 25 coefficients. The same behaviour was of course registered for $b = \log 2$, corresponding to the polynomial $2x + x^2$.

Taking $b = 2$, about twice as many LF coefficients could be determined as with $b = 1$ (by Theorem 2.4), and the outcome was quite revealing. The first 19 values Λ_k were again 0, but thereafter Λ_k increased slowly to $\Lambda_{44} = 0.00259959015$, a good indication of L-regularity. The difference in behaviour is understandable if the LF function itself is L-regular but not completely monotonic, since we are now testing the higher derivatives of the LF function by taking the points of evaluation twice as densely as before. It was this example that suggested L-regularity (instead of complete monotonity) as the correct indicator of regular growth. Of course, there is no guarantee that Λ_k will stay bounded past Λ_{44}, but the observed behaviour up to Λ_{45} strongly suggests that the LF function of $(1+x)^e - 1$ is indeed L-regular and (by Theorems 2.3 and 2.4) so is every

$$F(x) = \frac{1}{\beta}\big((1+\beta x)^{e^\sigma} - 1\big)$$

with $\beta > 0$ and $\sigma > 0$. We have thus produced a (presumably) regularly growing family of functions with asymptotics γx^τ (where $\tau = e^\sigma$ and $\gamma = \beta^{\tau-1}$) for every $\tau > 1$ and $\beta > 0$. In particular, the family of polynomials

$$\frac{1}{\beta}\big((1+\beta x)^n - 1\big), \quad \text{for } \beta > 0, \tag{3–1}$$

is regularly growing.

Passing on to polynomials not of the form (3–1), we took as a first example $F(x) = 3x + x^2$. Here the outcome was quite "unexpected": in sharp contrast to $2x+x^2$, the Λ-sequence grew dramatically from $\Lambda_1 > 0$, $\Lambda_4 = 0.938$ to $\Lambda_{19} = 3733.2$, showing unmistakably that the LF coefficients were not L-regular, hence $F(x)$ not regularly growing according to our criterion. The same behaviour was registered by, for example, $F(x) = 3x + 3x^2 + 2x^3$, with $\Lambda_5 = 1.2185$, $\Lambda_{24} = 4228931.0$. Indeed no polynomial different from those of the form (3–1) was found to be regularly growing. Looking for a reason the following conjecture has emerged from numerous experiments.

Conjecture 3.1. *If $F(x)$ has an L-regular LF function and $f(x) = O((F(x))^\delta)$ for some $0 \le \delta < 1$, the LF function of $F(x) + f(x)$ is not L-regular.*

If this conjecture is true, the strange behaviour of polynomials is of course perfectly understandable. Examples such as these show indeed the remarkable sensitivity of the Λ-test; they also suggest that "regularly growing" functions form a highly selective class of functions, which could conceivably serve as an appropriate scale for functional growth.

As a next example we took $F(x) = e^{cx} - 1$, for various values $c \ge 1$. With such fairly fast-growing functions the decrease of the Fourier coefficients (hence LF coefficients) is not nearly as rapid as for $x^2 + 2x$; in consequence we could compute many more LF coefficients with comparable accuracy and computational effort. Taking $c = 2$ the LF coefficients decreased from $\alpha_1 = -5.96067633273$ to $\alpha_{150} = -197.6143012223$, and Λ_k was found to be 0 for all $k < 150$: quite a remarkable outcome hinting at some strong analytic reason. A similar outcome was registered for other values of c, including $c = 1$, making it very plausible that $e^{cx} - 1$ satisfies the regularity criterion for all $c \ge 1$. Indeed the case $c = 1$ was in no way different (except for greatly increased computing time) from those with $c > 1$, and $e^x - 1$ had at least 180 completely monotonic LF coefficients, with

$$\alpha_1 = -6.66486681079445$$

and

$$\alpha_{180} = -227.21602223.$$

On the other hand the functions $e^x + x - 1$, $e^{2x} + x - 1$, $\frac{3}{2}e^x - \frac{1}{2}e^{-x}$, and so on all showed violent increases of Λ_k with Λ_{150} reaching values of order 10^{20} or more, in accordance with Conjecture 3.1. The last example shows that $f(x)$ in the conjecture can even tend to 0 fairly rapidly. In contrast, functions such as $2xe^x$, $(2x+2)e^x - 2$, and $(2x+x^2)e^x$ all seemed to have L-regular (not always completely monotonic) LF coefficients; this indicates that the growth of $f(x)$ relative to $F(\alpha)$ in Conjecture 3.1 cannot be too close to $F(x)$ itself.

Examples such as these can be produced at will, without adding anything new to what we have already displayed. There is more interest in examining functions with nonelementary growth, which therefore do not fit into Hardy's scale. To produce hopeful examples once again Abel's functional equation comes to our rescue. A simple case is provided by the inverse $B = A^{-1}$ of the principal Abel function of $x^2 + bx$, for various values of $b > 1$. The special interest of this example arises from the fact that there are two particular values of b for which the principal Abel function happens to be an elementary function (no others are known and probably do not exist). One is $b = 2$; the principal Abel function of $x^2 + 2x$ is

$$A(x) = \frac{1}{\log 2} \log\log(1+x),$$

with inverse $B(x) = \exp(e^{x \log 2}) - 1$. The other value is $b = 4$; the principal Abel function and its inverse are

$$A(x) = \frac{1}{\log 2} \log\log \tfrac{1}{2}\big(\sqrt{x} + \sqrt{x+4}\,\big),$$
$$B(x) = \big(\exp(e^{x\log 2}) - \exp(-e^{x\log 2})\big)^2,$$

as seen by substitution into (1–1).

We know already what to expect: the first clearly ought to be regularly growing, the second not according to Conjecture 3.1, since

$$B(x) = \exp(2e^{x\log 2}) + \exp(-2e^{x\log 2}) - 2.$$

Computation does indeed confirm both expectations; in the first case Λ_k was found to be 0 for $1 \le k < 150$, in the second case Λ_k grew to $\Lambda_{169} = 1.02 \times 10^{20}$. For no other values of b did we find L-regular LF coefficients. For $b = 2.1$ the LF coefficients of

$$F(x) = \frac{2}{B'(0)}(B(x) - B(0))$$

gave $\Lambda_{169} = 1.8 \times 10^{10}$ and for $b = 3$, for example, we obtained Λ_{169} of order 10^{33}. These results can be understood if we remember that $b = 2$ was the only value of b for which $x^2 + bx$ was regularly growing. This suggests another conjecture:

Conjecture 3.2. *The inverse of the principal Abel function of $F(x)$ (suitably normalised) is regularly growing if and only if $F(x)$ is regularly growing.*

By "suitably normalized" we understand here

$$\frac{c}{B'(a)}(B(x) - B(a)) \quad \text{for } a \in \mathbb{R},\ c \ge 1.$$

Abel's equation also supplies "nice" examples of functions which transcend Hardy's scale in the sense that they grow faster than any finite iterate of e^x. Hardy himself mentions such an example closely related to an Abel function in [Hardy 1910, p. 35], without making any reference to Abel. To produce such an example in $\mathcal{C}$ we took the inverse $B(x)$ of the principal Abel function of $e^x + e^{bx} - 2$ for various values of $b \ne 0$. If Conjectures 3.1 and 3.2 are valid for such very fast-growing functions, we should expect

$$F(x) = \frac{c}{B'(0)}(B(x) - B(0)), \quad \text{for } c \ge 1,$$

to be regularly growing when $b = 1$, and nonregularly growing when $b \ne 1$. This turns out to be the case: experimenting with $c = 1$ or $c = 2$ it was found that $\Lambda_k = 0$ for $1 \le k \le 250$ if $b = 1$ but λ_{250} is of order 10^{11} for $b = 2$, and of order 10^9 for $b = 0.5$. A number of other such examples support without exception the hypothesis that Conjectures 3.1 and 3.2 hold for arbitrarily fast growing functions.

We remark here that the practical computation of $B(x)$ for given $G \in \mathcal{C}^+$ presents no difficulties ($G \in \mathcal{C}_1$ is more awkward). For large negative x—say $x < -10$—$B(x)$ has an expansion

$$B(x) = \sum_{j=1}^{\infty} a_j e^{jx \log b},$$

where $a_1 = 1$ and the remaining expansion coefficients a_j, for $j > 1$, are calculated from $B(x+1) = G(B(x))$. From this series $B(x)$ and its derivatives are easily computed and so are the Taylor coefficients of $B(x)$ at $x = 0$.

The experimental results presented in this section clearly support two additional conjectures:

Conjecture 3.3. *The set $\mathcal{R}$ of regularly growing functions, that is, functions with L-regular LF coefficients, is nonempty.*

Conjecture 3.4. *$\mathcal{R}$ contains arbitrarily fast-growing functions.*

We conclude this paper with a somewhat sketchy account of the link between our Theorems 2.1 and 2.2 and Lévy's original criterion.

4. THE LÉVY–FOURIER COEFFICIENTS

Lévy's original suggestion [1928] for "croissance régulière" amounted to this: Given $F \in \mathcal{C}_1$, take any $a > 0$ and define

$$F_a(x) = \frac{1}{F'(a)}(F(x+a) - F(a)) \tag{4–1}$$

so that $F_a \in \mathcal{C}_1$. Let D, D_a be the respective principal iteration generators of F and F_a. Then we can define a second iteration generator D_a^* for F_a by demanding that

$$\lim_{x\to\infty} D_a^*(x)/D(x) = 1. \tag{4–2}$$

Lévy actually stated his condition in terms of the Abel functions A, A_a, but essentially it amounts to (4–2). Now Lévy proposed that for regularly growing functions D_a^* should be identical with D_a for all $a > 0$. Of course in those pre-computer times

Lévy had no means of checking on his criterion experimentally; otherwise he would have realised that the criterion as it stands is too restrictive and possibly cannot be satisfied for any F except the trivial $e^x - 1$, for which $F_a = F$ for all $a > 0$.

We shall modify Lévy's condition in several ways (apart from the trivial modification (4–2)), so as to make it workable and more accessible to experimentation. First, instead of demanding the equality of D_a and D_a^* we shall examine the discrepancy between the two iteration generators and formulate a criterion that refers to this discrepancy. Secondly, we do not insist on the condition $F'(0) = 1$, which can place uncomfortable demands on computing time, without offering much in return. We can then replace F_a in (4–1) by the simpler

$$F_a(x) = F(x + a) - F(a).$$

Finally, we carry out the comparison of D_a and D_a^* for infinitesimal $a = \varepsilon$, thereby linearizing the problem. This makes the examination of the discrepancy much simpler and hence more accessible.

So let's assume first that $F \in \mathcal{C}_c$ with $c > 1$, so that

$$F(x) = \sum_{j \geq 1} b_j x^j,$$

with $b_1 = c > 1$. Set, for infinitesimal ε,

$$F_\varepsilon(x) = F(x+\varepsilon) - F(\varepsilon) \simeq F(x) + \varepsilon F'(x) - c\varepsilon. \quad (4\text{–}3)$$

Here $\simeq$ means mod ε^2; that is, we neglect higher powers of ε. The factor $F'(x) - c$ of ε is of course

$$\left(\frac{\partial}{\partial \varepsilon} F_\varepsilon(x)\right)_{\varepsilon=0}.$$

To determine D_ε^*, start from a fixed (appropriately small) $x_0 > 0$ and define sequences $x_n = F(x_{n-1})$, $n \geq 1$ and $D_n = D(x_n)$ determined by

$$D_0 = D(x_0), \quad D_n = F'(x_{n-1}) D_{n-1}, \quad \text{for } n \geq 1,$$

according to (2–5). To determine the corresponding sequences $x_n^{(\varepsilon)}$ and $D_n^{(\varepsilon)} = D_\varepsilon(x_n^{(\varepsilon)})$ pertaining to F_ε, set

$$x_n^{(\varepsilon)} \simeq x_n - \varepsilon y_n \quad \text{for } n \geq 0,$$

where the initial value y_0 will be determined later, from the behaviour of y_n at $n \to \infty$.

Then

$$\begin{aligned} x_{n+1}^{(\varepsilon)} &\simeq F_\varepsilon(x_n - \varepsilon y_n) \simeq F_\varepsilon(x_n) - \varepsilon y_n F'(x_n) \\ &\simeq F(x_n) + \varepsilon(F'(x_n) - c - y_n F'(x_n)) \\ &\simeq x_{n+1} - \varepsilon y_{n+1}, \end{aligned}$$

giving

$$y_{n+1} = (y_n - 1)F'(x_n) + c.$$

Or, setting $t_n = 1 - y_n$,

$$\begin{aligned} t_{n+1} &= t_n F'(x_n) - (c-1), \\ t_n &= \frac{t_{n+1} + (c-1)}{F'(x_n)}. \end{aligned}$$

Hence, with the "initial value" of the sequence y_n chosen appropriately, namely $y_\infty = 1$, the recursion is satisfied by

$$t_n = (c-1)\left(\frac{1}{F'(x_n)} + \frac{1}{F'(x_n)F'(x_{n+1})} + \cdots\right).$$

Using the notation (2–9) for $t(x)$ we see that

$$t_n = 1 - y_n = (c-1)(t(x_n) - 1). \quad (4\text{–}4)$$

With the sequence $x_n^{(\varepsilon)} \simeq x_n - \varepsilon y_n$ we can now determine $D_n^{(\varepsilon)} = D_\varepsilon(x_n^{(\varepsilon)})$, where D_ε is the principal iteration generator of F_ε. Set

$$D_n^{(\varepsilon)} \simeq (1 + \varepsilon S_n) D_n;$$

then, since $F_\varepsilon'(x) \simeq F'(x) + \varepsilon F''(x)$,

$$\begin{aligned} &(1 + \varepsilon S_{n+1}) D_{n+1} \\ &\quad \simeq F_\varepsilon'(x_n - \varepsilon y_n)(1 + \varepsilon S_n) D_n \\ &\quad \simeq (1 + \varepsilon S_n)\big(F'(x_n) + \varepsilon(F''(x_n) - y_n F''(x_n))\big) D_n \\ &\quad \simeq (1 + \varepsilon S_n)\Big(1 + \varepsilon \frac{F''(x_n)}{F'(x_n)} t_n\Big) D_{n+1}, \end{aligned}$$

giving

$$S_{n+1} = S_n + (c-1)\frac{F''(x_n)}{F'(x_n)}(t(x_n) - 1)$$

by (4–4). With the notation (2–11) it results in

$$S_\infty = \lim_{n\to\infty} S_n = S_0 + (c-1)\sum_{n=0}^{\infty} \frac{F''(x_n)}{F'(x_n)}(t(x_n)-1)$$
$$= S_0 - (c-1)t'(x_0). \qquad (4\text{–}5)$$

Setting D^*_ε asymptotically equal to D according to the modified Lévy matching process (4–2), we see that S_∞ represents the discrepancy between the two iteration generators D_ε and D^*_ε of F_ε, that is, the quantity that we wish to identify. We still need to know S_0, and this is determined from the assumption (which we haven't exploited so far) that D_ε is the principal iteration generator of F_ε. This means that D_ε has the appropriate behaviour at $x = 0$, that is we can set

$$D_\varepsilon(x) \simeq D(x) + \varepsilon\big(cD'(x) + (c-1)E(x)\big), \qquad (4\text{–}6)$$

where of course $D(x)$ is the principal iteration generator of F. The form of the ε-term on the right hand side is for convenience.

Substituting for $F_\varepsilon(x)$ from (4–3) and for $D_\varepsilon(x)$ from (4–6) we get, using (2–5),

$$\begin{aligned} &D_\varepsilon(F_\varepsilon x)\\ &\quad\simeq D(F_\varepsilon x) + \varepsilon\big(cD'(Fx) + (c-1)E(Fx)\big)\\ &\quad\simeq D(Fx) + \varepsilon(F'x - c)D'(Fx)\\ &\qquad\qquad + \varepsilon\big(cD'(Fx) + (c-1)E(Fx)\big)\\ &\quad\simeq F'x\,Dx + \varepsilon(F''x\,Dx + F'x\,D'x + (c-1)E(Fx)), \end{aligned}$$

and

$$\begin{aligned} &F'_\varepsilon x\, D_\varepsilon x\\ &\quad\simeq (F'x + \varepsilon F''x)\big(Dx + \varepsilon(cD'x + (c-1)Ex)\big)\\ &\quad\simeq F'x\,Dx + \varepsilon\big(F''x\,Dx + cF'x\,D'x + (c-1)F'x\,Ex\big). \end{aligned}$$

Equating these two expressions we get

$$E(F(x)) = F'(x)E(x) + F'(x)D'(x)$$

that is equation (2–6) for the inhomogeneous iteration generator $E(x)$. All we have to observe now is that

$$\begin{aligned} &D_\varepsilon(x_0 - \varepsilon y_0)\\ &\quad\simeq D_\varepsilon(x_0) - \varepsilon y_0 D'(x_0)\\ &\quad\simeq D(x_0) + \varepsilon(cD'(x_0) + (c-1)E(x_0) - y_0D'(x_0)) \end{aligned}$$

by (4–6) and so

$$\begin{aligned} S_0 &= \frac{(c-y_0)D'(x_0) + (c-1)E(x_0)}{D(x_0)}\\ &= \frac{c-1}{D(x_0)}(t(x_0)D'(x_0) + E(x_0)). \end{aligned}$$

Combining this with (4–5) and using the notation of Theorem 2.1 we get

$$S_\infty = (c-1)\varphi(x_0).$$

This establishes the link between Theorem 2.1 and Lévy's matching process (in the modified form).

Our final remark concerns the value of e_1 in (2–16). We saw earlier that the LF coefficients are independent of e_1, nevertheless it is important to verify that with the value (2–16) of e_1 the control equation

$$\int_0^1 \psi(\sigma)\,d\sigma = 0 \qquad (4\text{–}7)$$

is indeed satisfied. This is important not only for its own interest but because we have used (4–7) repeatedly as a check for the accuracy of the computations in Section 3.

First note that $d_1 = \log c$ in (2–3); therefore $d_1^{(\varepsilon)} = \log c^{(\varepsilon)}$, where

$$\begin{aligned} D_\varepsilon(x) &= \sum_{j\ge 1} d_j^{(\varepsilon)} x^j,\\ F_\varepsilon(x) &= c^{(\varepsilon)}x + \sum_{j\ge 2} b_j^{(\varepsilon)} x^j. \end{aligned}$$

But

$$F_\varepsilon(x) \simeq \sum_{j\ge 1} b_j x^j + \varepsilon \sum_{j\ge 2} j b_j x^{j-1},$$

by (4–2); hence $b_1^{(\varepsilon)} = c^{(\varepsilon)} \simeq c + 2\varepsilon b_2$ and

$$d_1^{(\varepsilon)} = \log c^{(\varepsilon)} \simeq \log c + \frac{2\varepsilon}{c} b_2.$$

On the other hand,

$$D_\varepsilon(x) \simeq D(x) + \varepsilon c D'(x) + \varepsilon(c-1)E(x)$$

from (4–6); hence, by (2–3),

$$d_1^{(\varepsilon)} = d_1 + \varepsilon\left(\frac{2b_2\log c}{c-1} + (c-1)e_1\right).$$

Equating these two expressions for $d_1^{(\varepsilon)}$ we obtain

$$(c-1)e_1 = \frac{2b_2}{c} - 2b_2\frac{\log c}{c-1},$$

which immediately gives (2–16).

Now both D_ε and D_ε^* are iteration generators of F_ε, and so Lemma 1.1 tells us that $D_\varepsilon(x) = D_\varepsilon^*(x)\bigl(1+\varepsilon\Phi'(A(x))\bigr)$ for some periodic Φ with period 1. Hence $\psi(\sigma) = \Phi'(\sigma)$ and

$$\int_0^1 \psi = \Phi(1) - \Phi(0) = 0,$$

which is (4–7). This derivation leans heavily on the fact that ψ is the discrepancy between two iteration generators whereas Theorem 2.1 makes no reference to this interpretation. A more direct derivation would no doubt be of some interest.

The argument that leads to Theorem 2.1 breaks down if $b_1 = F'(0) = 1$, essentially because of the different form (2–4) of $D(x)$. There are two different approaches to this problem: we may either "infinitesimalize" Lévy's original process, or regard $F \in \mathcal{C}_1$ as a limiting case of functions $\mathcal{C}_c$, $c > 1$. The second approach leads to the formulation of Theorem 2.2 and we omit details. The first approach is more interesting, not only because of its closer links with Lévy's process but because it shows that the construction of a ψ with the required property is by no means unique and there are alternative forms which do not necessarily lead to a useful criterion. We sketch the steps that result in a second form of Theorem 2.2.

We want to compare the principal iteration generator of

$$F(x) = x + \sum_{j\ge 2} b_j x^j$$

with that of

$$\frac{F(x+a) - F(a)}{F'(a)}.$$

Taking an infinitesimal $a = \varepsilon$, we have now

$$\begin{aligned} F_\varepsilon(x) = \frac{F(x+\varepsilon) - F(\varepsilon)}{F'(\varepsilon)} &\simeq \frac{F(x) + \varepsilon F'(x) - \varepsilon}{1 + b\varepsilon} \\ &\simeq F(x) + \varepsilon\bigl(F'(x) - 1 - bF(x)\bigr), \end{aligned} \tag{4–8}$$

where $b = 2b_2$.

The sequence $x_n^{(\varepsilon)} \simeq x_n - \varepsilon y_n$ satisfies

$$\begin{aligned} x_{n+1}^{(\varepsilon)} &\simeq F_\varepsilon(x_n - \varepsilon y_n) \\ &\simeq F(x_n) + \varepsilon(F'(x_n) - 1 - bF(x_n) - y_n F'(x_n)), \end{aligned}$$

giving

$$y_{n+1} = (y_n - 1)F'(x_n) + 1 + bF(x_n).$$

Hence $t_n = 1 - y_n$ satisfies

$$\begin{aligned} t_{n+1} &= t_n F'(x_n) - bF(x_n), \\ t_n &= \frac{t_{n+1} + bF(x_n)}{F'(x_n)}, \\ t_n &= b\left(\frac{F(x_n)}{F'(x_n)} + \frac{F^2(x_n)}{F'(x_n)F'(F(x_n))} + \cdots\right). \end{aligned}$$

Or, defining

$$t^*(x) = \sum_{k\ge 1} \frac{F^k(x)}{{F^k}'(x)}, \tag{4–9}$$

we get

$$t_n = bt^*(x_n).$$

For S_n we obtain

$$\begin{aligned} &(1+\varepsilon S_{n+1})D_{n+1} \\ &\quad\simeq F_\varepsilon'(x_n - \varepsilon y_n)(1+\varepsilon S_n)D_n \\ &\quad\simeq (1+\varepsilon S_n) \\ &\qquad\times\bigl(F'(x_n) + \varepsilon(F''(x_n) - bF'(x_n) - y_n F''(x_n))\bigr)D_n, \end{aligned}$$

giving

$$S_{n+1} = S_n - b + t_n\frac{F''(x_n)}{F'(x_n)}.$$

Defining now

$$s(x) = \sum_{n\ge 0}\left(t(x_n)\frac{F''(x_n)}{F'(x_n)} - 1\right) \tag{4–10}$$

we obtain

$$S_\infty = S_0 + bs(x_0).$$

We have assumed the convergence of (4–9) and (4–10), which is not true for all $F \in \mathcal{C}_1$. But convergence is assured if $F = \exp F^* - 1$ with $F^* \in \mathcal{C}_1$ (just note that $F/F' = 1/(\log F)'$.) Therefore whatever regularity criterion we might be able to formulate for $F^* \in \mathcal{C}_1$ following Lévy's original process, we would have to exponentiate the function to be tested before subjecting it to the criterion.

Now for $D_\varepsilon(x)$ we set

$$D_\varepsilon(x) \simeq D(x) + \varepsilon(D'(x) + bE^*(x)). \tag{4–11}$$

Then, as before,

$$\begin{aligned}
&D_\varepsilon(F_\varepsilon x)\\
&\quad\simeq D(F_\varepsilon x) + \varepsilon\big(D'(Fx) + bE^*(Fx)\big)\\
&\quad\simeq D(Fx) + \varepsilon\big((F'x - bFx)D'(Fx) + bE^*(Fx)\big)\\
&\quad\simeq F'xDx + \varepsilon\bigg(\Big(1 - b\frac{Fx}{F'x}\Big)(F''xDx + F'xD'x)\\
&\qquad\qquad + bE^*(Fx)\bigg)
\end{aligned}$$

and

$$\begin{aligned}
&F'_\varepsilon x D_\varepsilon x\\
&\quad\simeq \big(F'x + \varepsilon(F''x - bF'x)\big)\big(Dx + \varepsilon(D'x + bE^*x)\big).
\end{aligned}$$

Equating these two expressions we get, after some cancellations, the equation

$$\begin{aligned}
E^*(Fx) &= F'x\,E^*x - F'x\,Dx + Fx\,D'x + \frac{Fx\,F''x}{F'x}Dx\\
&= F'x\,E^*x - F'x\,Dx + Fx\,D'(Fx)
\end{aligned} \tag{4–12}$$

for E^*x. Since

$$\begin{aligned}
&D_\varepsilon(x_0 - \varepsilon y_0)\\
&\qquad\simeq D(x_0) + \varepsilon(D'(x_0) + bE^*(x_0) - y_0 D'(x_0)),
\end{aligned}$$

we derive as before

$$S_0 = \frac{(1-y_0)D'(x_0) + bE^*(x_0)}{D(x_0)} = bS(x_0),$$

where

$$S(x) = \frac{t^*(x)D'(x) + E^*(x)}{D(x)}.$$

This leads to the next result:

Theorem 4.1. *Let $F(x) = \exp(F^*(x)) - 1$ with $F^* \in \mathcal{C}_1$, and let D, E^*, t^*, s be defined by* (2–5), (4–12), (4–9), *and* (4–10). *Set*

$$S(x) = \frac{1}{D(x)}\big(t^*(x)D'(x) + E^*(x)\big).$$

Then $\varphi^*(x) = S(x) + s(x)$ has the property that $\varphi^*(F(x)) = \varphi^*(x)$ for all $x > 0$.

This is the alternative form of Theorem 2.2; its proof is analogous to that of Theorem 2.1 and is omitted. The value of the missing coefficient e_2 is found to be

$$e_2 = \tfrac{1}{2}b_2;$$

its derivation is similar to the derivation of (2–16).

We could derive now LF coefficients of a second kind based on the φ^*-function of Theorem 4.1. It is easy to see that the φ^*-function of $F(x) = e^x - 1$ (corresponding to F^* the identity function) is identically 0. In fact if $F(x) = e^x - 1$ then $t(x) = 1$ identically in (4–9) and $s(x) = 0$. Hence $E^*(x) = -D'(x)$ by (4–12) and (2–5), $S(x) = 0$, hence $\varphi^*(x)$ is identically 0 in Theorem 4.1. This is as it should be since in this case F_ε is identical with F and the Lévy matching is trivial.

Taking the second iterate $\exp(e^x - 1) - 1$ of $e^x - 1$ the even Fourier coefficients are of course 0, but the odd ones are not. This alone makes it unlikely that a useful regularity criterion based on L-regularity could be obtained from Theorem 4.1.

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Received October 21, 1996; accepted in revised form July 25, 1997