A Family of Cubic Rational Maps and Matings of Cubic Polynomials

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CONTENTS

We study a family of cubic branched coverings and matings of cubic polynomials of the form $g \perp\!\!\!\perp f$, with $g = g_a : z \mapsto z^3 + a$ and $f = P_i$ for $i = 1, 2, 3$ or 4. We give criteria for matability or not of critically finite g_a with each P_i. The maps $g_a \perp\!\!\!\perp P_1$ illustrate features that do not occur for matings of quadratic polynomials: they never have Levy cycles but do sometimes have Thurston obstructions.

INTRODUCTION

A mating of two polynomials is a covering of the 2-sphere constructed in a certain simple way, which gives it interesting dynamics.

The precise construction is as follows. Let g and f be two monic polynomials of the same degree, such that all critical points have finite orbits. We first add to $\mathbb{C}$ a circle of directions at infinity, and extend both g and f to this circle in the natural way. Next we sew up two copies of $\mathbb{C}$ along the circles at infinity, with opposite orientations. This gives a topological sphere. We then define a branched covering $g \perp\!\!\!\perp f$ that coincides with g on one hemisphere and with f on the other. If this map, or a suitable modification of it, is Thurston-equivalent to a rational map (which is to say, it is equivalent to a rational map up to isotopy relative to the post-critical set and topological conjugacy), we say that g and f are *matable*, and that $g \perp\!\!\!\perp f$ is a *mating* of f and g.

In this paper we study matings of cubic polynomials $g \perp\!\!\!\perp f$ that occur in $\mathcal{A}$, the class of cubic branched coverings with one double critical point and one period-three orbit containing two simple critical points. By affine conjugation one can reduce to the case where g is of the form of $g_a : z \mapsto z^3 + a$ and f is among four specific cubic polynomials P_1,

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P_2, P_3, and P_4 (Lemma 2.1). We want to study the matability of these pairs of polynomials.

There exist already several necessary or sufficient conditions for two general polynomials g and f to be matable, namely by means of Thurston obstructions and Levy cycles on one hand, and by means of the topology of $S^2/\sim_{\mathrm{ray}}$ on the other hand, where $\sim_{\mathrm{ray}}$ is the equivalence relation on the sewed up sphere generated by external rays of g and f. We will introduce the notion of removable Levy cycles and formulate these various criteria.

Theorem 0.1. *The four properties*

(a) *g and f are not matable*,
(b) *$g \perp\!\!\!\perp f$ has an irreducible obstruction which is not a removable Levy cycle*,
(c) *$g \perp\!\!\!\perp f$ has a non-removable Levy cycle*,
(d) *$S^2/\sim_{\mathrm{ray}}$ is not homeomorphic to S^2*

can be related as follows:

$$\begin{array}{ccc} \text{(a)} & \overset{\Longrightarrow}{`\Longleftarrow\text{'}} & \text{(b)} \\ \Uparrow & & \Uparrow \\ \text{(d)} & \Longleftarrow & \text{(c)} \end{array}$$

where (a)'$\Longleftarrow$'(b) *means that* (a)$\Longleftarrow$(b) *if a suitable modification $s(g \perp\!\!\!\perp f)$ of $g \perp\!\!\!\perp f$* (*see Theorem* 3.12) *has a hyperbolic orbifold* (*there are counterexamples otherwise*).

Implication (c)$\Longrightarrow$(b) is trivial, and (d) $\Longrightarrow$ (a) was proved by M. Rees (see also [Shishikura 2000]). We will give a proof of the rest; other similar proofs exist in the literature.

We first show how to modify a branched covering in order to delete a maximal number of removable Levy cycles, without changing the dynamics too much (Theorem 3.12 and Corollary 3.13). We then apply this technique to matings, making use of the ray-equivalence relation (Proposition 3.16).

For quadratic matings and cubic matings related to Newton maps, we also have (b)$\Longrightarrow$(c) [Tan 1992; 1997]; therefore in this case the four conditions are in fact equivalent.

One major objective of this work is to show that (b), (c), and (d) are not always equivalent (K. Pilgrim [1994] has found other examples of this type). More precisely, we show that (a) $\Longleftrightarrow$ (b) for a mating $g \perp\!\!\!\perp f$ in $\mathcal{A}$ and that, for $g_a \perp\!\!\!\perp P_1$, $S^2/\sim_{\mathrm{ray}}$ is a sphere, there is no Levy cycles and there are a-values for which $g_a \perp\!\!\!\perp P_1$ satisfies (b) (and therefore (a)). We conjecture that g_a and P_1 are not matable if and only if for $s = a$ or $\bar{a}$, the two external rays $R_{g_s}(\frac{7}{26})$ and $R_{g_s}(\frac{9}{26})$ land at the same point. The set of such s values form a sublimb of the connectedness locus M_3. So far we can only prove that for s in one half of the sublimb union the largest copy of M_3 in the sublimb, and $a = s$ or $\bar{s}$, g_a and P_1 are not matable. We find also a values outside of the sublimb (in terms of the topological entropy of g_a on its Hubbard tree) for which g_a and P_1 are matable. But we are unable to give a complete proof of either direction of the conjecture. It might therefore not be a realistic conjecture.

For each other choice of f, we try also to find conditions on a so that g_a and f are matable. Two of the four possible candidates for f, denoted by P_3 and P_4, are non real polynomials with star-like Hubbard trees. We shall prove that each of them is matable with any critically finite g_a (Theorem 2.6). For $g_a \perp\!\!\!\perp P_2$, the remaining case, we show that (c) is equivalent to the fact that the two external rays $R_{g_a}(\frac{5}{8})$ and $R_{g_a}(\frac{7}{8})$ land at the same point (Theorem 2.5). We conjecture that this last condition is a consequence of (b), therefore the four properties (a), (b), (c), and (d) are equivalent.

In order to prove these results, we study at first the structure of Thurston obstructions for a general map in $\mathcal{A}$ (Theorem 2.2 and Proposition 2.3), and then apply this to matings, using techniques about Hubbard trees, external rays, etc.

Although our results are only partial, some of the tools introduced here might be useful for the study of other class of maps. We emphasise also that most phenomena about matings in $\mathcal{A}$ were first observed by computer experiments, and then proved or partially proved mathematically later on. Here the computer experiments have served us as a guide to rigorous mathematical results.

The paper is organized as follows: Sections 1A–1C recall definitions and results. In Section 2 we introduce the class $\mathcal{A}$ and the polynomials P_i, state our results about structures of Thurston obstructions for a map in $\mathcal{A}$ and, for each i, give criteria for g_a and P_i to be not matable. We give also an example of a mating in $\mathcal{A}$ which has a Thurston obstruction but no Levy cycles. Section 3A contains a sufficient condition for $S^2/\sim_{\rm ray}$ of a mating to be a sphere. In Sections 3B–3H we review and develop several techniques for the analysis of Thurston obstructions and (removable) Levy cycles together with their applications to matings. In Section 4 we prove our criteria about non-matability of g_a and P_i for $i = 2, 3, 4$ and show that for $g_a \perp\!\!\!\perp P_1$, the quotient $S^2/\sim_{\rm ray}$ is a sphere. In Section 5 we establish results regarding Thurston obstructions for a map in $\mathcal{A}$ and for $g_a \perp\!\!\!\perp P_1$.

In the appendix, we give interpretations of our results in the parameter space of rational maps in $\mathcal{A}$ along with several numerical observations.

1. PRELIMINARIES

1A. Thurston and Levy's Theory on Branched Coverings from S^2 to S^2

All branched coverings in this paper are assumed to be orientation-preserving and of degree greater than one.

Definition (postcritical set). Let $F : S^2 \to S^2$ be a branched covering of degree d. Set

$$\Omega_F = \{\text{critical points of } F\} \quad\text{and}\quad P_F = \bigcup_{n>0} F^n(\Omega_F).$$

The set P_F is called the *postcritical set* of F. We say that F is *postcritically finite*, if P_F is finite.

Definition (equivalence between branched coverings). We say two postcritically finite branched coverings F and G from S^2 to S^2 are *equivalent*, and write $F \sim G$, if some topological conjugate of G is isotopic to F rel P_F. More precisely, there exist two orientation-preserving homeomorphisms $\theta_1, \theta_2 : S^2 \to S^2$ such that $G \circ \theta_1 = \theta_2 \circ F$, $\theta_1 = \theta_2$ on P_F, $\theta_i(P_F) = P_G$ for $i = 1, 2$, and θ_1, θ_2 are isotopic relative to P_F.

There are some equivalent conditions for F, G to be equivalent. First, we summarize related results of Thurston and Levy. For details and proofs see [Thurston 1983; Douady and Hubbard 1993].

Definition (multicurve). Let $F : S^2 \to S^2$ be a postcritically finite branched covering, with degree $d \geq 2$. A simple closed curve in $S^2 - P_F$ is called *peripheral* if it bounds a disc containing at most one point of P_F. A *multicurve* Γ is a collection of finitely many disjoint nonperipheral simple closed curves in $S^2 - P_F$ such that no two curves are isotopic to each other in $S^2 - P_F$.

For a multicurve Γ of F, the *Thurston linear transformation* F_Γ is a linear map from $\mathbb{R}^\Gamma = \{\sum_{\gamma\in\Gamma} c_\gamma \gamma \mid c_\gamma \in \mathbb{R}\}$ to itself defined by

$$F_\Gamma(\gamma) = \sum_{\gamma'\subset F^{-1}(\gamma)} \frac{1}{\deg(F : \gamma' \to \gamma)} [\gamma']_\Gamma \quad \text{for } \gamma \in \Gamma,$$

where the sum is over all connected components γ' of $F^{-1}(\gamma)$, and $[\gamma']_\Gamma$ denotes the curve in Γ isotopic to γ' if it exists and $[\gamma']_\Gamma = 0$ otherwise. We denote by λ_Γ the leading eigenvalue of F_Γ.

A multicurve Γ with $\lambda_\Gamma \geq 1$ is called a *Thurston obstruction*. Initially, Thurston obstructions are defined for F-invariant multicurves. But the two definitions are equivalent, as shown in Lemma 3.5 below.

Theorem 1.1 (Thurston). *Suppose $F : S^2 \to S^2$ is a postcritically finite branched covering. If F has no Thurston obstructions then F is equivalent to a rational map. The rational map is unique up to conformal conjugacy. If F has a hyperbolic orbifold (see the remark below) and Thurston obstructions, then F is not equivalent to a rational map.*

Remark. The notion of orbifold can be found in [Douady and Hubbard 1993]. If a branched covering F is has a non-hyperbolic orbifold, then $F^{-1}(P_F) \subset \Omega_F \cup P_F$ and $\#P_F \leq 4$.

Definition (good, degenerate and removable Levy cycles). A multicurve $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ of F is called a *Levy cycle* if each $F^{-1}(\gamma_{i+1})$ contains a component γ'_{i+1} isotopic in $S^2 - P_F$ to γ_i and $F : \gamma'_{i+1} \to \gamma_{i+1}$ is of degree one for $i = 0, \ldots, n-1$, where $\gamma_0 = \gamma_n$. We say that Γ is a *good Levy cycle* if the connected components of $S^2 - \bigcup_{i=1}^n \gamma_i$ are $B_1, B_2, \ldots, B_m, C$, with B_i discs, and, in case $n = 1$, $C = \varnothing$ and $F : \gamma'_1 \to \gamma_1$ reverses the orientation; in case $n > 1$, one component C' of $F^{-1}(C)$ is isotopic to C and $F : C' \to C$ is of degree one. We say that Γ is a *degenerate Levy cycle* if the connected components of $S^2 - \bigcup_{i=1}^n \gamma_i$ are $B_1, B_2, \ldots, B_n, C$, with B_i discs, and each $F^{-1}(B_{i+1})$ has a component B'_{i+1} isotopic to B_i (rel P_F), and $F : B'_{i+1} \to B_{i+1}$ is of degree one for $i = 0, 1, \ldots, n-1$, where $B_0 = B_n$. We say that Γ is a *removable Levy cycle* if it is degenerate and for all $j \geq 1$ and all i, the components of $F^{-j}(B_i)$ are discs.

Here are some known results concerning degree two mappings:

Theorem 1.2 [Levy 1985]. *Suppose F is of degree 2 and postcritically finite. If it has a Thurston obstruction then it has a Levy cycle.*

Theorem 1.3 [Rees 1986; Tan 1992]. *Suppose F is of degree 2 and postcritically finite. If it has a non-removable Levy cycle then it has a good Levy cycle.*

We will see in this work many counterexamples of Theorem 1.2 for higher-degree mappings. In fact we are going to construct a family of cubic branched coverings which have nested obstructions (see definition below) but no Levy cycles. A counterexample of Theorem 1.3 can be easily found.

We will see in Section 3E that each removable Levy cycle can be literally 'removed' by modifying the map. On the other hand, one can not always remove simultaneously two such cycles if they are not disjoint. The following result, which will be made precise and proved in Section 3E, gives the condition under which one can remove all the Levy cycles and get a rational map:

Definition (irreducible obstruction). We say that a multicurve Γ is an *irreducible obstruction* if the matrix F_Γ is irreducible with $\lambda(\Gamma) \geq 1$.

Theorem and definition (weak equivalence). We say that F is *weakly equivalent* to a rational map if all irreducible obstructions of F are removable Levy cycles smaller or equal to an element of $\Sigma'(F)$ (see Section 3E for definition). In this case a suitable modification of F is equivalent to a rational map.

Definition (nested obstruction). A multicurve Γ for a postcritically finite map F is a *nested obstruction* if $S^2 - \Gamma$ has only two disc components and $\lambda(F_\Gamma) \geq 1$. For example a Levy cycle with one or two curves is a nested obstruction.

1B. Matings of Polynomials

In this section, we give definitions of matings.

Definition (formal mating). Let f and g be two monic polynomials of degree d. Set

$$\tilde{\mathbb{C}} = \mathbb{C} \cup \{\infty \cdot e^{2\pi i s} \mid s \in \mathbf{T} = \mathbb{R}/\mathbb{Z}\}.$$

We then extend f and g continuously to $\tilde{\mathbb{C}}$ by setting

$$f(\infty \cdot e^{2\pi i s}) = \infty \cdot e^{2d\pi i s},$$
$$g(\infty \cdot e^{2\pi i s}) = \infty \cdot e^{2d\pi i s}.$$

Set

$$S^2_{f,g} = \tilde{\mathbb{C}}_f \sqcup \tilde{\mathbb{C}}_g / \{(\infty \cdot e^{2\pi i s}, f) \approx (\infty \cdot e^{-2\pi i s}, g)\}.$$

The *formal mating* of f and g is defined to be the branched covering $f \perp\!\!\!\perp g : S^2_{f,g} \to S^2_{f,g}$ with

$$f \perp\!\!\!\perp g = f \text{ on } \tilde{\mathbb{C}}_f \quad \text{and} \quad f \perp\!\!\!\perp g = g \text{ on } \tilde{\mathbb{C}}_g.$$

When there is no ambiguity, we write S^2 instead of $S^2_{f,g}$.

Definition (matability). We say that two monic postcritically finite degree d polynomials f and g are *matable* if $f \perp\!\!\!\perp g$ is weakly equivalent to a rational map.

See [Douady and Hubbard 1984] or [Carleson and Gamelin 1993] for the definitions of the Julia set J_f, the filled-in Julia set K_f and the external rays of a polynomial f. Recall that if f is postcritically finite, then K_f is connected and locally connected, and all external rays land at K_f. For $\theta \in \mathbf{T}$, let us denote by $R_f(\theta)$ the closure in $\tilde{\mathbb{C}}_f$ of the external ray of angle θ.

In $S^2_{f,g}$, the external rays $R_f(\theta)$ and $R_g(-\theta)$ are connected at the point $(\infty \cdot e^{2\pi i\theta}, f)$.

Definition (ray-equivalence). For x and y in $\tilde{\mathbb{C}}_f$, we define $\sim_f$ to be the equivalence relation generated by $x, y \in R_f(\theta)$ for some θ. The relation $\sim_g$ on $\tilde{\mathbb{C}}_g$ is defined similarly. In $S^2_{f,g}$, define the *ray-equivalence* $\sim_{\text{ray}}$ to be the equivalence relation generated by $\sim_f$ on $\tilde{\mathbb{C}}_f$ and $\sim_g$ on $\tilde{\mathbb{C}}_g$. Denote by $[x]$ the ray-equivalence class (or for short: the ray class) of $x \in S^2$.

A Levy cycle for a mating is closely related to a periodic cycle of ray classes.

Theorem 1.4 [Tan 1992]. *Set $F = f \perp\!\!\!\perp g$. Each Levy cycle Γ corresponds to a unique periodic cycle of ray classes $[x_0], \dots, [x_{m-1}]$ (they are called limit set of Γ) and a finitely connected subset X_i of $[x_i]$ for each i such that $F : [x_i] \to [x_{i+1 \bmod m}]$, $X_i \to X_{i+1 \bmod m}$ are homeomorphisms, and Γ is isotopic to the boundary of a tubular neighborhood of $X_0 \cup \cdots \cup X_{m-1}$. In particular,*

1. *if* Γ *is degenerate, then*

$$\#[x_i] \cap P_F \geq \#X_i \cap P_F \geq 2,$$

and each component of X_i *is a tree;*
2. *if* Γ *is not degenerate, each* X_i *(and* $[x_i]$*) contains a closed curve;*
3. *if* Γ *is a good, then* X_0 *is connected,* $F(X_0) = X_0$, $F([x_0]) = [x_0]$. *If* $\#\Gamma > 1$, *then* $\pi_1(X_0)$ *is isomorphic to the fundamental group of the unique non-disc component of* $S^2 - \Gamma$. *If* $\#\Gamma \leq 2$, *then* X_0 *is a simple closed curve,* $F : X_0 \to X_0$ *reverses the orientation and* X_0 *contains exactly two fixed points of* F *(if* F *is quadratic,* $[x_0]$ *contains exactly two fixed points, even when* $\#\Gamma > 2$*).*

1C. M. Rees' Theory of Matings

Denote by $\gamma_f(\theta)$ the landing point of $R_f(\theta)$ on J_f. Note that

$$S^2_{f,g}/\sim_{\text{ray}} = K_f \sqcup K_g/\{\gamma_f(\theta) \approx \gamma_g(-\theta)\}.$$

Theorem 1.5 [Rees 1992; Shishikura 2000]. *Suppose* f *and* g *are two monic postcritically finite polynomials, both with degree* d. *If* f *and* g *are matable then the following two conditions are satisfied:*

(i) *the quotient space* $S^2_{f,g}/\sim_{\text{ray}}$ *is homeomorphic to* S^2;
(ii) *the quotient map*

$$[f \perp\!\!\!\perp g] : S^2_{f,g}/\sim_{\text{ray}} \to S^2_{f,g}/\sim_{\text{ray}}$$

is topologically conjugate to a rational map.

This theorem is presented and proved in case of $d = 2$ and f and g both hyperbolic polynomials in [Rees 1992]. Shishikura gave a proof of the general case in [Shishikura 2000]. Note that a ray class of $f \perp\!\!\!\perp g$ containing at least two postcritical points will represent one postcritical point for the quotient mapping. This is the reason for which we have to modify the formal mating in order to study the equivalence between a mating and a rational map.

Condition (i) is a purely topological one, but the condition (ii) involves conformal structure problems. For quadratic matings and matings related to cubic Newton methods, (i) implies also the matability of the polynomials [Rees 1986; Tan 1992; 1997]. But our study in this paper will show that this is not always true.

2. STATEMENT OF THE RESULTS AND EXAMPLES

2A. The Class $\mathcal{A}$ and Related Matings

We are mainly interested in a special class $\mathcal{A}$ of cubic branched coverings and related matings.

Notation. We denote by $\mathcal{A}$ the set of cubic branched coverings F satisfying: F has a double critical point w and a period-three cycle $x \mapsto y \mapsto z \mapsto x$ containing two simple critical points x and y. Denote by a the critical value $F(w)$.

We want to study matings in $\mathcal{A}$. It turns out that they all have a simple form, as indicated in the next lemma. First note that cubic polynomials with a double critical point are always of the form $g_a : z \mapsto z^3 + a$. Next, there are exactly four pairs of monic cubic polynomials P_1, $\tilde{P}_1$, P_2, $\tilde{P}_2$, P_3, $\tilde{P}_3$, P_4, $\tilde{P}_4$ for which there is a 3-periodic orbit containing two simple critical points, with $\tilde{P}_i(t) = -P_i(-t)$ for all $t \in \mathbb{C}$. Denote by x, y the two simple critical points with orbit $x \mapsto y \mapsto z \mapsto x$. We choose P_1 to be a monic real cubic polynomial with x, y, z real and $y < x < z$. These properties determine P_1 uniquely up to translation by a real vector. We choose P_2 to be a monic cubic polynomial for which the imaginary axis $i\mathbb{R}$ is preserved, $x, y, z \in i\mathbb{R}$ and $iz < iy < ix$. For P_2, the fixed point $\alpha' \in [y, z]$ has external angles $\frac{1}{8}, \frac{3}{8}$.

An efficient way to describe a polynomial is by its Hubbard tree. According to Douady and Hubbard, each postcritically finite monic polynomial g has a forward invariant *Hubbard tree*, which is the convex hull of P_g in K_g [Douady and Hubbard 1984; 1985; Poirier 1993]. The Hubbard tree of $P_1, \ldots, P_4$ are given in Figure 1.

Lemma 2.1. *The matings in* $\mathcal{A}$ *are exactly maps of the form* $g \perp\!\!\!\perp f$, *with* $g = g_a : z^3 + a$ *and* $f \in \{P_i, \tilde{P}_i, i = 1, 2, 3, 4\}$. *Moreover each mating is topologically conjugate to* $g_a \perp\!\!\!\perp f$, *for some value* a *and* $f \in \{P_1, P_2, P_3, P_4\}$.

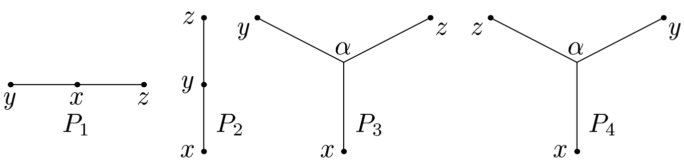


FIGURE 1. The Hubbard trees of P_1, P_2, P_3, P_4.

Proof. Clearly the branched coverings of the form $g_a \amalg P_i$ and $g_a \amalg \tilde{P}_i$ belong to $\mathcal{A}$. On the other hand, if a mating $g \amalg f$ is in $\mathcal{A}$, one of the polynomials, say, f, must have a 3-periodic orbit containing two simple critical points, and the other polynomial, g, must have a double critical point. So g is of the form $z^3 + a$ and f is among P_i and $\tilde{P}_i$. We notice also that $g_a \amalg \tilde{P}_i$ is conjugate to $g_{-a} \amalg P_i$. □

2B. Results

In order to study which matings in $\mathcal{A}$ are equivalent to rational maps, we need to study at first possible obstructions for a general map in $\mathcal{A}$. Here are our results:

Theorem 2.2. *Let Γ be an irreducible obstruction for a postcritically finite map F in $\mathcal{A}$. Then exactly one of the following statements holds.*

1. *Γ is a removable Levy cycle.*
2. *Some preimages of Γ contains a good Levy cycle with at most two elements.*
3. *Γ is a nested obstruction but not a Levy cycle, with a and y contained in one disc component, say $D_+(\Gamma)$, of $S^2 - \Gamma$, z in the other disc component $D_-(\Gamma)$ of $S^2 - \Gamma$ and $x \notin D_+(\Gamma) \cup D_-(\Gamma)$.*

Proposition 2.3. *Assume that Γ' is a multicurve for a postcritically finite map F in $\mathcal{A}$, such that $S^2 - \Gamma'$ has only two disc components $D_+(\Gamma')$ and $D_-(\Gamma')$ with $a, y \in D_+(\Gamma')$, $z \in D_-(\Gamma')$ and $x \notin D_+(\Gamma') \cup D_-(\Gamma')$.*

For any Γ a submulticurve of Γ', denote by $D_\pm(\Gamma)$ the disc component of $S^2 - \Gamma$ containing $D_\pm(\Gamma')$. Set $A_0(\Gamma) = F^{-1}(D_+(\Gamma))$.

Then Γ' is a Thurston obstruction if and only if there is a submulticurve Γ of Γ' such that, up to isotopy, $D_+(\Gamma)$ is contained in a component of $F^{-1}(D_-(\Gamma))$, $A_0(\Gamma)$ contains no curve of Γ and separates $D_+(\Gamma)$ and $D_-(\Gamma)$, and $D_-(\Gamma)$ is separated from $A_0(\Gamma)$ by a component of $F^{-1}(\partial A_0(\Gamma))$. In this case $\lambda_\Gamma = 1$ and $\mu_\Gamma = \frac{1}{2}(1 + \sqrt{5})$, where μ_Γ is the leading eigenvalue of the unweighted Thurston linear transformation defined by

$$F_{\#,\Gamma} : \gamma \mapsto \sum_{\gamma' \subset F^{-1}(\gamma)} [\gamma']_\Gamma \quad \text{for } \gamma \in \Gamma.$$

We now turn to results and conjectures about matings. It is a case by case study. We write $\theta \sim_{g_a} \eta$ if the two external rays for g_a of angles θ and η land at the same point.

Theorem 2.4. *Let g_a be postcritically finite. Then $S^2_{g_a, P_1}/\sim_{\text{ray}}$ is homeomorphic to S^2. Moreover, of the following conditions, 1 and 2 are equivalent, 3 implies 1 and 4, and both 1 and 4 imply 5.*

1. *g_a and P_1 are not matable.*
2. *$g_a \amalg P_1$ has a nested obstruction which is not a Levy cycle.*
3. *For β_0 the fixed point of g_a with external angle 0, either β_0 and $g_a(a)$ are in the same component of $J_a - \{w\}$ and g_a^3 is renormalizable (i.e. g_a^3 is a cubic polynomial-like mapping with connected Julia set), or $\beta_0, g_a(a)$ and $g_a^2(a)$ are in the same component of $J_a - \{w\}$.*
4. *Either $\frac{7}{26} \sim_{g_a} \frac{9}{26}$ or $\frac{19}{26} \sim_{g_a} \frac{17}{26}$.*
5. *The topological entropy of g_a on its Hubbard tree is at least $\log\left(\frac{1}{2}(1 + \sqrt{5})\right)$.*

We conjecture that 1 is equivalent to 4. We consider the above theorem as a partial proof for the direction $4 \Longrightarrow 1$ and the direction $4 \Longleftarrow 1$. At present we don't have a complete proof of either direction of the conjecture.

The part $1 \Longrightarrow 5$ relies on a recent result of Shishikura relating μ_Γ to the topological entropy. It is easy to see from examples that 5 does not imply 4.

In the appendix we give parametric interpretations for Conditions 3 and 4.

Theorem 2.5. *Let g_a be postcritically finite. Then conditions* B, C, *and* D *are equivalent and each implies* A.

A. *g_a and P_2 are not matable.*
B. $\frac{5}{8} \sim_{g_a} \frac{7}{8}$.
C. *$g_a \amalg P_2$ has a good Levy cycle with at most two elements.*
D. *$g_a \amalg P_2$ has a non-removable Levy cycle.*

We also conjecture that A and B are equivalent.

Theorem 2.6. *If g_a is postcritically finite, then g_a and P_i are matable, for $i = 3, 4$.*

2C. An Example of a Nested Obstruction

Consider $g = g_c : z \mapsto z^3 + c$ such that the double critical point $w = 0$ has orbit $w \mapsto c \mapsto c^3 + c \mapsto w$ and that the two rays $R_g(\frac{7}{26})$ and $R_g(\frac{9}{26})$ land at the same point. These conditions determine c uniquely.

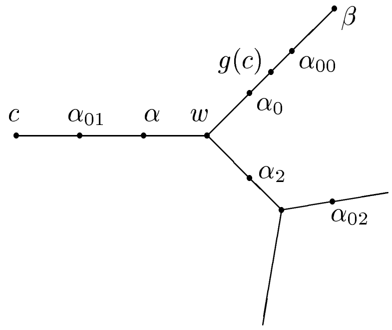


FIGURE 2. Extended Hubbard tree for g.

Figure 2 shows an extension of the Hubbard tree of g. The points α and β are fixed points,

$$\{\alpha,\ \alpha_0,\ \alpha_2\} = g^{-1}(\alpha),$$

and $\{\alpha_{01},\ \alpha_{00},\ \alpha_{12}\} = g^{-1}(\alpha_0)$. The external angles of these points are

$$\geqslant\beta = \{0\},$$

$$\geqslant\alpha = \left\{\tfrac{1}{8}, \tfrac{3}{8}\right\},\ \geqslant\alpha_0 = \left\{\tfrac{1}{24}, \tfrac{19}{24}\right\},\ \geqslant\alpha_2 = \left\{\tfrac{11}{24}, \tfrac{17}{24}\right\},$$

$$\geqslant\alpha_{01} = \left\{\tfrac{19}{72}, \tfrac{25}{72}\right\},\ \geqslant\alpha_{00} = \left\{\tfrac{1}{72}, \tfrac{67}{72}\right\},\ \geqslant\alpha_{02} = \left\{\tfrac{43}{72}, \tfrac{49}{72}\right\}.$$

Figure 3 shows an extension of the Hubbard tree of P_1 (see also Figure 1). Here β', β'' and α' are fixed points, $\{\beta', \beta_1', \beta_2'\} = P_1^{-1}(\beta)$, $\{\alpha',\ \alpha_0',\ \alpha_2'\} = P_1^{-1}(\alpha')$, $\{\alpha_{01}',\ \alpha_{00}',\ \alpha_{02}'\} = P_1^{-1}(\alpha_0')$ and $x_*,\ y_*,\ z_*$ are the roots of basins of attraction of $x,\ y,\ z$, respectively (i.e. repelling periodic orbit of period 3 on the boundary of the basins). The external angles of these points are

$$\geqslant\beta' = \{0\},\ \geqslant\beta'' = \left\{\tfrac{1}{2}\right\},\ \geqslant\beta_1' = \left\{\tfrac{1}{3}\right\},\ \geqslant\beta_2' = \left\{\tfrac{2}{3}\right\},$$

$$\geqslant x_* = \left\{\tfrac{3}{13}, \tfrac{10}{13}\right\},\ \geqslant y_* = \left\{\tfrac{4}{13}, \tfrac{9}{13}\right\},\ \geqslant z_* = \left\{\tfrac{1}{13}, \tfrac{12}{13}\right\},$$

$$\geqslant\alpha' = \left\{\tfrac{1}{4}, \tfrac{3}{4}\right\},\ \geqslant\alpha_0' = \left\{\tfrac{1}{12}, \tfrac{11}{12}\right\},\ \geqslant\alpha_2' = \left\{\tfrac{5}{12}, \tfrac{7}{12}\right\},$$

$$\geqslant\alpha_{01}' = \left\{\tfrac{11}{36}, \tfrac{25}{36}\right\},\ \geqslant\alpha_{00}' = \left\{\tfrac{1}{36}, \tfrac{35}{36}\right\},\ \geqslant\alpha_{02}' = \left\{\tfrac{13}{36}, \tfrac{23}{36}\right\}.$$

We now construct a Thurston obstruction $\{\delta_1, \delta_2\}$ for $F = g \perp\!\!\!\perp P_1$, which is made of external rays and some part of the equator $E = \{(\infty \cdot e^{2\pi i s}, g) \mid s \in \mathbf{T} = \mathbb{R}/\mathbb{Z}\}$.

For $t', t \in]0, 1[$, set

$$E[t', t] = \{\, (\infty \cdot e^{2\pi i s}, g) \mid s \in [\min\{t', t\}, \max\{t', t\}]\,\}.$$

For $i = g, P$, if two external rays $R_i(\theta), R_i(\theta')$ land at a common point $u \in K_i$, set

$$R_i(u; \theta, \theta') = R_i(\theta) \cup R_i(\theta').$$

Now we define

$$\delta_1 = E\left[\tfrac{1}{8}, \tfrac{1}{4}\right] \cup R_P\left(\alpha'; -\tfrac{1}{4}, -\tfrac{3}{4}\right) \cup E\left[\tfrac{3}{4}, \tfrac{3}{8}\right] \cup R_g\left(\alpha; \tfrac{3}{8}, \tfrac{1}{8}\right).$$

δ_1 is clearly a nonperipheral simple closed curve in $S^2_{g,P_1} - P_F$. We will see that:

(i) $F^{-1}(\delta_1)$ contains two components δ_2 and δ_3 with $\deg(F : \delta_2 \to \delta_1) = 1$, $\deg(F : \delta_3 \to \delta_1) = 2$. The curve δ_2 is nonperipheral, non-isotopic to δ_1 and disjoint with δ_1. The curve δ_3 is isotopic to δ_1 rel P_F.

(ii) $F^{-1}(\delta_2)$ contains two components δ_4 and δ_5 with $\deg(F : \delta_4 \to \delta_2) = 1$, $\deg(F : \delta_5 \to \delta_2) = 2$. δ_4 is peripheral and δ_5 is isotopic to δ_1 rel P_F.

Proof. An easy calculation shows that

$$\begin{aligned}
\delta_2 = {} & E\left[\tfrac{1}{24}, \tfrac{1}{12}\right] \cup R_P\left(\alpha_0'; -\tfrac{1}{12}, -\tfrac{11}{12}\right) \cup \\
& E\left[\tfrac{11}{12}, \tfrac{19}{24}\right] \cup R_g\left(\alpha_0; \tfrac{19}{24}, \tfrac{1}{24}\right), \\
\delta_3 = {} & E\left[\tfrac{3}{8}, \tfrac{5}{12}\right] \cup R_P\left(\alpha_2'; -\tfrac{5}{12}, -\tfrac{7}{12}\right) \cup \\
& E\left[\tfrac{7}{12}, \tfrac{11}{24}\right] \cup R_g\left(\alpha_2; \tfrac{11}{24}, \tfrac{17}{24}\right) \cup \\
& E\left[\tfrac{17}{24}, \tfrac{3}{4}\right] \cup R_P\left(\alpha'; -\tfrac{3}{4}, -\tfrac{1}{4}\right) \cup \\
& E\left[\tfrac{1}{4}, \tfrac{1}{8}\right] \cup R_g\left(\alpha; \tfrac{1}{8}, \tfrac{3}{8}\right), \\
\delta_4 = {} & E\left[\tfrac{1}{72}, \tfrac{1}{36}\right] \cup R_P\left(\alpha_{00}'; -\tfrac{1}{36}, -\tfrac{35}{36}\right) \cup \\
& E\left[\tfrac{35}{36}, \tfrac{67}{72}\right] \cup R_g\left(\alpha_{00}; \tfrac{67}{72}, \tfrac{1}{72}\right), \\
\delta_5 = {} & E\left[\tfrac{25}{72}, \tfrac{13}{36}\right] \cup R_P\left(\alpha_{02}'; -\tfrac{13}{36}, -\tfrac{23}{36}\right) \cup \\
& E\left[\tfrac{23}{36}, \tfrac{43}{72}\right] \cup R_g\left(\alpha_{02}; \tfrac{43}{72}, \tfrac{49}{72}\right) \cup \\
& E\left[\tfrac{49}{72}, \tfrac{25}{36}\right] \cup R_P\left(\alpha_{01}'; -\tfrac{25}{36}, -\tfrac{11}{36}\right) \cup \\
& E\left[\tfrac{11}{36}, \tfrac{19}{72}\right] \cup R_g\left(\alpha_{01}; \tfrac{19}{72}, \tfrac{25}{72}\right).
\end{aligned}$$

We get (i) and (ii) from Figure 4. □

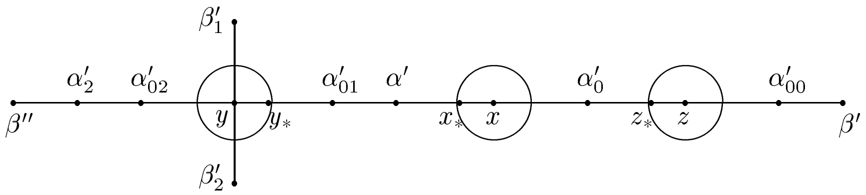


FIGURE 3. Extended Hubbard tree for P_1.

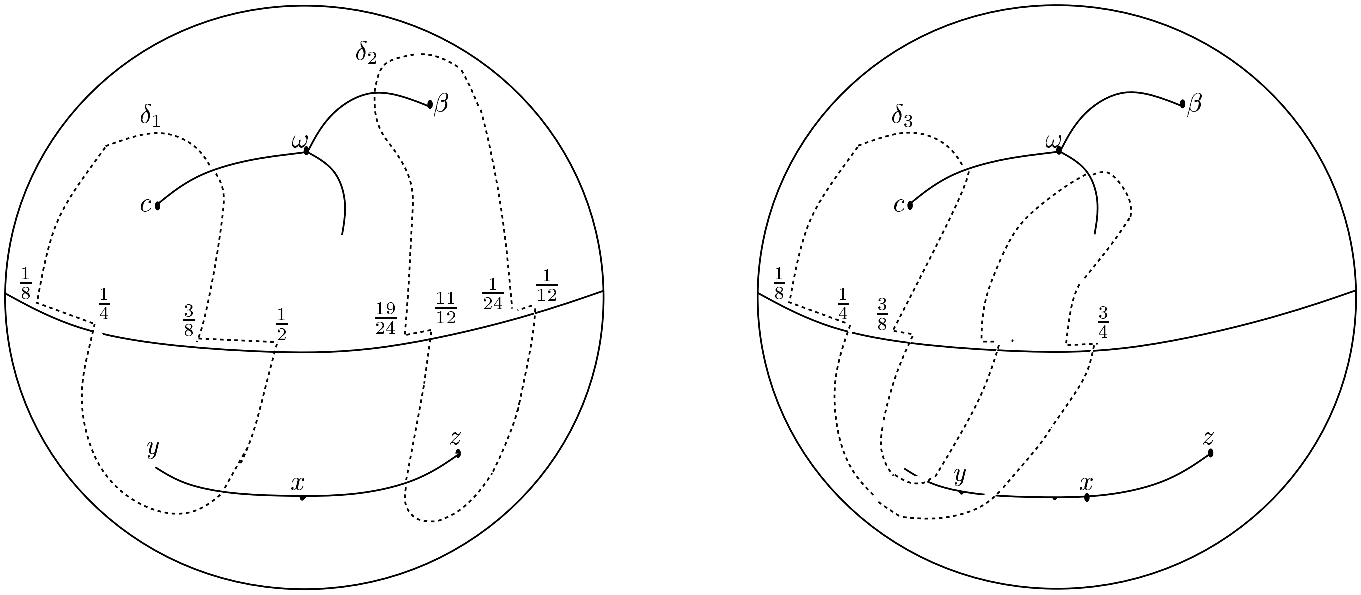


FIGURE 4. The curves δ_1, δ_2, and δ_3.

Now $\Gamma = \{\delta_1, \delta_2\}$ is a multicurve whose Thurston linear transformation $F_\Gamma : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$ is given, in the basis δ_1, δ_2, by the matrix

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix}$$

whose leading eigenvalue λ_Γ equals 1, with eigenvector $\binom{1}{1}$. Clearly Γ is an irreducible obstruction but not a removable Levy cycle. It is also a nested obstruction. By definition, F is not weakly equivalent to a rational map; that is, g, P_1 are not matable.

3. GENERAL ANALYSIS ON BRANCHED COVERINGS AND MATINGS

Section 3A studies the ray-equivalence of a mating. Sections 3B–3H contain useful techniques to deal with Thurston obstructions, intersection problems of a multicurve with a Hubbard tree, weak equivalence to a rational map, etc. We will see some applications in the next section.

3A. The Quotient is a Sphere

A mating $F = f \perp\!\!\!\perp g$ posesses a ray-equivalence relation, which gives more detailed information about Thurston obstructions and Levy cycles, such as results in Theorem 1.4 and Theorem 1.5.

In this section we want to analyze some properties of the ray-equivalence relation of a mating, and give some conditions for $S^2/\sim_{\text{ray}}$ to be a topological 2-sphere.

First a general result about a Hubbard tree; for a proof, see [Tan 1997].

Proposition 3.1. *For a postcritically finite polynomial f, every point $x \in J_f$ with more than one external ray falls eventually into the Hubbard tree of f.*

We say that an equivalence relation $\sim$ in a compact metric space X is *closed* if the graph of $\sim$ in $X \times X$ is a closed set. This is equivalent to say that any sequences $x_n \to x$, $y_n \to y$ such that $x_n \sim y_n$ for each n satisfy $x \sim y$.

Proposition 3.2. *Suppose f, g are degree-d monic polynomials with locally connected Julia set. Set $F = f \perp\!\!\!\perp g$. If there is $K < \infty$ such that $\#[x] \cap E \leq K$ for every $x \in S^2_{f,g}$, ray-equivalence is closed.*

Proof. An external ray $R(\theta) = R_f(\theta) \cup R_g(-\theta) \subset S^2_{f,g}$ is a connected arc. Since the Julia set of both f and g is locally connected, the rays move continuously. So, as $\theta_n \to \theta_0$ as $n \to \infty$, we have $R(\theta_n) \to R(\theta_0)$ with respect to the Hausdorff distance on the space of closed subsets of $S^2_{f,g}$.

Suppose $x_n, y_n \in S^2_{f,g}$ such that $x_n \sim_{\text{ray}} y_n$ for all n, and $x_n \to x_0$, $y_n \to y_0$. We need to prove $x_0 \sim_{\text{ray}} y_0$.

For each n, there is a collection of angles $\theta_{n,1}$, $\theta_{n,2}$, $\ldots$, θ_{n,k_n} such that

$$x_n \in R(\theta_{n,1}),\ y_n \in R(\theta_{n,k_n}),\ R(\theta_{n,i}) \cap R(\theta_{n,i+1}) \neq \varnothing$$

for $i = 1, 2, \ldots, k_n - 1$.

By assumption, we have $k_n \leq K$ for each n. Taking a subsequence if necessary, we may assume that k_n is a constant k for all n. By taking again subsequences, we may assume that $\theta_{n,i} \to \theta_i$ (as $n \to \infty$) for some $\theta_i \in \mathbf{T}$ and for each $i = 1, 2, \ldots, k$.

By the continuity of external rays $R(\theta)$, we have

$$R(\theta_{n,i}) \to R(\theta_i), \text{ as } n \to \infty,$$

for $i = 1, 2, \ldots, k$. It follows that $x_0 \in R(\theta_1)$, $y_0 \in R(\theta_k)$, and $R(\theta_i) \cap R(\theta_{i+1}) \neq \varnothing$ for $i = 1, 2, \ldots, k-1$. Hence

$$\bigcup_{i=1}^{k} R(\theta_i)$$

is connected, and therefore $x_0 \sim_{\text{ray}} y_0$. □

Corollary 3.3. *If, moreover, no equivalence class of $\sim$ separates $S^2_{f,g}$, then $S^2_{f,g}/\sim$ is homeomorphic to S^2.*

Proof. This is a simple application of the Moore's theorem [1925]: If G is a partition of S^2 into compact, connected, nonseparating sets that gives rise to a closed equivalent relation, the quotient S^2/G is homeomorphic to S^2. □

Proposition 3.4. *The induced quotient map $[f \perp\!\!\!\perp g]$ from $S^2_{f,g}/\sim$ to itself is again a branched covering.*

Proof. Denote by V the union of the finitely many ray classes containing critical values of f. Then $f^{-1}(V)$ consists again of finitely many ray classes and $f : S^2 - f^{-1}(V) \to S^2 - V$ is an unbranched covering. Let $y \in \pi(S^2 - V)$ and Y a simply connected open neighborhood of y. Since $\pi : S^2 - V \to \pi(S^2 - V)$ is proper, $\pi^{-1}(Y)$ is again simply connected [Douady and Douady 1979]. Thus each connected component of $f^{-1}\pi^{-1}(Y)$ is simply connected. So is each connected component of $\pi f^{-1}\pi^{-1}(Y)$. This proves that $[f \perp\!\!\!\perp g]$ is a covering of S^2 minus finitely many points. It extends then naturally to a branched covering of S^2. □

3B. Irreducible Obstructions

Let $F : S^2 \to S^2$ be a postcritically finite branched covering. Recall that, for a multicurve

$$\Gamma = \{\gamma_1, \ldots, \gamma_n\},$$

the corresponding matrix $F_\Gamma = (a_{ij})$ is defined by

$$a_{ij} = \sum_{\gamma' \in F^{-1}(\gamma_j),\ \gamma' \sim \gamma_i} \frac{1}{\deg(F : \gamma' \to \gamma_j)}.$$

See [Gantmacher 1959] for the definition and properties of irreducible nonnegative matrices.

Definition. A multicurve Γ is called *irreducible* if F_Γ is irreducible. Γ is called *an irreducible obstruction* if F_Γ is irreducible and $\lambda(F_\Gamma) \geq 1$. In other words, for any $(i,j) \in \{1, \ldots, k\}^2$, $k = \#\Gamma$, there is an integer n, a component γ' of $F^{-n}(\gamma_j)$ isotopic to γ_i, and for $1 \leq m \leq n$, $F^m(\gamma')$ is isotopic to a curve in Γ. Levy cycles are examples of irreducible obstructions. A multicurve Γ is *F-invariant*, if for any $\gamma \in \Gamma$, each connected component of $F^{-1}(\gamma)$ is either peripheral or isotopic to a curve in Γ rel P_F.

Lemma 3.5. *Any Thurston obstruction contains an irreducible obstruction. Any irreducible obstruction is isotopically contained in a F-invariant Thurston obstruction. As a consequence, in the case that F has a hyperbolic orbifold, F is not equivalent to a rational map if and only if F has an irreducible obstruction.*

Proof. Let Γ be a Thurston obstruction for F. Then Γ can be considered as disjoint union of submulticurves Γ_m such that F_{Γ_m} is irreducible. Moreover, $\lambda(F_\Gamma) = \max_m\{\lambda(F_{\Gamma_m})\}$. Any Γ_m with $\lambda(F_{\Gamma_m}) \geq 1$ is an irreducible obstruction.

Now let Γ_0 be an irreducible obstruction for F. Let Γ_1 be the set of isotopy classes of curves in $F^{-1}(\Gamma_0)$. Then $\Gamma_0 \subset \Gamma_1$ (isotopically). To show that Γ_1 is again a multicurve we should check that two classes in Γ_1 do not intersect (isotopically). Since curves in Γ_0 are disjoint from each other, so are curves in $F^{-1}(\Gamma_0)$, so are isotopy classes in Γ_1. Now we can take inductively Γ_n to be the set of isotopy classes of $F^{-1}(\Gamma_{n-1})$. Then Γ_n is a multicurve containing Γ_{n-1}. There is a n such that $\Gamma_{n+1} = \Gamma_n$, i.e. Γ_n is an F-invariant multicurve. It remains to show $\lambda(F_{\Gamma_n}) \geq 1$. Let $F_{\Gamma_n} = (a_{ij})_{m \times m}$, $F_{\Gamma_0} = (b_{ij})_{k \times k}$. We have $k \leq m$. Number curves in Γ_n so that the first k ones are in Γ_0. Define a new matrix $(c_{ij})_{m\times m}$ as follows: $c_{ij} = b_{ij}$ if $i, j \leq k$, and $c_{ij} = 0$ elsewhere. Then $a_{ij} \geq c_{ij}$ for each (i,j). By Perron-Frobenius theory on nonnegative matrices, we have $\lambda((a_{ij})) \geq \lambda((c_{ij})) = \lambda((b_{ij})) = \lambda(F_{\Gamma_0}) \geq 1$. So Γ_n is an F-invariant Thurston obstruction. □

3C. Disc-Components in the Complement

Let F be a postcritically finite branched covering. Let A, B be two subsets of S^2. We say that A is *isotopically contained in* B if there is a homeomorphism ψ of S^2, isotopic to the identity rel P_F, such that $\psi(A) \subset B$. If there is such a ψ with $\psi(A) = B$, we say that A is *isotopic* to B.

Lemma 3.6. *Let Γ be an irreducible multicurve for F (for example, a Levy cycle). Then*

(a) *Each connected component of $S^2 - F^{-1}(\Gamma)$ is exactly a connected component of $F^{-1}(A)$, where A is some connected component of $S^2 - \Gamma$.*
(b) *Any connected component of $S^2 - F^{-1}(\Gamma)$ is isotopically contained in a connected component of $S^2 - \Gamma$.*
(c) *Let D' be a disc-component of $F^{-1}(D)$ for D a component of $S^2 - \Gamma$, such that $\partial D'$ is isotopic to a curve γ_i in Γ, then D' is isotopic to a disc-component of $S^2 - \Gamma$ with boundary γ_i. And if the components of $F^{-1}(D)$ are discs, at least one of them must be isotopic to a disc-component of $S^2 - \Gamma$.*

Proof. Statement (a) is due to general properties of a branched covering.

For (b), since any curve in Γ is isotopic to a curve in $F^{-1}(\Gamma)$, there is a homeomorphism ψ isotopic to the identity rel P_F mapping Γ into $F^{-1}(\Gamma)$. So ψ^{-1} maps a component of $S^2 - F^{-1}(\Gamma)$ into a component of $S^2 - \Gamma$.

For (c), suppose that D' is isotopically contained in A and that A is not a disc-component of $S^2 - \Gamma$. Since each boundary curve of A is nonperipheral, $\partial D'$ can not be isotopic to any of them. So A is a disc, $\partial D'$ is isotopic to ∂A and D' is isotopic to A. Assume now that the components of $F^{-1}(D)$ are discs. Since at least one curve γ' of $F^{-1}(\partial D)$ is isotopic to a curve in Γ, the unique disc-component D' of $S^2 - F^{-1}(\Gamma)$ bounded by γ' is isotopic to a disc-component of F. □

The following result combines work of S. Levy, M. Rees, the authors and others.

Proposition 3.7. *Γ is a removable Levy cycle if and only if Γ is an irreducible obstruction and there is a disc-component D of $S^2 - \Gamma$, such that for all n, the connected components of $F^{-n}(D)$ are discs. In this case $F^{-1}(\Gamma)$ contains a unique subset Γ' isotopic to Γ such that $F : \Gamma' \to \Gamma$ is a homeomorphism. If Γ is an irreducible obstruction and each disc component of $S^2 - \Gamma$ contains at most one critical value, then Γ is a removable Levy cycle.*

Proof. Necessity is a consequence of the definition. To prove sufficiency, set $\gamma = \partial D$. Given any $\gamma_j \in \Gamma$, by irreducibility, there is an integer n such that a curve γ' in $F^{-n}(\gamma)$ is isotopic to γ_j. Since γ' bounds a disc-component D' of $S^2 - F^{-n}(\Gamma)$, by the above lemma, D' is isotopic to a disc-component D_j of $S^2 - \Gamma$ bounded by γ_j. Thus $S^2 - \Gamma$ has at most one non-disc-component.

For simplicity, assume that $\#\Gamma = k > 1$ (the case $\#\Gamma = 1$ is left to the reader). Then $S^2 - \Gamma$ has exactly k disc-components $D_1, \ldots, D_k$, with $\gamma_i = \partial D_i$. We claim that $F_\Gamma = (b_{ij})$, with

$$b_{ij} = \sum_{\substack{D' \text{ isotopic to } D_i,\\ F(D')=D_j}} \frac{1}{\deg(F : D' \to D_j)}.$$

Let γ' be a component of $F^{-1}(\gamma_j)$. It bounds a unique disc-component D' of $S^2 - F^{-1}(\Gamma)$ such that $F(D') = D_j$. Moreover $\deg(F : \gamma' \to \gamma_j) = \deg(F : D' \to D_j)$. By the above lemma, γ' is isotopic to γ_i if and only if D' is isotopic to D_i. By definition of F_Γ, we get the claim.

Now for each j, there is at most one D' in $F^{-1}(D_j)$ isotopic to D_i. So the sum in b_{ij} contains at most one term. On the other hand, for each i, there is at most one j such that $b_{ij} \neq 0$, because $F^{-1}(D_j)$ and $F^{-1}(D_{j'})$ are isotopically disjoint whenever $j \neq j'$. By the irreducibility and the fact $\lambda(\Gamma) \geq 1$, we conclude that, after a suitable numeration, $b_{i(i+1)} = 1$, $b_{k1} = 1$ and $b_{ij} = 0$ elsewhere. This proves that $F^{-1}(\Gamma)$ contains a unique subset Γ' satisfying the properties in the proposition and Γ is a degenerate Levy cycle. Since the components of $F^{-n}(D_1)$ are discs for all n, Γ is also removable.

In the last case of the proposition, take D a disc-component. Since it contains at most one critical value, all components of $F^{-1}(D)$ are discs, each is isotopically contained in a component of $S^2 - \Gamma$, therefore contains at most one critical value. By induction, all components of $F^{-n}(D)$ are discs. □

From this we get again a result of M. Rees and a result of S. Levy:

Corollary 3.8. *Any irreducible obstruction of a quadratic branched covering contains a Levy cycle.*

Proof. Let Γ be an irreducible obstruction for F.

A quadratic map F has exactly two critical values. If some curve $\gamma \in \Gamma$ separates the two critical values of F, then we are in the case to apply the above proposition. So Γ must be a removable Levy cycle (by looking at the degree, we can even prove that $\lambda(\Gamma) < 1$, a contradiction. So in fact this situation does not occur). If the two critical values of F are contained in the same component of $S^2 - \Gamma$, then $F^{-1}(\gamma)$ for each $\gamma \in \Gamma$ consists of two curves and each of them is mapped by F to γ with degree one. This implies every periodic cycle in Γ is a Levy cycle. □

3D. Geometric Intersection Number

Let $f : S^2 \to S^2$ be a postcritically finite branched covering. We say that η is a *nontrivial open arc* in $S^2 - P_f$ if $\eta = h(]0,1[)$, where $h : [0,1] \to S^2$ is a continuous mapping, injective on $]0,1[$, with $h(0), h(1) \in P_f$ and $h(]0,1[) \cap P_f = \varnothing$, moreover h is not homotopic to a constant map relative to its boundary values. Denote by $[\eta]$ the isotopy class (rel P_f) of η.

Denote by $\mathcal{L}$ the set of isotopy classes of non-peripheral simple closed curves and nontrivial open arcs in $S^2 - P_f$, and by $\mathcal{R}$ the real linear space generated by $\mathcal{L}$. For $[\eta], [\zeta] \in \mathcal{L}$, we define the *geometric intersection number* by:

$$[\eta] \cdot [\zeta] = \inf\{\, \#\eta' \cap \zeta' \mid \eta' \in [\eta],\ \zeta' \in [\zeta]\,\}.$$

Note that $[\eta] \cdot [\eta] = 0$. This geometric intersection number can be extended bilinearly to $\mathcal{R} \times \mathcal{R}$.

We define a linear transformation $f_\# : \mathcal{R} \to \mathcal{R}$ by

$$f_\#([\eta]) = \sum_{\eta' \subset f^{-1}(\eta)} [\eta'],$$

where the sum is taken over all connected components of $f^{-1}(\eta)$, and again $[\eta']$ denotes the isotopy class of η', which is zero if η' is a) a peripheral closed curve, b) an open arc with at least one end point out of P_f, or c) an open arc isotopic to a point (rel P_f). It is clear that $f_\#([\eta])$ does not depend on the choice of the representative of $[\eta]$, and $(f_\#)^n = (f^n)_\#$.

Assume that a finite subset $\{[\gamma_1], \dots, [\gamma_k]\}$ of $\mathcal{L}$ admits a representative $\Gamma = \{\gamma_1, \dots, \gamma_k\}$ such that $\gamma_i \cap \gamma_j = \varnothing$ for $i \neq j$. We define two linear transformations $f_\Gamma : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$ and $f_{\#,\Gamma} : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$ by

$$f_\Gamma(\gamma) = \sum_{\gamma' \subset f^{-1}(\gamma)} \frac{1}{\deg(f : \gamma' \to \gamma)} [\gamma']_\Gamma,$$

$$f_{\#,\Gamma}(\gamma) = \sum_{\gamma' \subset f^{-1}(\gamma)} [\gamma']_\Gamma$$

for every $\gamma \in \Gamma$, where $[\gamma']_\Gamma$ denotes the element in Γ isotopic to γ' if it exists and 0 otherwise. Note that $\deg(f : \eta' \to \eta) = 1$ whenever η is an arc. We have $(f_{\#,\Gamma})^n \le (f^n)_{\#,\Gamma}$, and $(f_\Gamma)^n(\gamma) \le (f^n)_\Gamma(\gamma)$, since some n-th preimage γ' of γ might come back to Γ but $f(\gamma') \notin \Gamma$.

For Γ as above, set $\tilde{\Gamma}(f^n)$ to be the union of those components of $f^{-n}(\Gamma)$ that are isotopic to elements of Γ (the case $n = 1$ being written simply $\tilde{\Gamma}$). If Γ is irreducible, each component of Γ is isotopic to some (not necessarily unique) component of $\tilde{\Gamma}(f^n)$.

A different form of the next theorem appeared in the preprint version of this paper. The generalization given here is proved in [Pilgrim and Tan 1998].

Theorem 3.9. *Let $f : S^2 \to S^2$ be a postcritically finite branched covering. Assume that two finite subsets of $\mathcal{L}$ admit representatives $\Gamma = \{\gamma_1, \dots, \gamma_k\}$ and $\Lambda = \{\lambda_1, \dots, \lambda_l\}$ such that $\gamma_i \cap \gamma_j = \lambda_i \cap \lambda_j = \varnothing$ for $i \neq j$. Assume furthermore that*

(0) $\#(\Gamma \cap \Lambda) = \Gamma \cdot \Lambda$;
(1) f_Γ *has neither zero row nor zero column, with leading eigenvalue at least* 1;
(2) $\tilde{\Lambda}$ $(\subset f^{-1}(\Lambda))$ *has a subset* $\tilde{\Lambda}'$ *isotopic to* Λ *and* $f : \tilde{\Lambda}' \to \Lambda$ *is a homeomorphism.*

Then either

1. $\Gamma \cdot \Lambda = 0$ *and* $\Gamma \cdot f^{-n}(\Lambda) = 0$, $f^{-n}(\Gamma) \cdot \Lambda = 0$ *for all* $n \ge 1$; *or*
2. $\Gamma \cdot \Lambda \neq 0$ *and*
   a. $\tilde{\Gamma}$ *is isotopic to* Γ, $\tilde{\Lambda}$ *is isotopic to* Λ, *the mappings* $f : \tilde{\Gamma} \to \Gamma$ *and* $f : \tilde{\Lambda} \to \Lambda$ *are homeomorphisms (so* $\tilde{\Lambda}' = \tilde{\Lambda}$*),* $\tilde{\Gamma} \cap (f^{-1}(\Lambda) - \tilde{\Lambda}) = \varnothing$ *and* $\tilde{\Lambda} \cap (f^{-1}(\Gamma) - \tilde{\Gamma}) = \varnothing$. *More precisely, each*

component of Γ is isotopic to a unique component of $\tilde{\Gamma}$, each component of Λ is isotopic to a unique component of $\tilde{\Lambda}$, and

i. *for each $\gamma \in \Gamma$, there is exactly one component $\gamma' \subset f^{-1}(\gamma)$ such that $\gamma' \cap \tilde{\Lambda} \neq \varnothing$, moreover, γ' is the unique component of $f^{-1}(\gamma)$ isotopic to an element of Γ;*

ii. *for each $\lambda \in \Lambda$, there is exactly one component λ' of $f^{-1}(\lambda)$ such that $\lambda' \cap \tilde{\Gamma} \neq \varnothing$, moreover, λ' is the unique component of $f^{-1}(\lambda)$ isotopic to an element of Λ.*

b. *The transformations $f_{\#,\Gamma}$ and $f_{\#,\Lambda}$ are transitive permutations of the basis vectors.*

c. *The above results remain true if we replace f by f^n, for any $n \geq 1$ (though transitivity may fail).*

d. *For any λ isotopic to a component of $f^{-n}(\Lambda)$ for some $n > 1$ but not isotopic to a component of Λ, $\Gamma \cdot \lambda = 0$. Similarly, for any γ isotopic to a component of $f^{-n}(\Gamma)$ for some $n > 1$ but not isotopic to a component of Γ, $\Lambda \cdot \gamma = 0$.*

Corollary 3.10. *If Λ is a Levy cycle and Γ an irreducible obstruction with $\Gamma \cdot \Lambda \neq 0$, then Γ is again a Levy cycle.*

Corollary 3.11. *Suppose that in a mating $f \perp\!\!\!\perp g$, one polynomial, say g, has a "star-like" Hubbard tree H_g, that is, H_g has only one branch point α and $g : H_g \to H_g$ is a homeomorphism. Then any irreducible obstruction of $f \perp\!\!\!\perp g$ is a Levy cycle, whose limit set (according to Theorem 1.4) coincides with the ray class $[\alpha]$. Moreover, if the rays of α form a single orbit by g then either there is θ such that $R_f(\theta) \cup R_g(-\theta)$ links α to a fixed point of f, or $[\alpha]$ is a tree.*

Proof. Consider H_g as a periodic cycle of isotopy classes of arcs. For Γ an irreducible obstruction of $f \perp\!\!\!\perp g$, we must have $\Gamma \cdot H_g \neq 0$ (otherwise Γ reduces to an obstruction for f). So by the above theorem Γ is a Levy cycle, with limit set passing through H_g. The only possibility is $[\alpha]$, since α is the unique point of H_g in the Julia set.

Denote by d the degree of f and g. By assumption, the map $\theta \mapsto d\theta$ acts as a cyclic permutation on the set $-\theta_1, \dots, -\theta_k$ of external angles of α. Denote by α_i the landing point of $R_f(\theta_i)$. If $\alpha_i = \alpha_j$ for some $i \neq j$, then by the cyclic permutation property and the fact that F preserves the cyclic order at branch points of $[\alpha]$, we have $\alpha_1 = \alpha_2 = \cdots = \alpha_k$ and it is a fixed point of f. Assume now the α_i's are pairwise distinct. Fix an index j and $\eta \neq \theta_j$ (if any) an external angle of α_j. Then the ray $R_f(\eta) \cup R_g(-\eta)$ lands at a periodic point $u \in J_g - H_g$ (since $H_g \cap J_g = \{\alpha\}$). The point u has no other external rays (Proposition 3.1). So $[\alpha]$ is a tree. □

3E. Removing Levy Cycles

Let $F : S^2 \to S^2$ be a postcritically finite branched covering. We describe here how to modify F in order to delete simultaneously a maximal number of removable Levy cycles.

Recall that each removable Levy cycle Γ with n curves decomposes the sphere into n disc-components $B_1, \dots, B_n$ and one extra component C (which is not a disc unless $n = 1$), such that each $F^{-1}(B_j)$ has a component B'_j isotopic to $B_{j-1 \pmod n}$, with $\deg(F : B'_j \to B_j) = 1$. Moreover for all $k > 0$, all components of $F^{-k}(B_1)$ are discs. Therefore for all $k > 0$ and $j = 1, \dots, n$, all components of $F^{-k}(B_j)$ are discs. Set $\mathcal{B}(\Gamma) = \bigcup_j B_j$.

Denote by $\Sigma = \Sigma(F)$ the set of isotopy classes of removable Levy cycles for F. We define a partial order on Σ: we say that $\xi_1 < \xi_2$ if there are representatives Γ_1, Γ_2 of ξ_1, ξ_2 such that $\mathcal{B}(\Gamma_1) \subset \mathcal{B}(\Gamma_2)$.

If the geometric intersection number $\xi_1 \cdot \xi_2$ is zero for two elements of Σ, either one is smaller than the other, or there are representatives Γ_1, Γ_2 of ξ_1, ξ_2 such that $S^2 - \Gamma_1 \cup \Gamma_2$ has exactly one non-disc-component, and $\#\Gamma_1 + \#\Gamma_2$ disc-components.

Denote by $\Sigma' = \Sigma'(F)$ the set of $\xi \in \Sigma$, satisfying that ξ is maximal in Σ and $\xi \cdot \Gamma = 0$ for any Levy cycle Γ (see Sections 3F–3H for examples and counterexamples).

Denote by $\mathcal{G}_d$ the set of equivalence classes of postcritically finite branched coverings of degree d.

Theorem 3.12. *There is a mapping $s : \mathcal{G}_d \to \mathcal{G}_d$ with these properties:*

(a) *it fixes every class which contains a rational map;*

(b) *if $s([F]) = [G]$, there is a bijection between the two sets $A_F = \{$isotopy classes of irreducible obstructions of F not smaller than or equal to an* *element of* $\Sigma'(F)\}$ *and* $A_G = \{$*isotopy classes of irreducible obstructions of* $G\}$;

(c) $\Sigma'(G) = \varnothing$ *for* $[G] \in s(\mathcal{G}_d)$ *and* $s \circ s = s$.

Proof. For any $\xi \in \Sigma'(F)$, we have $\xi \cdot \Gamma = 0$ for any irreducible obstruction Γ, since by Corollary 3.10, any Γ with $\xi \cdot \Gamma \neq 0$ is a Levy cycle.

I. Construction of s. We define $s([F]) = [F]$ if $\Sigma'(F) = \varnothing$. Assume that $\Sigma' = \Sigma'(F) \neq \varnothing$.

For any $\xi \in \Sigma'$, the set $\xi \cap P_F$ is defined to be $\mathcal{B}(\Gamma) \cap P_F$, for any Γ a representative of ξ. The set $\xi \cap P_F$ is not empty, independent of the choice of Γ, and contains only periodic non-superattracting elements of P_F. For $\xi, \xi' \in \Sigma'$ distinct, we have $(\xi \cap P_F) \cap (\xi' \cap P_F) = \varnothing$. Since $\#P_F$ is finite, the set Σ' contains only finitely many elements $\xi_1, \dots, \xi_k$.

Step 1: The mapping H and the integer n. For each i, choose Γ_i a representative of ξ_i such that Γ_i, Γ_j are disjoint for $i \neq j$. Denote by $\Gamma = \bigcup_{i=1}^k \Gamma_i$, and $\mathcal{B} = \bigcup_{i=1}^k \mathcal{B}(\Gamma_i)$. From the proof of Proposition 3.7, the set Γ' of curves in $F^{-1}(\Gamma)$ homotopic to some curve in Γ is isotopic to Γ, and $F : \Gamma' \to \Gamma$ is a homeomorphism.

Let $\Psi : S^2 \to S^2$ be a homeomorphism isotopic to the identity rel P_F, such that $\Psi(\Gamma) = \Gamma'$. Set $H = F \circ \Psi$. Then H is Thurston equivalent to F, $P_H = P_F$, $H(\Gamma) = \Gamma$, and $H : \mathcal{B} \to \mathcal{B}$ is a homeomorphism.

Note that $\Gamma \subset H^{-1}(\Gamma)$ and $\#H^{-1}(\Gamma) \geq 2$. By induction $H^{-(n-1)}(\Gamma) \subset H^{-n}(\Gamma)$. There is a minimal integer n such that

$$(H^{-(n+1)}(\mathcal{B}) - H^{-n}(\mathcal{B})) \cap (P_H \cup \Omega_H) = \varnothing,$$

where Ω_H is the set of critical points of H.

Step 2: Marking points. In each component B of $\mathcal{B}$ we mark a point y such that y has the same period as B (such a point exists according to Brouwer's fixed point theorem), and H maps marked point to marked point. Inductively, for $j = 1, \dots, n$, we mark a point y in each component B of $H^{-j}(\mathcal{B}) - H^{-(j-1)}(\mathcal{B})$ such that $H(y)$ is the marked point of $H(B)$.

Step 3: Modifications. We will modify the map in each marked disc (B, y) of $H^{-n}(\mathcal{B})$ containing a critical point (such a disc is surely not periodic, that is, $B \notin \mathcal{B}$). Set $(B', y') = H((B, y))$. Denoting by D the unit disk, let $\varphi : \overline{B} \to \overline{D}$ be an orientation-preserving homeomorphism with $\varphi(y) = 0$, and likewise $\varphi' : \overline{B'} \to \overline{D}$ with $\varphi'(y') = 0$. Extend the boundary mapping $g = \varphi' \circ H_{\partial B} \circ \varphi^{-1}$ to a mapping (denoted again by g) of the whole disk by $t \cdot w \mapsto t \cdot g(w)$, $0 \leq t \leq 1$. It is a covering branched only at 0, and its degree coincides with $\deg(H : \partial B \to \partial B')$.

Now we define $G : S^2 \to S^2$ to be: on each disc B of $H^{-n}(\mathcal{B})$ containing a critical point, set $G_B = (\varphi')^{-1} \circ g \circ \varphi$. Elsewhere set $G = H$.

Step 4: Independence of [G] on the choices in the construction. At first another choice of Γ in its isotopy class and Ψ will give always a mapping H in the class of F. The integer n in Step 1 remains unchanged. Next another choice of the marked orbit and the maps φ, φ' will give a different extension of $H|_{\partial B}$ for B a component of $H^{-n}(\mathcal{B})$ containing a critical point. So this gives a different mapping G_1. However, for a circle covering $h : S^1 \to S^1$, any two extensions of h in the unit disc as a branched covering such that $h^{-1}(0) = 0$ are topologically conjugate by a homeomorphism $\psi : \overline{D} \to \overline{D}$ with $\psi|_{S^1 \cup \{0\}} = \mathrm{id}$. In our case, there is a homeomorphism Φ of the sphere, which is the identity on $S^2 - H^{-n}(\mathcal{B})$, mapping the first choice of the marked orbit to the second one, such that $\Phi^{-1} \circ G_1 \circ \Phi$ coincides with G everywhere, except on $H^{-(n+1)}(\mathcal{B}) - H^{-n}(\mathcal{B})$. The exceptional set is a disjoint union of finite discs away from P_G. So $\Phi^{-1} \circ G_1 \circ \Phi$ and G are isotopic rel P_G. Thus G_1 and G are Thurston equivalent.

Now the mapping $s : [F] \to [G]$ is well defined.

II. Properties of s. Any rational map R which is postcritically finite has no Levy cycle [Bielefeld et al. 1992], so $s([R]) = [R]$.

Next, let $F, H, G, \mathcal{B}$ be as above. Since $P_G \cap H^{-n}(\mathcal{B})$ is contained in the set of marked points, curves in $\partial H^{-n}(\mathcal{B})$ are peripheral for G. So for $\xi \in A_G$ and $\tilde{\Gamma}$ a representative of ξ such that $\tilde{\Gamma} \cap H^{-n}(\mathcal{B}) = \varnothing$, we have $G^{-1}|_{\tilde{\Gamma}} = H^{-1}|_{\tilde{\Gamma}}$. So $\tilde{\Gamma}$ is again an obstruction for H and then for F. It gives rise to a uniquely determined $\xi' \in A_F$.

Now let $\xi' \in A_F$, so ξ' is not smaller than any element of $\Sigma'(F)$. There is a representative $\tilde{\Gamma}$ of ξ' such that $\tilde{\Gamma} \cap \mathcal{B} = \varnothing$. So $\tilde{\Gamma}$ is an obstruction for H. Now since $\tilde{\Gamma} \cdot \partial\mathcal{B} = 0$, by Theorem 3.9 (with $\Lambda = \partial\mathcal{B}$), we have $\tilde{\Gamma} \cdot H^{-n}(\partial\mathcal{B}) = 0$. Moreover $\tilde{\Gamma}$ is not isotopically contained in $H^{-n}(\mathcal{B})$. So we can choose $\tilde{\Gamma}$ in its isotopy class such that $\tilde{\Gamma} \cap H^{-n}(\mathcal{B}) = \varnothing$. In other words, $\tilde{\Gamma}$ is in the identical part of H^{-1} and G^{-1}. As a consequence, $\lambda(\tilde{\Gamma}, G) = \lambda(\tilde{\Gamma}, H) = \lambda(\tilde{\Gamma}, F) \geq 1$ (this shows in particular that curves in $\tilde{\Gamma}$ are not peripheral for G). Thus $\tilde{\Gamma}$ is an irreducible obstruction for G, and gives rise to a uniquely determined $\xi \in A_G$.

Finally, let $\tilde{\Gamma}$ be a maximal removable Levy cycle of G such that $\tilde{\Gamma} \cdot \Gamma = 0$ for any irreducible obstruction Γ of G (i.e., $[\tilde{\Gamma}] \in \Sigma'(G)$). It is again a removable Levy cycle for F and gives rise to a $\xi' \in A_F$, by the preceding paragraph. Moreover $\tilde{\Gamma} \cdot \Gamma = 0$ for any irreducible obstruction Γ of F. By definition of $\Sigma'(F)$ and the fact that $\xi' \notin \Sigma'(F)$, there is a removable Levy cycle ξ'' for F strictly larger than ξ'. This contradicts the maximality of $\tilde{\Gamma}$ for G.

As a consequence, $s \circ s = s$. □

Definition. Two branched coverings F, F' are *weakly equivalent* if $s([F]) = s([F'])$. We say that F is *weakly equivalent* to a rational map R if $s([F]) = [R]$.

Notation. We denote by $s(F)$ a representative of $s([F])$.

Corollary 3.13. *If all irreducible obstructions for F are removable Levy cycles smaller than or equal to an element of $\Sigma'(F)$, F is weakly equivalent to a rational map. On the other hand, in case that $s(F)$ has a hyperbolic orbifold, if F is weakly equivalent to a rational map then all irreducible obstructions for F are removable Levy cycles smaller than or equal to an element of $\Sigma'(F)$.*

Proof. Set $G = s(F)$. In the first case G has no irreducible obstructions; by Lemma 3.5 it has no Thurston obstructions. Therefore by Theorem 1.1 G is equivalent to a rational map. In the second case, G has no Thurston obstructions (Theorem 1.1). We can then apply part (b) of the theorem above. □

We conclude this section with applications to several problems about matings.

3F. A Criterion for a Branched Covering to Be a Mating

Every mating is a branched covering. But many branched coverings can not be realized as a mating, for instance with mixed critical orbits. As an example, choose a degree two branched covering with one critical point periodic and the other falling eventually into the first orbit. But this mixed-orbit condition is far from sufficient, even in degree two case there are rational maps whose both critical points are periodic with disjoint orbits but which is not equivalent to a mating (see [Wittner 1986]). In fact the key point for a branched covering to be equivalent to a mating is that there is a curve is S^2 playing isotopically the role of the equator for a mating:

Theorem 3.14 [Thurston 1983; Levy 1985; Wittner 1986]. *Let F be a postcritically finite branched covering of degree d. Assume that F has no degenerate Levy cycle. Then F is equivalent to the mating of two polynomials f, g if and only if there is a closed curve $\gamma \subset S^2 - P_F$ such that $F^{-1}(\gamma) = \gamma'$ is again a single closed curve and γ' is isotopic to γ rel P_F with the same orientation. Moreover given such a γ the two polynomials f, g are uniquely determined.*

It happens also that there are several "equators" for the same branched covering. This phenomenon is called *shared mating* in [Wittner 1986]. Hence the mapping $f, g \mapsto f \perp\!\!\!\perp g$, from the space of pairs of monic polynomials of degree d to the space of degree-d branched coverings, is neither surjective nor injective.

3G. Levy Cycles around Periodic Ray Classes

This part intends to complete the picture of Theorem 1.4.

Lemma 3.15. *Let F be a mating. Let $[x]$ be a periodic ray class such that $[x]$ contains either a closed loop or at least two postcritical points. Then each boundary curve of a tubular neighborhood of $[x]$ generates a Levy cycle.*

Proof. All we need to prove is that such a curve γ is not peripheral (the periodicity and the degree-one property are guaranteed by that of $[x]$). But if γ were peripheral, the disc component D of $S^2 - [x]$ containing γ would be periodic, i.e. there would be an integer $k > 0$ such that $F^{-k}(D)$ has a component which coincides with D, i.e. $F^k(D) = D$. But $D \cap E$ consists of finitely many intervals (where E is the equator), and F^k is expanding on E. So

$$F^k(D \cap E) \neq D \cap E,$$

which is a contradiction. □

3H. How to Find Removable Levy Cycles in a Mating

Proposition 3.16. *Let $F = f \amalg g$ be a postcritically finite mating. The set $\Sigma'(F)$ consists of boundary curves of tubular neighborhoods of periodic cycles $\{[x_1],\ [x_2], \ldots,\ [x_m]\}$ of ray classes such that each $[x_i]$ contains at least two postcritical points and no ray class in $\bigcup_{n\geq 0} F^{-n}([x_i])$ contains a closed loop. If $S^2/\sim_{\rm ray}$ is homeomorphic to S^2 then F has no non-removable Levy cycles. Moreover f and g are matable if all irreducible obstructions of F are removable Levy cycles. And, in case that $s(F)$ has a hyperbolic orbifold, if f and g are matable then all irreducible obstructions of F are removable Levy cycles.*

Proof. We will only sketch the proof here. The details are left to the reader.

It is very easy to see that the boundary curves Γ of a tubular neighborhood of such a periodic cycle of ray classes form a removable Levy cycle. To see that Γ is maximal among removable Levy cycles, we apply Theorem 1.4. To see that Γ is (geometrically) disjoint from any other Levy cycle we apply Lemma 3.15 and then Theorem 1.4. To see that every element of $\Sigma'(F)$ is in this form we apply again Theorem 1.4.

Assume now that F has a non-removable Levy cycle. Then the limit set of the cycle $\{[x_1], \ldots, [x_m]\}$ satisfies that for some i and some n, the set $F^{-n}([x_i])$ contains a closed loop. So one of the ray classes contains a closed loop. Therefore $S^2_F/\sim_{\rm ray}$ is not homeomorphic to S^2.

Recall that by definition f and g are matable if F is weakly equivalent to a rational map, that is, if $s(F)$ is equivalent to a rational map.

Due to Theorem 1.4 and Lemma 3.15, if all irreducible obstructions of F are removable Levy cycles, each of them must be smaller than or equal to an element of $\Sigma'(F)$. So F is weakly equivalent to a rational map, by Corollary 3.13.

On the other hand, in case that $s(F)$ has a hyperbolic orbifold, if F is weakly equivalent to a rational map then by Corollary 3.13 again all irreducible obstructions of F are removable Levy cycles. $\square$

Now we give an example of a maximal removable Levy cycle which is not geometrically disjoint from an irreducible obstruction. Consider f to be a cubic polynomial with one fixed critical point and the second iterate β of the other critical point fixed (so that the second critical point behaves like 0 for z^2-2). Consider now $F = f \amalg \tilde{f}$, with $\tilde{f}(z) = -f(-z)$. Then the ray class $[\beta]$ consists of two fixed rays $R(0)$ and $R(1/2)$. A curve around each ray is a maximal removable Levy cycle, but intersects always a curve around the other ray.

4. PROOF OF THE RESULTS: FIRST PART

If $F \in \mathcal{A}$ is postcritically finite, then it is easy to check (by definition of s) that $s(F) \in \mathcal{A}$ and $s(F)$ has a hyperbolic orbifold.

Proof of Theorem 2.6. Denote by P one of P_3, P_4. Then P is star-like, with α the unique branched point in its Hubbard tree (see Figure 1). Denote by $-\theta_1, -\theta_2, -\theta_3$ the external angles of α. They form a single orbit by $\theta \mapsto 3\theta$. According to Corollary 3.11, an irreducible obstruction for $g_a \amalg P$ is a Levy cycle Γ with limit set $[\alpha]$, and, either $[\alpha]$ is a tree, or $R_{g_a}(\theta_1)$, $R_{g_a}(\theta_2)$ and $R_{g_a}(\theta_3)$ all land at the same point in J_{g_a} which is also a fixed point of g_a. This latter case does not occur for the following reasons: the map g_a has a unique fixed point α' having more than one external angles, and $\alpha' \in [a, w]$. A simple calculation shows that the set of external rays of α' is disjoint from either $\left[0, \frac{1}{3}\right]$ or $\left[\frac{2}{3}, 1\right]$. But $\{\theta_1, \theta_2, \theta_3\}$ intersects both $\left[0, \frac{1}{3}\right]$ and $\left[\frac{2}{3}, 1\right]$. So $\alpha' \notin [\alpha]$.

As a consequence, $[\alpha]$ is a tree. Furthermore, by induction each component of $F^{-n}([\alpha])$ is again a tree and contains at most one critical value (that is, a, since the other two critical values y and z do not have external rays). Therefore each component of $F^{n+1}([\alpha])$ is a tree. Hence any irreducible obstruction is a removable Levy cycle surrounding $[\alpha]$. So $g_a \amalg P$ is weakly equivalent to a rational map (Proposition 3.16). $\square$

Proof of Theorem 2.5. The part D $\Longrightarrow$ A is proved in Proposition 3.16 for general matings (recall that $s(g_a \amalg P_2)$ has always a hyperbolic orbifold).

B $\Longrightarrow$ C. In this case the closed curve $R(\frac{5}{8}) \cup R(\frac{7}{8})$ or a tubular neighborhood of it forms a good (thus non-removable) Levy cycle. C $\Longrightarrow$ D is trivial.

It remains to show D $\Longrightarrow$ B. By Theorem 2.2 (to be proved later), the map $g_a \amalg P_2$ has a good Levy cycle with at most two curves. Its limit set X_0 is contained in a ray class and is a simple closed curve with two fixed points (Theorem 1.4). Therefore $\alpha \in X_0$, where α is the fixed point in H_{P_2} (see Figure 1). The two external angles of α are $-\frac{5}{8}, -\frac{7}{8}$. No other two-periodic rays land at a point of H_{P_2}. Similar to the proof of Corollary 3.11, one can show that either $R_a\left(\frac{5}{8}\right)$ and $R_a\left(\frac{7}{8}\right)$ land at the same point or $[\alpha]$ is a tree. But the latter case does not occur since $X_0 \subset [\alpha]$. So $\frac{5}{8} \sim_{g_a} \frac{7}{8}$, proving B. □

The rest of this section is devoted to the proof of the equivalence 1 $\iff$ 2 in Theorem 2.4. First we need to study in more detail the ray-equivalence relation for $g_a \perp\!\!\!\perp P_1$.

According to Douady and Hubbard [1984], for any postcritically finite polynomial f with K as its filled-in Julia set, there is a way to define a (unique) regular arc $[u, v] \subset K$ for any pair of points in K so that the convex hull $[u_1, \dots, u_l] \subset K$ of any finite subset is a topologically finite tree, and $f([u_1, \dots, u_l]) = [f(u_1), \dots, f(u_l), f(w_1), \dots, f(w_k)]$, where $w_1, \dots, w_k$ are the critical points of f in $[u_1, \dots, u_l]$. The Hubbard tree is precisely the convex hull of the post-critical set.

As in Section 2C, set $P = P_1$, and for $i = a, P$, denote by $\gamma_i(\theta)$ the landing point of $R_i(\theta)$ on K_i. Let $F = g_a \perp\!\!\!\perp P$.

Set $\hat{H}_a = [H_a, \beta_0, \beta_1, \beta_2]$ with $\beta_i = \gamma_a(i/3)$ and $\hat{H}_P = [H_P, \beta'_0, \beta'_1, \beta'_2]$ with $\beta'_i = \gamma_P(i/3)$. These are the *extended Hubbard trees*. Note that $\hat{H}_P = H_P \cup [y, \beta'_1] \cup [y, \beta'_2] \cup [\beta'_0, z]$ (see Figure 3). For $i = a, P$, we have $F(H_i) \subset H_i$ and $F(\hat{H}_i) \subset \hat{H}_i$.

Note that each nontrivial ray class $[u]$ is completely determined by $[u] \cap (J_a \sqcup J_P)$. For example, the extremities of $[u]$ are points in $[u] \cap (J_a \sqcup J_P)$ having only one external angle. The other points in $[u] \cap (J_a \sqcup J_P)$ have at least two external rays. For this reason, we give

Definition (simple and multiple point). A point u in $J_a \sqcup J_P \subset S^2_{g_a,P}$ is a *simple point* if u has only one external angle, and a *multiple point* if u has at least two external angles.

Lemma 4.1. *If u is multiple and $u \neq w$, $F(u)$ is also multiple. If $w \notin [u]$, then $F : [u] \to [F(u)]$ is a homeomorphism.*

Proof. In $u = x$ or y it has no external rays and $[u]$ is trivial. If u is not a critical point of F, the mapping F is a local homeomorphism, and F sends external rays to external rays. So $F(u)$ has the same number of external angles as u. The rest follows. □

Lemma 4.2. *For $i = a, P$, there is an integer m_i such that for any $u \in J_i$, the number of external angles $\#\!>\!u$ is bounded by m_i. Moreover $m_P = 2$ and no point in H_P has external angles in $\left[\frac{1}{3}, \frac{2}{3}\right]$.*

Proof. For a multiple point $u \in J_P$, since all critical points of P are in $\mathrm{int}(K_P)$, the orbit of u contains no critical points. There is k such that $F^k(u) \in H_P$ (Proposition 3.1) and F^k is a homeomorphism in a neighborhood of u. So $\#\!>\!u = \#\!>\!F^k(u)$. Since H_P contains no branched points, $\#\!>\!u = m_P = 2$. Moreover since $[\beta'_1, \beta'_2] \cap H_P = \{y\}$ and y has no external rays (Figure 3), no point in H_P has angle in $\left[\frac{1}{3}, \frac{2}{3}\right]$.

It is known that there is m_a such that $\#\!>\!u \leq m_a$ for any $u \in H_a$. For a multiple point $u \in J_a$, either there is a unique k such that $F^k(u) = w$ or the orbit of u does not contain w and there is k such that $F^k(u) \in H_a$ (Proposition 3.1). In both cases $\#\!>\!u = \#\!>\!F^k(u) \leq m_a$. □

Lemma 4.3. *Every multiple point u in J_a has a forward image in $[w, a]$.*

Proof. Assume that none of the iterates of u meets w (otherwise the lemma is trivial). Iterate u enough times so that the set of external angles of $F^n(u)$ meets at least two components of $\mathbf{T} - \left\{0, \frac{1}{3}, \frac{2}{3}\right\}$. Then $F^n(u) \in [\beta_0, w] \cup [\beta_1, w] \cup [\beta_2, w]$. We have $F([\beta_i, w]) = [\beta_0, w] \cup [w, a]$, for $i = 0, 1, 2$, and $F : [\beta_0, w] \to [\beta_0, a]$ is expanding. So there is $n' \geq n$ such that $F^{n'}(u) \in [w, a]$. □

Proposition 4.4. *Each ray class $[u]$ for F contains at most $m_a + 1$ points of J_a and no closed loops.*

Proof. If all points in $[u] \cap J_a$ are simple we are done since a point in $[u] \cap J_P$ has at most two external rays.

Claim. *Suppose that u is a multiple point of J_a and that $F^n(u) \neq w$ for all n. Let θ, θ' be two distinct angles of u and $v = \gamma_P(-\theta)$, $v' = \gamma_P(-\theta')$. Then at least one of v, v' is a simple point. A consequence of this is that, if there is $\eta \in \angle u$ with $\gamma_P(-\eta)$ multiple, then $\gamma_P(-\eta')$ is simple for any $\eta' \in \angle u$ with $\eta' \neq \eta$.*

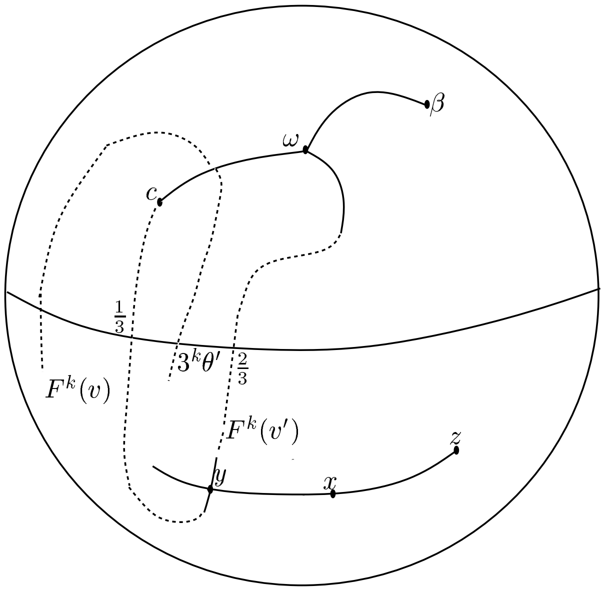


FIGURE 5. No point in H_P has an external angle in $\left[\frac{1}{3}, \frac{2}{3}\right]$.

Suppose by contradiction that both v and v' are multiple.

Note that w disconnects $\hat{H}_a$ into three components W_0, W_1, W_2 with $\beta_i \in W_i$, and $[w, a] - \{w\}$ is in one of W_1, W_2.

Since $F^n(u) \neq w$ for all n, $F^n(u), F^n(v)$ and $F^n(v')$ are multiple and $F^n(v) \neq F^n(v')$ for all n. Take k large enough such that $F^k(u) \in [w, a] - \{w\}$ (see Lemma 4.3) and such that $F^n(u) \in H_a$ and $F^n(v)$, $F^n(v') \in H_P$ for all $n \geq k$ (Proposition 3.1). We may assume $k = 0$. Now there is $l \geq 0$ minimal such that $\{3^l\theta, 3^l\theta'\}$ intersects at least two components of $\mathbf{T} - \{0, \frac{1}{3}, \frac{2}{3}\}$. In case $l = 0$, we have $u \in [w, a] - \{w\}$ so $u \in [w, a] \cap [\beta_i, w]$ for $i = 1$ or 2. Therefore one of the angle, say θ, is in $\left[\frac{1}{3}, \frac{2}{3}\right]$. So $-\theta \in \left[\frac{1}{3}, \frac{2}{3}\right]$. But $v = \gamma_P(-\theta)$ is in H_P. This is impossible since no point in H_P has angles in $\left[\frac{1}{3}, \frac{2}{3}\right]$ (Lemma 4.2; see also Figure 5). In case $l > 0$, if $F^l(u) \in W_1 \cup W_2$, then one of the angles, say $3^l\theta$ is in $\left[\frac{1}{3}, \frac{2}{3}\right]$ with $F^l(v) \in H_P$. Again this is impossible. Assume now $l > 0$ and $F^l(u) \in W_0$. We will show this does not occur. In $S^2_{g_a,P}$, the two rays $R_a(3^l\theta)$ and $R_a(3^l\theta')$ together with a portion of the equator form a Jordan curve. Denote by D the disc bounded by this curve containing β_0. Then $F^{-1}D$ has three discs each contains one of $\beta_0, \beta_1, \beta_2$. So the curve $R_a(3^{l-1}\theta) \cup F^{l-1}(u) \cup R_a(3^{l-1}\theta')$ together with a portion of the equator separates already $\{\beta_0, \beta_1, \beta_2\}$. This contradicts the fact that l is minimal. This ends the proof of the claim.

Now consider the critical value class $[a]$. If it is nontrivial a is eventually periodic (Misiurewicz case) and is never mapped to w.

For any two distinct angles θ_1, θ_2 of a we have $\gamma_P(-\theta_1) \neq \gamma_P(-\theta_2)$ otherwise $\gamma_P(-\theta_1)$ and $\gamma_P(-\theta_1)$ are both multiple contradicting the above claim.

Let θ be an angle of a. The point $\gamma_P(-\theta)$ is either simple or it has only one other angle $-\eta$. In the latter case let $u' = \gamma_a(\eta)$. Since η has the same preperiod as θ, u' is not mapped to w under the iterates of F. If u' is multiple, for any $\theta' \neq \eta$ external angle of u', by the above claim the point $\gamma_P(\theta')$ is simple. In any case $\#([a] \cap J_a) \leq 1 + \#{>}a$, and $[a]$ contains no loop.

An easy consequence is that $\#([w] \cap J_a) \leq 1 + \#{>}w = 1 + 3\#{>}a$, $\#([u] \cap J_a) = \#([w] \cap J_a)$ if $F^l(u) = w$ for some $l \geq 0$, and these ray classes don't contain closed loops.

Now assume $[u]$ contains no preimage of w. We claim that $\#([u] \cap J_a) \leq 2$ and $[u]$ contains no closed loop. Again this is easy if all points in $[u] \cap J_a$ are simple. Assume that u is a multiple point in J_a. Then by the above claim there is at most one external angle of u, say θ, such that $\gamma_P(-\theta)$ is multiple, and this point has at most one more angle $-\eta$. Now the point $u' = \gamma_a(\eta)$ satisfies the condition of the claim. So any other angle than η of u' must lead to a simple point in J_P. As a consequence $[u] \cap J_a = \{u, u'\}$ and $[u]$ contains no loops. ☐

Corollary 4.5. *Each ray class of F intersects the equator E at at most $m_a(m_a + 1)$ points.*

Proof. Each point of $[u] \cap J_a$ has at most m_a external rays and $[u] \cap J_a$ has at most $m_a + 1$ points. ☐

The next result is part of Theorem 2.4.

Corollary 4.6. *The quotient $S^2_{g_a,P}/\sim_{\text{ray}}$ is again a sphere.*

Proof. The previous two results put us in the situation of Proposition 3.2 and Corollary 3.3. ☐

Proof of Theorem 2.4, part 1 $\iff$ 2. We need to show that g_a and P_1 are not matable if and only if $g_a \perp\!\!\!\perp P_1$ has a nested obstruction which is not a Levy cycle. The "if" part follows from Proposition 3.16. For the other direction, since $S^2_{g_a,P}/\sim_{\text{ray}}$ is a sphere, the

Levy cycles of $g_a \amalg P_1$ are all removable (Proposition 3.16). Now we can apply Theorem 2.2 (to be proved later) to conclude that $g_a \amalg P_1$ must have a nested obstruction. □

5. PROOF OF THE RESULTS: SECOND PART

Proof of Theorem 2.2. It is not difficult from the following argument to see that the three cases are mutually distinct.

Recall that $\lambda(\Gamma) \geq 1$ by assumption. We will apply repeatedly Lemma 3.6.

Step 1. Suppose there is a disc-component D of $S^2 - \Gamma$ such that $D \cap \{x, y, z\} = \varnothing$. Then each connected component of $F^{-n}(D)$ is disjoint from $\{x, y, z\}$, so contains at most one critical value. So by induction on n all preimage components of D are discs. By Proposition 3.7, Γ is a removable Levy cycle.

Step 2. Suppose that there are exactly three disc-components D_x, D_y, D_z of $S^2 - \Gamma$, with $x \in D_x$, $y \in D_y$, $z \in D_z$. We claim that this is impossible.

If $a \notin D_y \cup D_z$, each component of $S^2 - \Gamma$ contains at most one critical value. By Proposition 3.7, we have $\Gamma = \{\gamma_x, \gamma_y, \gamma_z\}$. But in this case $\lambda(\Gamma) < 1$ (by looking at the degree).

Now assume $a \in D_z$. Then $F^{-1}(D_z)$ is connected, containing y, but none of x, z. So $F^{-1}(D_z)$ is isotopically contained in D_y. As a consequence, one connected component of $F^{-1}(\gamma_z)$ is isotopic to γ_y, the other is not in Γ. Thus $\Gamma = \{\gamma_x, \gamma_y, \gamma_z\}$, and, by looking at the degree, $\lambda(\Gamma) < 1$. The case $a \in D_y$ is similar.

Step 3. Suppose that there are exactly two disc-components D_1, D_2 with $\#(D_1 \cap \{x, y, z\}) = 1$ and $\#(D_2 \cap \{x, y, z\}) = 2$. We claim then Γ is a good Levy cycle with at most two curves.

Proof. Set $\gamma_i = \partial D_i$. We claim at first that $a \in D_1$ and $D_1 \cap \{x, y, z\} \subset \{y, z\}$. If not, $x \in D_1$ and D_1 contains at most one critical value. So $F^{-1}(D_1)$ are discs. One of them contains z, but not x, y, the others are disjoint from $\{x, y, z\}$. Thus no curve in $F^{-1}(\partial D_1)$ is isotopic to a curve in Γ. This contradicts the irreducibility.

One possibility is $a, z \in D_1$ and $x, y \in D_2$. In this case, $F^{-1}(D_2)$ consists of two discs one of which is strictly contained in D_2, so the other must be isotopic to D_1, with degree 1. On the other hand, $F^{-1}(D_1)$ is an annulus isotopically contained in D_2. The connected component γ' of $F^{-1}(\gamma_1)$ separating z and y is isotopic to γ_2, and $\deg(F : \gamma' \to \gamma_1) = 1$. Thus $\Gamma = \{\gamma_1, \gamma_2\}$ is a Levy cycle. Let A be the complement of $\overline{D}_1 \cup \overline{D}_2$. Either $A = \varnothing$ or a component of $F^{-1}(A)$ is isotopic to A of degree 1. In any case Γ is a good Levy cycle with at most two curves.

The other possibility is $a, y \in D_1$ and $x, z \in D_2$. A similar analysis will show that $\Gamma = \{\gamma_1, \gamma_2\}$ with $\lambda(\Gamma) = 1/2$, which is excluded by the hypothesis.

Step 4. Suppose that there are exactly two disc-components D_1, D_2 with

$$\#(D_1 \cap \{x, y, z\}) = \#(D_2 \cap \{x, y, z\}) = 1.$$

We claim that either $a, y \in D_1$ and $z \in D_2$ or $a, z \in D_1$ and $x \in D_2$. In the latter case, some preimages of Γ contain a good Levy cycle with at most two curves.

Proof. Similarly to step 2 above, one of the two discs, say D_1, must contain two critical values. One is a, the other is one of y, z. So $F^{-1}(D_2)$ are discs. So at least one component D' of $F^{-1}(D_2)$ is isotopic to one of D_1, D_2. It can not be isotopic to D_2 because of the periodic cycle $\{x, y, z\}$, so it must be isotopic to D_1. Thus if $y \in D_1$, then $z \in D_2$; if $z \in D_1$ then $x \in D_2$.

Suppose $x \in D_2$. Set $\gamma_1 = \partial D_1$ and $\gamma_2 = \partial D_2$. Complete Γ into an F-invariant multicurve Γ', as in the proof of Lemma 3.5. Define a subset

$$S_1 = \{\gamma_1\} \cup \{\gamma \in \Gamma' \mid \gamma \text{ separates } \gamma_1 \text{ and } \{y, x\}\}$$

(see the remark below). This set has a natural order. For any $\gamma \in S_1$, we have $F^{-1}(\gamma) = \gamma^* \cup \gamma^{**}$, with $[\gamma^*]_{\Gamma'} \in S_1$, $[\gamma^{**}]_{\Gamma'} \notin S_1$, $\deg(F : \gamma^* \to \gamma) = 1$, and the mapping $\gamma \mapsto [\gamma^*]_{\Gamma'}$, $S_1 \to S_1$ is weakly decreasing. So there is a periodic cycle of period at most 2.

Step 5. Now assume $a, y \in D_1$, $z \in D_2$ and $x \notin D_1 \cup D_2$. By definition, Γ is a nested obstruction. We just need to show that it's not a Levy cycle. This is part of Proposition 2.3. □

Remark. We say that a curve $\gamma \in S^2$ *separates* two sets U and V if U is contained in one component of $S^2 - \gamma$ and V is contained in the other component of $S^2 - \gamma$. Recall that $[\eta]_\Gamma$ denotes the curve in Γ isotopic to η, and $[\eta]_\Gamma = 0$ by convention, if no such curve exists.

Proof of Proposition 2.3. Assume at first $\lambda(\Gamma') \geq 1$. Let $\Gamma \subset \Gamma'$ be an irreducible obstruction. Denote by $D_\pm, A_0$ the sets $D_\pm(\Gamma), A_0(\Gamma)$ respectively. An application of Lemma 3.6 would show that, up to isotopy, D_+ is a component of $F^{-1}(D_-)$, A_0 does not contain curves of Γ and separates D_+ and D_-.

Let $B_\pm$ be the discs of $S^2 - \bar{A}_0$ containing $D_\pm$ (up to isotopy) respectively. Note that $x \in A_0$, $z \in B_-$, $y \in B_+$ and $B_+ \cup B_- = F^{-1}(S^2 - \bar{D}_+)$.

We decompose Γ into $S_+ \sqcup S_-$ with

$$S_+ = \{\gamma \in \Gamma \mid \gamma \text{ separates } \{a,y\} \text{ and } \{x,z\}\},$$
$$S_- = \{\gamma \in \Gamma \mid \gamma \text{ separates } \{a,y,x\} \text{ and } \{z\}\}.$$

Take $\gamma \in \Gamma$. Denote by Δ the component of $S^2 - \gamma$ containing z, and by $\Delta_\pm$ the two components of $F^{-1}(\Delta)$, with $\deg(F : \Delta_- \to \Delta) = 1$ and $\deg(F : \Delta_+ \to \Delta) = 2$. Then $\Delta_\pm$ are discs. Moreover, for $\gamma_\pm = \partial\Delta_\pm$ we have $\deg(F : \gamma_- \to \gamma) = 1$ and $\deg(F : \gamma_+ \to \gamma) = 2$.

For $\gamma \in \Gamma$, if γ_- is in Γ then it is in S_-, and if γ_+ is in Γ then it is in S_+. Moreover if $\gamma \in S_-$, then γ_- is not in Γ, since $z \notin \Delta_-$. Clearly Γ does not contain a Levy cycle. Let

$$v = \sum_{\gamma\in\Gamma} c_\gamma \gamma$$

be a positive eigenvector (i.e. every c_γ is positive) of F_Γ with the eigenvalue $\lambda = \lambda(\Gamma)$. Such a positive eigenvector exists, since by the assumption Γ is an irreducible multicurve. Write

$$v_i = \sum_{\gamma\in S_i} c_\gamma \gamma, \quad |v_i| = \sum_{\gamma\in S_i} c_\gamma,$$

where $i = +, -$. Using the notation above, we have

$$F_\Gamma(v_+) = \sum_{\gamma\in S_+} c_\gamma[\gamma_-]_\Gamma + \sum_{\gamma\in S_+} \tfrac{1}{2}c_\gamma[\gamma_+]_\Gamma,$$
$$F_\Gamma(v_-) = \sum_{\gamma\in S_-} \tfrac{1}{2}c_\gamma[\gamma_+]_\Gamma,$$

and

$$\sum_{\gamma\in S_+} \tfrac{1}{2}c_\gamma[\gamma_+]_\Gamma + \sum_{\gamma\in S_-} \tfrac{1}{2}c_\gamma[\gamma_+]_\Gamma = \lambda v_+,$$
$$\sum_{\gamma\in S_+} c_\gamma[\gamma_-]_\Gamma = \lambda v_-.$$

Hence

$$\left.\begin{aligned} \tfrac{1}{2}(|v_+| + |v_-|) &= \sum_{\gamma\in S_+} \tfrac{1}{2}c_\gamma + \sum_{\gamma\in S_-} \tfrac{1}{2}c_\gamma \\ &\geq \lambda \sum_{\gamma\in S_+} c_\gamma = \lambda|v_+|, \\ |v_+| &= \sum_{\gamma\in S_+} c_\gamma \\ &\geq \lambda \sum_{\gamma\in S_-} c_\gamma = \lambda|v_-|. \end{aligned}\right\} \tag{5–1}$$

A necessary and sufficient condition for both inequalities to be equalities is that

$(*)$ for $\gamma \in S_+$, both γ_+ and γ_- are isotopic to curves in Γ, and for $\gamma \in S_-$, γ_+ is isotopic to a curve in Γ.

Adding two times the first inequality in (5–1) to the second, we obtain

$$2|v_+| + |v_-| \geq \lambda(2|v_+| + |v_-|).$$

Hence $\lambda \leq 1$. But $\lambda \geq 1$ by assumption. It follows that $\lambda = 1$, and equalities hold in (5–1), proving $(*)$. Applying $(*)$ to ∂B_+, which is a component of ∂A_0, we conclude that $(\partial B_+)^-$ is isotopic to a curve in Γ, therefore separates A_0 from D_-.

We turn to the second half of the proposition. Let Γ be a submulticurve of Γ' with the given properties. Define

$$S_+ = \{\gamma \in \Gamma \mid \gamma \text{ separates } A_0(\Gamma) \text{ and } D_+(\Gamma)\},$$
$$S_- = \{\gamma \in \Gamma \mid \gamma \text{ separates } A_0(\Gamma) \text{ and } D_-(\Gamma)\}.$$

Then $\Gamma = S_+ \sqcup S_-$.

By assumption, For $\gamma \in S_+$, we have $F^{-1}(\gamma) = \gamma_+ \cup \gamma_-$, with $\gamma_+ \in S_+$, $\deg(F : \gamma_+ \to \gamma) = 2$ and $\gamma_- \in S_-$, $\deg(F : \gamma_- \to \gamma) = 1$.

For $\gamma \in S_-$, we have $F^{-1}(\gamma) = \gamma_+ \cup \gamma_-$, with $\gamma_+ \in S_+$, $\deg(F : \gamma_+ \to \gamma) = 2$ and $\gamma_- \notin \Gamma$.

Define $h : \mathbb{R}^\Gamma \to \mathbb{R}^2$ by $h(\gamma) = \binom{1}{0}$ if $\gamma \in S_+$ and $h(\gamma) = \binom{0}{1}$ if $\gamma \in S_-$. Define $G, G_\# : \mathbb{R}^2 \to \mathbb{R}^2$ to be the linear maps with matrices

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

respectively. Then one can check easily that $h \circ F_\Gamma = G \circ h$ and $h \circ F_{\#,\Gamma} = G_\# \circ h$. Set $\lambda = \lambda_\Gamma$. Let v be a nonnegative nonzero eigenvector of F_Γ for λ, that is: $F_\Gamma(v) = \lambda \cdot v$. Hence $\lambda \cdot h(v) = G(h(v))$. Since h has only nonnegative coefficients, $h(v)$ is also nonnegative nonzero. So λ is also a positive eigenvalue for G. By looking at left eigenvectors and the fact that h is surjective every eigenvalue of G is also an eigenvalue of F_Γ. As a consequence they have the same leading eigenvalue, which is 1. Similarly, $G_\#$ and $F_{\#,\Gamma}$ have the same leading eigenvalue, which is $\frac{1}{2}(1+\sqrt{5})$. □

Proof of Theorem 2.4, $3 \Rightarrow 2$. We start by introducing some notation. Assume that g_a is postcritically finite. We define $\theta^\pm = \theta^\pm(a)$ with $\theta^- < \theta^+$ to be two angles such that the corresponding two external rays land at the same point $b = b(a)$ and their union separates a from the rest of the critical orbit. If w is periodic, we take b to be the unique periodic point on the boundary of the immediate attracting basin of a which has at least two arguments and whose period divides that of a. If w is strictly preperiodic, we take b a point arbitrarily close to a having two arguments $\theta^- < \theta^+$ with the required property. It is known that such a point b always exists, and once b is chosen the two arguments $\theta^- < \theta^+$ are uniquely determined.

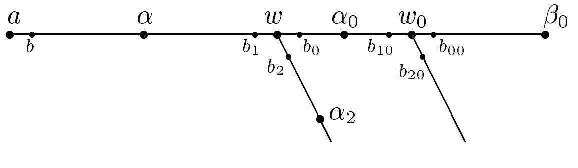
For $i = 0, 1, 2$, denote by β_i the point in J_a with external angle $i/3$. Let

$$\tilde{H} = g_a^{-1}[H_a, \beta_0, \beta_1, \beta_2],$$

where H_a is the Hubbard tree and $[H_a, \beta_0, \beta_1, \beta_2]$ denotes the convex hull of H_a and β_i (the extended Hubbard tree).

Note that w disconnects $\tilde{H}$ into three components W_0, W_1, W_2, numbered in a way such that $\beta_i \in W_i$. We have $a \in W_1 \cup W_2$. Since $g_a \perp\!\!\!\perp P_1$ and $g_{\bar{a}} \perp\!\!\!\perp P_1$ are conjugate by an orientation reversing homeomorphism, one has a Thurston obstruction if and only if the other one has. So replace g_a by $g_{\bar{a}}$ if necessary, we may assume in the following $a \in W_1$.

For any $z \in [H_a, \beta_0, \beta_1, \beta_2] - \{a\}$, denote by z_i the point $g_a^{-1}(z) \cap W_i$. For b the landing point of rays of angle $\theta^\pm$, we have $b_i \in [H_a, \beta_0, \beta_1, \beta_2] \cap W_i$, for $i = 0, 1, 2$. Furthermore $b_{i0} \in \tilde{H}$. Also $w_0 \in W_0$, and w_0 is the unique point in $[w, \beta_0]$ which is mapped to w. We may choose b (and therefore $\theta^\pm$) so that the b_i's are not separated from w in J_a (or H_a) by any points in the forward orbit of a, and similarly for the b_{i0}'s and w_0. The situation is this:



First assume that $\beta_0, g_a(a) \in W_0$ and g_a^3 is renormalizable. In particular, α, the fixed point in $[a, w]$, is not in the postcritical set, and has external angles $\frac{1}{8}$ and $\frac{3}{8}$. Moreover the point α_2 does not separate the period-three orbit of the renormalized Julia set (the small Julia sets). So we can define the curves δ_1, δ_2 exactly in the same way as in Section 2C. Since δ_1 does not meet the small Julia sets, no preimages of it does. Therefore δ_3, δ_5 as in Section 2C are homotopic to δ_1. So δ_1, δ_2 form a nested obstruction.

Now consider the case where $\beta_0, g_a(a), g_a^2(a) \in W_0$. One can choose $\theta^\pm$ so that neither b_{10} nor b_{20} belong to $[g_a(b), g_a(a)]$.

Recall from Section 2C that for P_1 the points x_*, y_*, z_* denote the roots of basins of attraction of x, y, z respectively, and $\geqslant y_* = \{\frac{4}{13}, \frac{9}{13}\}$, $\geqslant z_* = \{\frac{1}{13}, \frac{12}{13}\}$ (see Figure 3).

In $S^2_{g_a, P_1}$, we are going to modify the subsets

$$\gamma'_- := R_a(3\theta^+) \cup R_a(3\theta^-) \cup \{g_a(b)\} \cup R_{P_1}\left(\tfrac{1}{13}\right) \cup R_{P_1}\left(\tfrac{12}{13}\right) \cup \{z_*\},$$

$$\gamma'_+ := R_a(\theta^+) \cup R_a(\theta^-) \cup \{b\} \cup R_{P_1}\left(\tfrac{4}{13}\right) \cup R_{P_1}\left(\tfrac{9}{13}\right) \cup \{y_*\}$$

in a neighborhood of the equator, so that they become two Jordan curves γ_- and γ_+ whose homotopy classes rel P_F do not depend on the modification.

To do this first we take a closed annular neighborhood N of the equator of $S^2_{g_a, P_1}$ bounded by an equipotential e^u of g_a and an equipotential e^l of P_1 (here the subscripts u and l mean "upper" and "lower" boundary of N). We denote by (θ, u) the point in e^u with angle θ. Define (θ, l) similarly.

We know that $3\theta^\pm$ are two arguments of $g_a(b)$. As $g_a(b) \in W_0$ and β_2, the point in J_a with external angle $\frac{2}{3}$, is in W_2, we know that $\left(\frac{2}{3}, u\right)$, $(3\theta^-, u)$, $(3\theta^+, u)$ are placed counterclockwise in e^u.

On the other hand, the points $\left(\frac{1}{3}, l\right)$, $\left(\frac{1}{13}, l\right)$, $\left(\frac{12}{13}, l\right)$ are placed clockwise in e^l.

Note that $\left(R_a\left(\frac{2}{3}\right) \cup R_{P_1}\left(\frac{1}{3}\right)\right) \cap N$ is a vertical segment connecting $\left(\frac{2}{3}, u\right)$ to $\left(\frac{1}{3}, l\right)$. Now we can modify $\gamma'_- \cap N$ so that it consists of two affine segments disjoint from $\left(R_a\left(\frac{2}{3}\right) \cup R_{P_1}\left(\frac{1}{3}\right)\right) \cap N$ and connecting $(3\theta^-, u)$ to $\left(\frac{1}{13}, l\right)$ and $(3\theta^+, u)$ to $\left(\frac{12}{13}, l\right)$ respectively. This, together with $\gamma'_- - N$, defines our Jordan curve γ_-. See Figure 6.

Turning to the modification of γ'_+, we will use γ_- as reference rather than $R_a\left(\frac{2}{3}\right) \cup R_{P_1}\left(\frac{1}{3}\right)$, which is no longer good. The points $(3\theta^-, u)$, $(3\theta^+, u)$, (θ^-, u),

(θ^+, u) appear counterclockwise on e^u and the points $\left(\frac{1}{13}, l\right)$, $\left(\frac{12}{13}, l\right)$, $\left(\frac{9}{13}, l\right)$, $\left(\frac{4}{13}, l\right)$ appear clockwise on e^l. So we can modify $\gamma'_+ \cap N$ so that it consists of two affine segments disjoint from γ_- and connecting (θ^-, u) to $\left(\frac{9}{13}, l\right)$ and (θ^+, u) to $\left(\frac{4}{13}, l\right)$ respectively. This, together with $\gamma'_+ - N$, defines our Jordan curve γ_+. See Figure 6.

By construction, γ_- and γ_+ are disjoint; moreover one disc component D_+ in the complement of γ_+ contains only a and y as postcritical points, while a disc component D_- in the complement of γ_- contains only $F^2(w)$ and z as postcritical points. Set $A_0 = F^{-1}(D_+)$.

Note that $S^2 - D_+$ is a disc and that it contains D_-, the connected set

$$s = [b, \beta_0] \cup R_a(0) \cup R_{P_1}(0) \cup [\beta', y_*],$$

and a unique critical value, which is z. By looking at components of $F^{-1}(s)$ one can prove easily that $b_1, b_2 \in \partial^+ A_0$ (the component of ∂A_0 separating A_0 and a), the point b_0 is in the other component of ∂A_0 and D_+ is homotopically contained in a component of $F^{-1}(D_-)$. Iterating backward once more we can show that b_{10}, b_{20} are in the same component of $F^{-1}(\partial^+ A_0)$.

Now the fact that $b_0, b_1, b_2 \notin [g_a(a), \beta_0]$ implies that A_0 is essentially contained in the annulus A bounded by γ_+ an γ_-, and $b_{10}, b_{20} \notin [g_a(a), \beta_0]$ implies that the component of $F^{-1}(\partial^+ A_0)$ containing b_{10} is essentially contained in A. Now define Γ to be the set of curves γ such that γ is essentially contained in $A - A_0$ and is homotopic to a curve of $F^{-n}(\gamma_-)$ for some $n \geq 0$. By Proposition 2.3, Γ is a nested obstruction and not a Levy cycle. $\square$

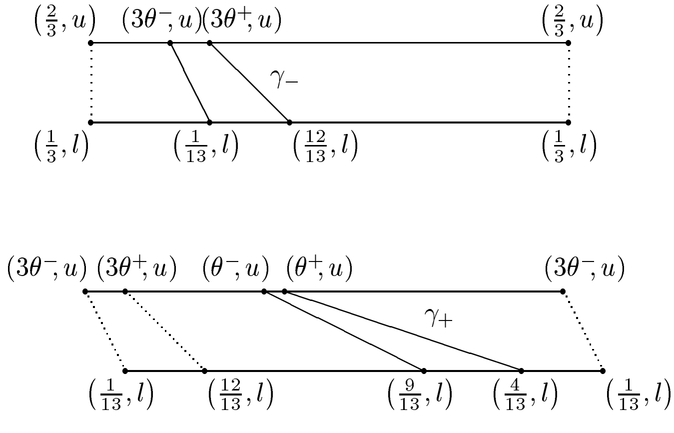


FIGURE 6. Modifications in N.

Proof of Theorem 2.4, $4 \Rightarrow 5$. There is a polynomial g_c for which w is three-periodic and $\theta^\pm(c) = \frac{7}{26}, \frac{9}{26}$. Its Hubbard tree is the segment $[c, g(c)]$ in Figure 2 and the entropy is $\log\left(\frac{1}{2}(1+\sqrt{5})\right)$. By monotonicity of entropies (see [Douady 1995], for example), any g_a with $\frac{7}{26} \sim_{g_a} \frac{9}{26}$ will have an entropy on its Hubbard tree at least $\log\left(\frac{1}{2}(1+\sqrt{5})\right)$. $\square$

Proof of Theorem 2.4, $2 \Rightarrow 5$. We will need the following (unpublished) result of M. Shishikura: Let Γ be a Thurston obstruction for a mating $F = g \amalg f$. Then

$$\mu_\Gamma \leq e^{h_{\mathrm{top}}(H_g)}. \tag{5–2}$$

where $h_{\mathrm{top}}(H_g)$ denotes the topological entropy of the map $g : H_g \to H_g$.

Now let $F = g_a \amalg P_1$ and Γ' be a nested obstruction. By Proposition 2.3, Γ' contains a subobstruction Γ with $\mu_\Gamma = \frac{1}{2}(1+\sqrt{5})$. Put this into (5–2) we get

$$e^{h_{\mathrm{top}}(H_g)} \geq \tfrac{1}{2}(1+\sqrt{5}),$$

which is Condition 5 of the theorem. $\square$

Proof of Theorem 2.4, $3 \Rightarrow 4$. Assume $a \in W_1$ (otherwise consider $g_{\bar{a}}$). If $g_a(a) \in W_0$ and g_a^3 is renormalizable, the small Julia set containing a has a fixed point (for g_a^3) in $[a, w]$ with external angles $\frac{7}{26}, \frac{9}{26}$. If $g_a(a), g_a^2(a) \in W_0$, then $w_0 \in [a, g_a(a)]$. Since g_a maps each segment in the sequence

$$[\alpha_0, w_0] \to [\alpha, w] \to [\alpha, a] \to [\alpha, g_a(a)]$$

homeomorphically onto the next (Figure 5), and since $[\alpha, g_a(a)]$ contains $[\alpha_0, w_0]$, there is a 3-periodic point z in $[\alpha_0, w_0]$. An easy calculation shows that $g_a^2(z)$ has external angles $\frac{7}{26}, \frac{9}{26}$. $\square$

APPENDIX: MATINGS SEEN IN PARAMETER SPACE AND SOME NUMERICAL OBSERVATIONS

The rational maps in $\mathcal{A}$ form, up to Möbius transformation, a one-parameter family given by:

$$\begin{aligned} F_t(z) &= \frac{(3t-2)z^3 - (t^4 - 3t^2 + 5t - 2)(3z-2)}{(3t-2)z^3 - t^3(3z-2)} \\ &= 1 - \frac{(t-1)^3(t+2)(3z-2)}{(3t-2)z^3 - t^3(3z-2)}, \end{aligned}$$

with the parameter $t \in \mathbb{C} - \{-2, \frac{2}{3}, 1\}$. Here $0, \infty, 1, t$ correspond to w, x, y, z; in other words, 0 is a free double critical point, ∞ and 1 are two simple critical points, with orbit $\infty \mapsto 1 \mapsto t \mapsto \infty$.

Figure 7 shows the t-plane in the window $-5 \leq \operatorname{Re} t \leq 5$, $-5 \leq \operatorname{Im} t \leq 5$, and then again under magnification near the point $t = -2$.

We first describe several major hyperbolic components. The unique unbounded light grey component consists of t-values such that 0 is in the immediate basin of ∞ for F_t. The central dark grey component G consists of t values such that 0 is in the immediate basin of t for F_t. The largest black component B symmetric with respect to $\mathbb{R}$ represents t values such that 0 is in the immediate basin of 1 for F_t. Note that $0 \in G$ and the degenerate parameter $t = 1$ is in B. (In the zoom-in of Figure 8 the set G touches the left edge and B touches the right edge.)

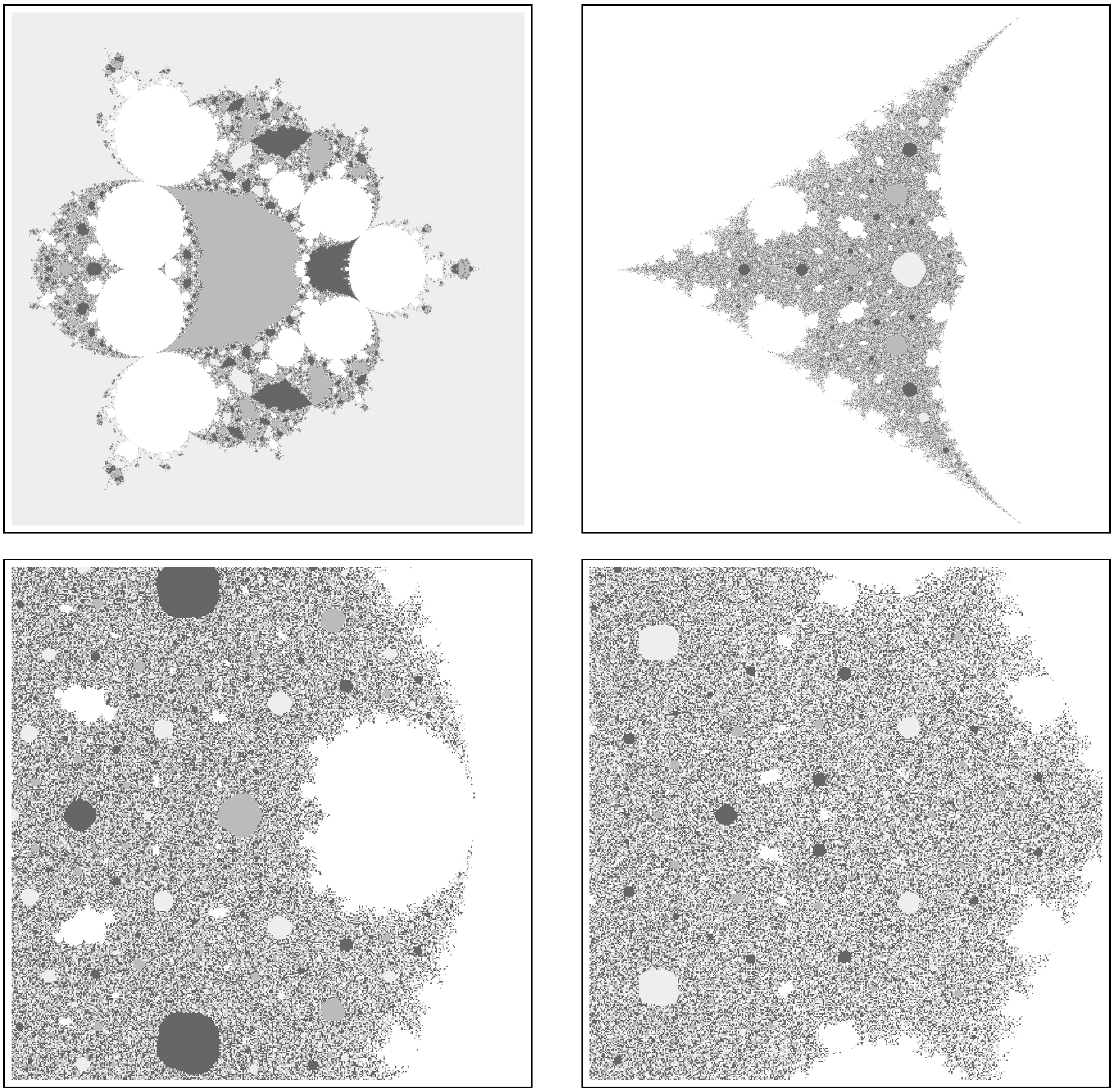


FIGURE 7. The family $\{F_t\}$. On the top left is the region $-5 \leq \operatorname{Re} t \leq 5$, $-5 \leq \operatorname{Im} t \leq 5$. The remaining panes zoom in at -2 with increasingly greater magnifications. In each case, white pixels represent values of t for which the orbit $\{F_t^n(0)\}$ is not attracted by the orbit of ∞. We color a t-value light grey, dark grey, or black if $F_t^{3p}(0)$, $F_t^{3p+1}(0)$, $F_t^{3p+2}(0)$, respectively, is contained in the basin of ∞, for some integer $p \geq 0$.

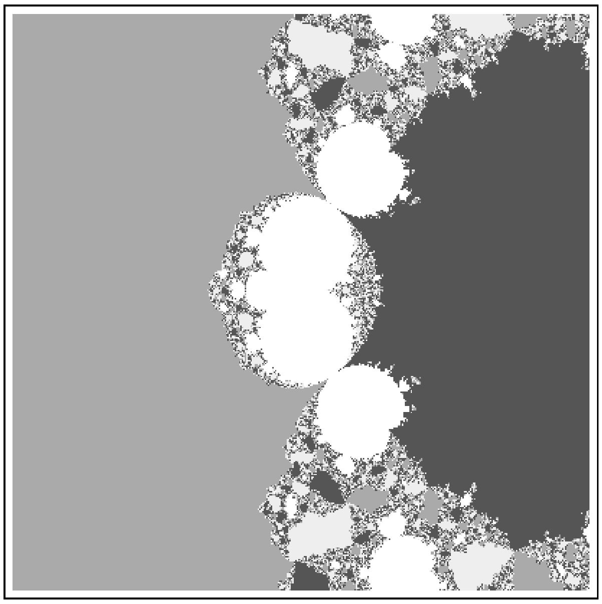


FIGURE 8. The family $\{F_t\}$ near the point $\frac{2}{3}$.

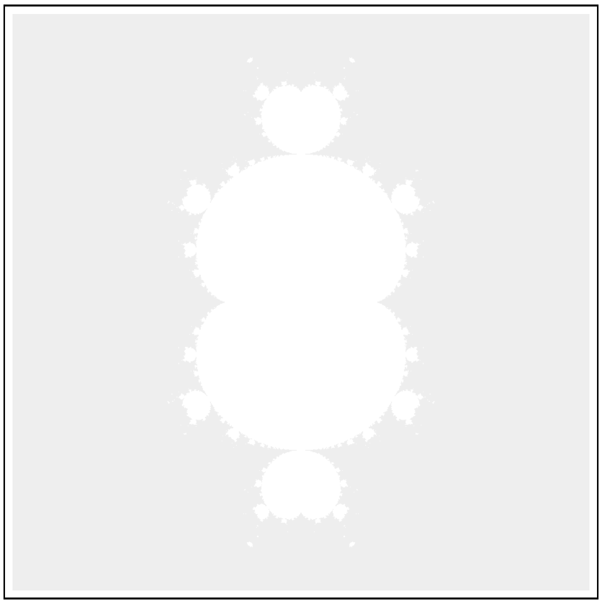


FIGURE 9. The connected locus of the family $z^3 + a$.

There are four "simplest" (but not always largest) white components, defined as follows.

There are four distinguished t-values for which 0 is fixed under F_t. They are the solutions of $t^4 - 3t^2 + 5t - 2 = 0$. Two of them, denoted t_1, t_2 with $t_1 < t_2$, are real, and two, denoted by t_3, t_4 with $Im t_3 > 0$, are complex conjugate. For $i = 1, 2, 3, 4$, the polynomial F_{t_i} is conjugate to P_i.

Denote by W_i the white component containing t_i, for $i = 1, 2, 3, 4$. Then W_1 is the largest white component on the left of G. The boundary of W_1 has two cusps, the right one corresponding to the left corner of Figure 7, top right. The set W_2 is the largest white component sitting between G and B. Figure 8 gives a closer look of it. The boundary of W_2 has three cusps, with the unique real cusp at the degenerate parameter $t = \frac{2}{3}$. W_3 and W_4 are the two big white components above and below B. Their boundaries have two cusps.

We now give an interpretation of our result about matings. Define

$$M_3 = \{a \in \mathbb{C} \mid g_a^n(0) \not\to \infty \ \text{ as } \ n \to \infty\}$$

(see Figure 9) and

$$M' = \{a \in \mathbb{C} \mid g_a \text{ is postcritically finite } \} \subset M_3.$$

Recall that, for $i = 1, 2, 3, 4$, we defined

$$A_i = \{\, a \in M' \mid g_a \text{ and } P_i \text{ are matable}\,\}.$$

For $a \in A_i$ the mating $g_a \perp\!\!\!\perp P_i$ is weakly equivalent to $F_{t(a,i)}$ for a unique $t = t(a,i)$ (Theorem 1.1 of Thurston). In particular, for $a = 0$, we have $g_0 \perp\!\!\!\perp P_i \sim F_{t_i}$ for $i = 1, 2, 3, 4$. We denote by m_i the mapping $A_i \to \mathbb{C}$, $a \mapsto t(a,i)$. Therefore $m_i(0) = t_i \in W_i$.

Theorem 2.6 shows that $A_3 = A_4 = M'$. Denote by L_t the limb of M_3 with internal angle t and by $\gamma_{M_3}(\theta)$ the landing point in M_3 (if it exists) of the external ray of angle θ. Then (as proved in the case of the Mandelbrot set) $L_{\frac{3}{4}}$ coincides with the set of $a \in M_3$ such that $\frac{5}{8} \sim_{g_a} \frac{7}{8}$.

Theorem 2.5 shows $A_2 \subset M' - L_{\frac{3}{4}}$ and we conjecture that $A_2 = M' - L_{\frac{3}{4}}$.

The set A_1 is dealt with by Theorem 2.2.

Similarly to the study of the Mandelbrot set, one can show that $\gamma_{M_3}\big(\frac{7}{26}\big) = \gamma_{M_3}\big(\frac{9}{26}\big)$, it is a root of a hyperbolic component $W(c)$ with center c as in Section 2C and $\gamma_{M_3}\big(\frac{8}{26}\big)$ is the other root of $W(c)$.

Next, if we denote by $L_{\frac{7}{26},\frac{9}{26}}$ the component of $M_3 - \{\gamma_{M_3}\big(\frac{7}{26}\big), \gamma_{M_3}\big(\frac{9}{26}\big)\}$ containing c and by $M(c)$ the small copy of M_3 centered at c, then $c \in W(c) \subset M(c) \subset L_{\frac{7}{26},\frac{9}{26}}$, the set $M(c)$ corresponds to the set of a such that g_a^3 is renormalizable and $\frac{7}{26} \sim_{g_a} \frac{9}{26}$, and the set $L_{\frac{7}{26},\frac{9}{26}}$ corresponds to the set of $a \in M_3$ such that $\frac{7}{26} \sim_{g_a} \frac{9}{26}$.

Furthermore, the union of the internal rays of $W(c)$ with angles 0 and 1/2 separates $L_{\frac{7}{26},\frac{9}{26}}$ into two halves, named by $L_{\frac{7}{26},\frac{8}{26}}$ and $L_{\frac{8}{26},\frac{9}{26}}$ respectively, with the last one containing $\gamma_{M_3}\bigl(\frac{1}{3}\bigr)$.

Condition 4 of Theorem 2.2 corresponds to

$$L_{\frac{7}{26},\frac{9}{26}} \cup L_{\frac{17}{26},\frac{19}{26}} = L_{\frac{7}{26},\frac{9}{26}} \cup \bar{L}_{\frac{7}{26},\frac{9}{26}}.$$

We conjecture that $A_1 = M' - L_{\frac{7}{26},\frac{9}{26}} \cup L_{\frac{17}{26},\frac{19}{26}}$. Condition 3 of Theorem 2.2 corresponds to

$$M' \cap (M(c) \cup \overline{M}(c) \cup L_{\frac{8}{26},\frac{9}{26}} \cup L_{\frac{17}{26},\frac{18}{26}}),$$

and Theorem 2.2 shows that

$$A_1 \subset M' - (M(c) \cup \overline{M}(c) \cup L_{\frac{8}{26},\frac{9}{26}} \cup L_{\frac{17}{26},\frac{18}{26}}).$$

Condition 5 is more difficult to describe. One can show, however, that any a in the segment from 0 to $\gamma_{M_3}\bigl(\frac{7}{26}\bigr)$ has topological entropy smaller than $\log\bigl(\frac{1}{2}(1+\sqrt{5}\,)\bigr)$, and therefore, by Theorem 2.2, is in A_1.

Our computer observation strongly suggests that the mappings m_i are continuous and can be extended to a considerably large and connected subset of M_3. Moreover it seems that $m_4(L_{\frac{7}{8}}) = m_3(L_{\frac{5}{8}})$ and $m_1(L_{\frac{1}{4}} - L') = m_1(-(L_{\frac{1}{4}} - L'))$. Using shared mating techniques, we can actually show that

$$\begin{aligned} m_1(L_{\frac{7}{8}} \cap M') &\subset m_3(M'),\\ m_2(L_{\frac{7}{8}} \cap M') &\subset m_3(M'),\\ m_4(L_{\frac{7}{8}} \cap M') &\subset m_3(M'). \end{aligned}$$

The period-two hyperbolic component D of M_3 attached to the main component W_0 with internal angle $\frac{3}{4}$ does not seem to have a corresponding image (under m_2) attached to W_2. We think that $t = \frac{2}{3}$ corresponds to the cutting point (or the root) of D of M_3. This supports the conjecture that $A_2 = M' - L_{\frac{3}{4}}$.

The degenerate parameter $t = -2$ is much more mysterious. The component W_1 looks very much like W_0. However there is a period 2 component $W_{1,2}$ which is attached to W_1 at two points, each corresponding to a root of $W_{1,2}$. This suggests that $W_{1,2}$ can be considered as a self shared mating, and can be indeed proved. The point $t = -2$ is surrounded by $W_{1,2}$ and W_1 (see the zoom-ins in Figure 7), and we see no white component attached to $t = -2$.

This is the first time in our experience that we see a singularity in an analytic family which is not on the boundary of any structurally stable component. We think that this should be related to the presence of non Levy cycle obstructions in $\mathcal{A}$.

There is another way to construct rational maps from polynomials, named by 'captures'. It has been intensively studied by B. Wittner, M. Rees, J. Head, Tan Lei and others. For the precise definition see [Tan 1997]. It is farely easy to show that, for a postcritically finite map F_t, if the basins of x, y and z are attached at a common fixed point, then F_t is equivalent to either a mating or a capture. One may ask if this is still true for any postcritically finite F_t (it is indeed the case for cubic Newton maps; see [Tan 1997]). Note that our family $\{F_t\}$ is somewhat similar to the family of quadratic rational maps with one critical point three-periodic (they both have one free critical point and a period-three orbit containing the remaining critical points). We therefore believe that, as known in the quadratic family due to Wittner, there are maps in the form of F_t with w periodic and not equivalent to a mating (they are by definition not equivalent to a capture).

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