Conjecturally Optimal Coverings of an Equilateral Triangle with Up to 36 Equal Circles

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CONTENTS

This paper presents a computational method to find good, conjecturally optimal coverings of an equilateral triangle with up to 36 equal circles. The algorithm consists of two nested levels: on the inner level the uncovered area of the triangle is minimized by a local optimization routine while the radius of the circles is kept constant. The radius is adapted on the outer level to find a locally optimal covering. Good coverings are obtained by applying the algorithm repeatedly to random initial configurations.

The structures of the coverings are determined and the coordinates of each circle are calculated with high precision using a mathematical model for an idealized physical structure consisting of tensioned bars and frictionless pin joints. Best found coverings of an equilateral triangle with up to 36 circles are displayed, 19 of which are either new or improve on earlier published coverings.

1. INTRODUCTION

During the last few decades, packing circles in various shapes in the plane or on a sphere has been a popular research subject of discrete geometry. One tries to find a configuration of n equal nonoverlapping circles inside the area so that the radius of the circles is maximized. For example, packings have been searched in a square, in an equilateral triangle, in a circle, and on a sphere; see [Croft et al. 1991; Melissen 1997b] for references. In the searches computers often play an important role.

On the other hand, the dual problem of determining good *coverings* of an area with n equal circles so that the radius of the circles is as small as possible has received much less attention. Covering problems are intriguing mathematical problems, but they also have a few interesting practical applications. Examples include finding good locations of sprinklers on cultivated ground (to minimize the differences between the amount of watering), finding locations of

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floor drains on a garage floor (to minimize the altitude difference on the floor), or even placing n pieces of jalapeño on a pizza so that the distance from any point on the pizza to the nearest piece of jalapeño is minimal (there are no empty-looking areas on the pizza).

Early results in the literature on coverings of geometrical shapes concern mostly the problem of covering a sphere by circles (circular caps); see [Hardin et al. n.d.; Melissen 1997b] and their references. Recently, however, there has been a growing interest in studying good coverings of several other shapes, mainly in the plane.

The earliest computer results for circle coverings in the plane were obtained by Zahn [1962], who searched for good coverings of a circle by smaller, equal circles. Zahn made the problem discrete by dividing the big circle into small cells and then applying local optimization methods to cover as many of these cells as possible while keeping the radius of the small circles constant.

The first computational results on coverings of a square appeared in [Tarnai and Gáspár 1995], with up to 10 circles. The authors found locally optimal coverings by simulating systems consisting of shrinking, tensioned bars and pin joints. Melissen and Schuur [1996] improved the coverings with 6 and 8 circles and presented a new covering with 11 circles. Their approach is based on a simulated annealing algorithm using the Voronoi tessellation of the square with respect to the centers of the circles. Extending the work of [Tarnai and Gáspár 1995], Lengyel and Veres [1996] presented coverings of a square with up to 23 circles. Their coverings with 12 to 21 circles were later surpassed by Nurmela and Östergård [2000], who also gave new coverings with 24 to 30 circles. Coverings of a rectangle have been published with up to 7 circles [Heppes and Melissen 1997; Melissen and Schuur 2000].

Coverings of an equilateral triangle with up to 18 circles (including optimality proofs in the smallest cases) were presented in [Melissen 1997a; 1997b]; this work extends those results. A recent survey on circle coverings can be found in [Melissen 1997b, Chapter 5].

In Section 2 we present our algorithm for finding good circle coverings of an equilateral triangle. A method for numerically determining the structure of a covering is presented in Section 3. Using these algorithms, in Section 4 we improve the previous covering of a triangle with 13 circles and present new, conjecturally optimal coverings with 19 to 36 circles. All the (conjecturally) optimal coverings with 2 to 36 circles are depicted to show the structure of each covering.

2. COVERING ALGORITHMS USED

In this work we use the algorithm in [Nurmela and Östergård 2000] with only minor changes. We give here a brief description of the algorithm.

The algorithm consists of two nested levels: on the inner level, the radius of the circles is constant and a local optimization routine is used to minimize the uncovered area of the triangle by moving the circles. We use the SUMSL subroutine of [Gay 1983], an implementation of the BFGS secant method, with analytically calculated first partial derivatives.

The outer level of the algorithm adjusts the radius of the circles. We start from a random initial circle configuration and try to cover the triangle using the BFGS routine and a fixed radius r. Depending whether or not a covering is found we decrease or increase the radius r, respectively, and restart the BFGS routine from the best covering found during this run of the algorithm, or from the random initial configuration if no coverings have yet been found.

The control of the radius r is not trivial, because we have to make compromises in the convergence criteria: the BFGS routine may stop with a partial covering even though a covering would be obtainable with some extra iterations. That is, failure in finding a covering does not imply that a covering with equal or smaller radius cannot be found later, after additional BFGS runs and adjustments of r. We get a series of radii, $r_1, r_2, \ldots$. After the ith BFGS run with radius r_i we calculate the next radius r_{i+1} according to the following two alternatives (using control parameters $\alpha_1, \alpha_2, \ldots$):

1. Covering with r_i was found: $r_{i+1} = \alpha_i r_i$; let
$$\alpha_{i+1} = \max(\alpha_i^2, 0.9).$$
2. Covering with r_i was not found: let j be the largest integer such that $j < i$ and a covering with r_j was found. Now $r_{i+1} = \frac{1}{2}(r_i + r_j)$. Let $\alpha_{i+1} = \frac{1}{2} + \frac{1}{2}(r_i/r_{i+1})$.

By selecting those radii that produced a covering (alternative 1), we get a monotonically decreasing series of radii corresponding to a series of coverings with gradually shrinking circles. These coverings approximate a locally optimal covering.

The first radius, r_1, is selected so that it is slightly larger than the radius in the best known covering. A suitable value for α_1 is for example 0.9, if r_1 is not very close to the radius in the conjecturally optimal covering. Note that if the first BFGS run with r_1 does not produce a covering, we can reject this initial configuration and stop this run of the algorithm (it is probable that the algorithm will converge to a poor local optimum). Furthermore, if at any stage of the algorithm a BFGS run converges to only a partial covering when the radius r is clearly larger than in the best currently known covering, we can stop the run (we are close to a local optimum different from the global optimum). This greatly speeds up the search procedure, because we can often very fast reject initial solutions leading to a poor local optimum.

Another way to speed up the algorithm is to limit the number of iterations in the BFGS routine in the early stages of the algorithm. Namely when r is not (yet) very accurate, it is unnecessary to calculate the local optimum in the BFGS routine with high precision.

A successful optimization run with $n \leq 36$ that finds a covering not exceeding the prescribed value of r takes at most a few minutes of CPU-time on a current Pentium PC running Linux operating system when the following stopping criterion is used: we stop optimization when a cover with r_i is not found and $r_j - r_i < 10^{-7}$, where r_j corresponds to the best covering found during this optimization run. If the initial solution is poor, so that the optimization algorithm is likely to converge to a covering inferior to the best known covering, then the run can in most cases be ended in a couple of seconds.

To find good coverings we simply apply the algorithm repeatedly starting from random initial configurations and selecting the best covering found. This is a simple stochastic global optimization algorithm called the *multistart* algorithm. The quality of the results depends on the distribution of the initial solutions and on the number of optimization runs performed.

3. FINDING THE STRUCTURE OF A COVERING

The covering algorithm of the previous section can provide the coordinates of a covering only with an accuracy limited by the precision of the computations. It is necessary to compute the positions of the circles with high precision to validate the assumed structure of the covering. To find the coordinates of a locally optimal covering with greater accuracy, we identify the points where at least three circles (or two circles and the triangle boundary, or one circle and a corner of the triangle) appear to intersect. By inserting edges between these points and the centers of the circles, we get the *graph* of the covering. By requiring that each edge of the graph has length equal to r we get a system of nonlinear equations, which has a solution at a locally optimal covering. However, this system of equations is usually underdetermined and the value of r cannot be determined from the underlying graph alone, see [Melissen 1997b; Nurmela and Östergård 2000].

In order to obtain a system with a unique solution in the locally optimal covering we use a mathematical model for an idealized physical structure with tensioned bars and frictionless pin joints; compare [Nurmela and Östergård 2000; Tarnai and Gáspár 1995]. Each edge in the graph of the covering corresponds to a bar and each vertex corresponds to a pin joint. Now if each bar is tensioned, the system starts to move towards a locally optimal covering, provided that the tensions are adjusted dynamically so that all bars have equal length. In a configuration corresponding to a locally optimal covering, all pin joints (and the triangle) are in rest while most (but not necessarily all) bars have positive tension.

We write down the equations of motion for each pin joint and the triangle and add these equations to the system of equations formed earlier. We now also have new variables corresponding to the tensions in the bars. At no stage do we impose symmetry constraints on the solution of the system; any symmetries are investigated only after the system is solved numerically.

The total tension of the structure is not determined, so we arbitrarily set the tension of a suitably selected bar equal to a constant. The optimization algorithm in Section 2 provides us with very good initial values for the coordinates of the pin joints, but the tensions in the bars are not known. The initial tensions are solved in a separate step before solving the whole system (this requires finding an approximate solution to an overdetermined linear system of equations [Nurmela and Östergård 2000]).

In the numerical solution we use the modified Newton–Raphson method of [Ben-Israel 1966], which works even when the system is overdetermined. All systems in this paper were solved with such an accuracy that the maximal error in the equations was less than 10^{-100} (in the computations the side of the triangle was equal to 1 and one of the tensions was fixed equal to 1), using Mathematica. It is not necessary to perform all the calculations with high precision [Nurmela and Östergård 2000]; this makes the computations much faster.

The largest system of equations in this paper has 383 variables (the covering with 36 circles). That covering is similar to the optimal packing of 36 circles. From the structure of the optimal packing it is easy to construct a system of 108 equations, which has a unique solution in a finite neighborhood of the optimal packing. Since the number of equations is so much smaller in the packing problem (compare [Nurmela and Östergård 1997; 2000]), one may suggest that a covering problem with n circles is in some sense more difficult than the corresponding packing problem with the same number of circles.

In order to determine whether the solution found is unique in a finite neighborhood of the initial solution we solve the system several times with slightly perturbed initial solutions; compare [Graham et al. 1998]. Since in all the coverings in this work the solution converged each time to the same solution (within the high precision used), we conclude that the coverings with presented structure exist and are true (at least locally) optimal coverings with high probability. We now also have very accurate numerical values for the corresponding values of r.

4. RESULTS

For completeness we show all the best known coverings for $2 \leq n \leq 36$ in Figures 1–4, since the structures of the coverings have not been shown explicitly before. In these figures each pin joint is denoted by a black dot (center of a circle) or a small circle (intersection). Each bar is denoted by a line segment.

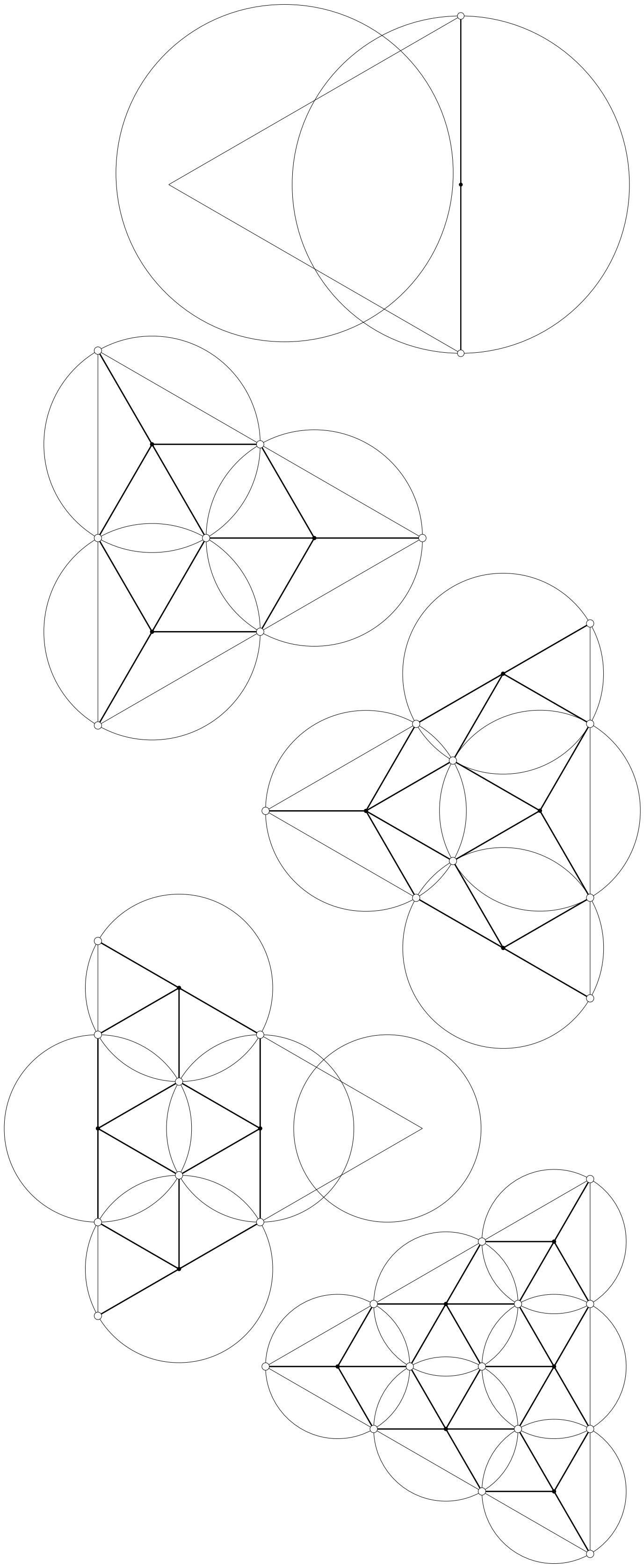

FIGURE 1. Coverings for $n = 2$ to $n = 6$.

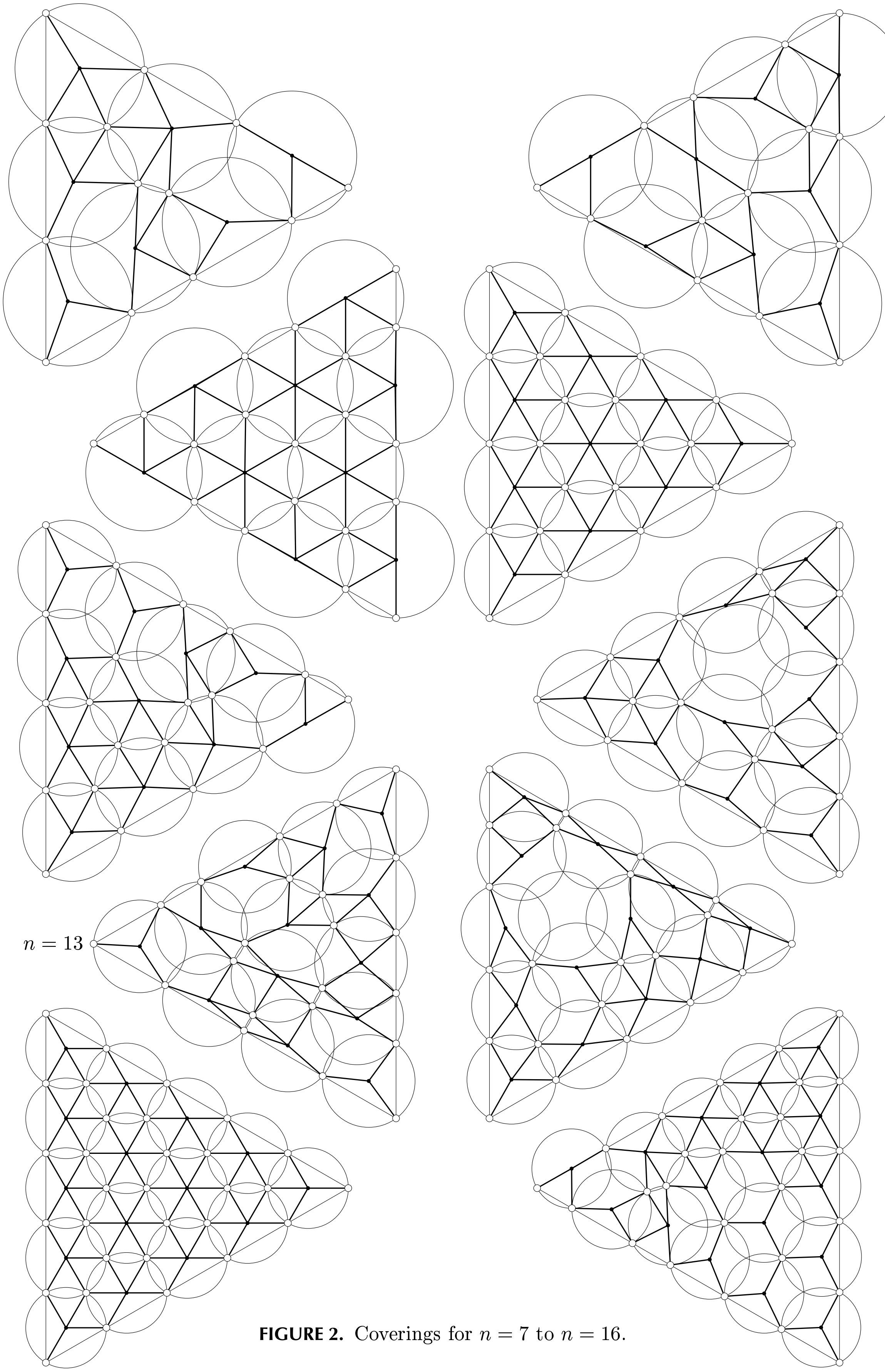

FIGURE 2. Coverings for $n = 7$ to $n = 16$.

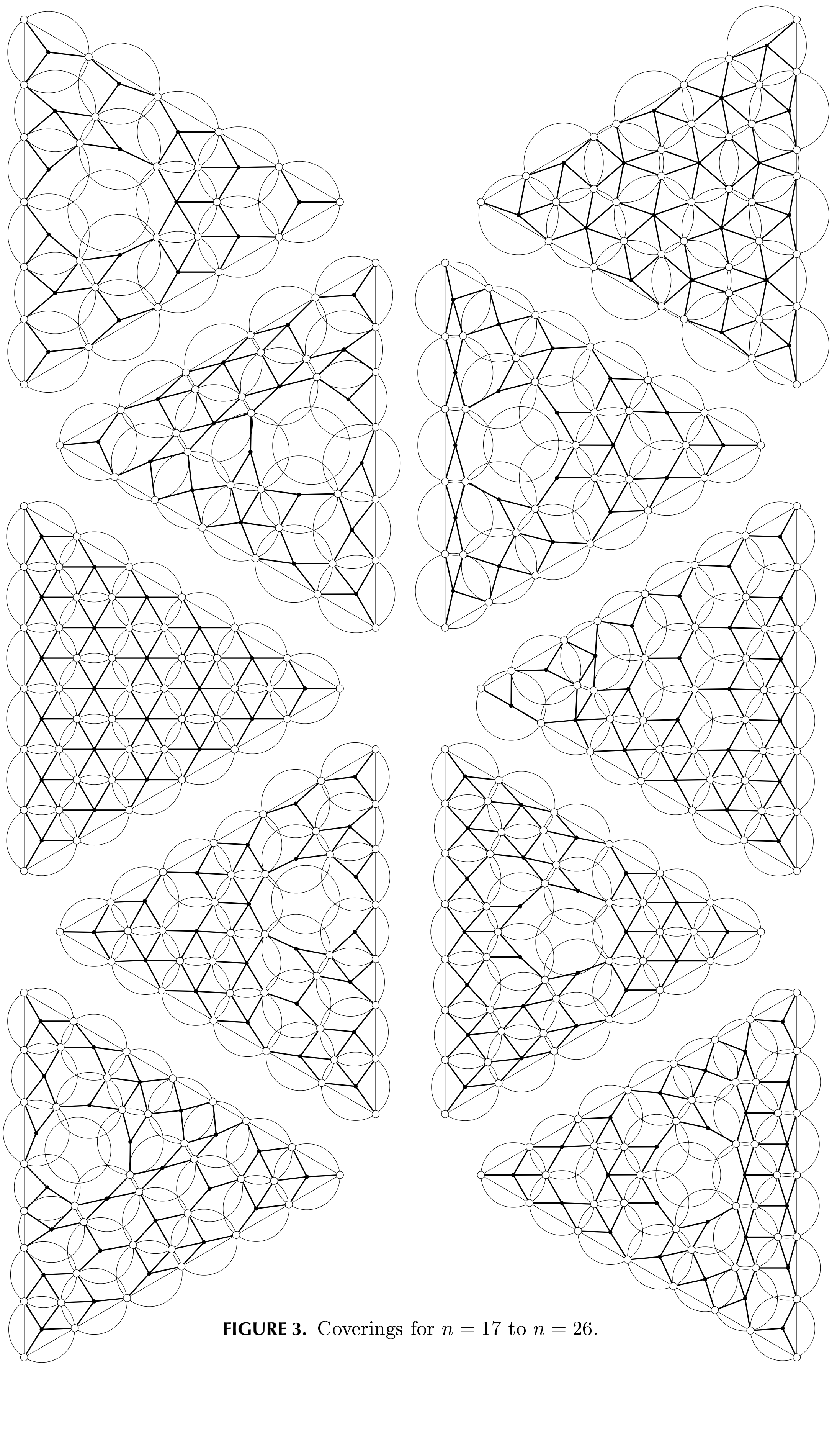

FIGURE 3. Coverings for $n = 17$ to $n = 26$.

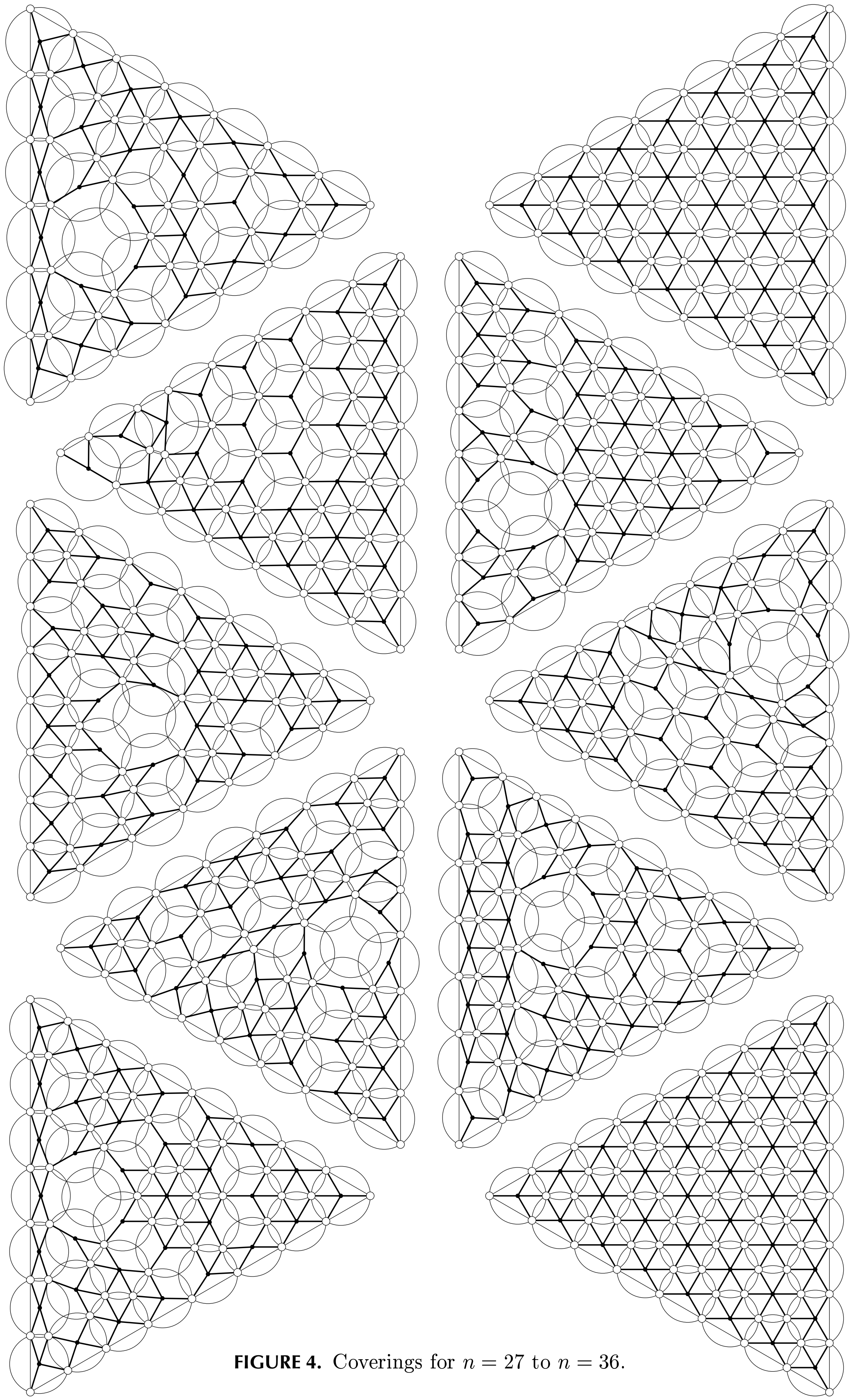

FIGURE 4. Coverings for $n = 27$ to $n = 36$.

The optimal covering for $n = 1$ is of course the circumscribed circle. The coverings for $2 \leq n \leq 12$ and $14 \leq n \leq 18$ appear in [Melissen 1997b], where it is also shown that those with 2, 3, 4, 5, 6, 9, and 10 circles are optimal [Melissen 1997b]. The covering with 13 circles in Figure 2 improves on the covering given in [Melissen 1997b]. The remaining coverings for $19 \leq n \leq 36$ are new.

For each $n \leq 34$ where the optimal covering is not known we applied the algorithm in Section 2 with independent random initial configurations (with uniform distribution of the circle centers within the triangle) until the best covering had been found 80 times. This required 300000 initial configurations for $n = 18$. For other values of n, fewer configurations sufficed (computation times for one initial configuration of course increase when n is increased).

For $n = 35$ the algorithm was run 50000 times, which produced the best covering 3 times. It seems that over a million initial configurations would be needed to find 80 times the covering, which is beyond our computational resources. However, since the structure of the covering with 35 circles is very similar to that with 20 circles (and resembles those of 14 and 27), we think it is probable that the covering with 35 circles is optimal.

Only 10000 runs were performed for $n = 36$, but since 36 is a triangular number it seems probable that the obvious covering is the best possible. No searches were performed for $n > 36$, although—with smaller probability of finding true global optima—the algorithm can be used also for larger n. The computer search for the coverings in this paper took a total of about two months of CPU time on a current Pentium PC.

Table 1 shows some data relative to the conjecturally optimal coverings of Figures 1–4: the radius of the circles, the radius normalized with respect to coverings by triangular numbers of circles, and the symmetry group.

When n is a triangular number $n = k(k+1)/2$, where $k = 1, 2, \ldots$, it seems very probable that the obvious covering — a piece of the hexagonal lattice in the plane — is the best possible; see also [Melissen 1997b]. However, proofs for $n = 3$, 6, and 10 in [Melissen 1997b] cannot be generalized for triangular

n	radius	norm. rad.	G	n	radius	norm. rad.	G
1	0.5773502691896257645	1	D_3	19	0.1061737927289732618	1.04540	C_1
2	0.5	1.35234	D_1	20	0.1032272183417310354	1.04493	D_1
3	0.2886751345948128823	1	D_3	21	0.0962250448649376274	1	D_3
4	0.2679491924311227065	1.10098	D_1	22	0.0951772351261450917	1.01418	C_1
5	0.25	1.16981	D_1	23	0.0937742911094478264	1.02338	C_1
6	0.1924500897298752548	1	D_3	24	0.0923541375945022204	1.03115	D_1
7	0.1852510855786008545	1.05080	C_1	25	0.0906182448311340175	1.03414	C_1
8	0.1769926664029649641	1.08250	C_1	26	0.0887829248953373781	1.03467	D_1
9	0.1666666666666666667	1.08888	C_3	27	0.0868913397937031505	1.03325	C_1
10	0.1443375672974064411	1	D_3	28	0.0824786098842322521	1	D_3
11	0.1410544578570137366	1.03027	C_1	29	0.0818048133956910115	1.01056	C_1
12	0.1373236156889236662	1.05236	C_1	30	0.0808828500258641436	1.01737	C_1
13	0.1326643857765088351	1.06239	C_1	31	0.0798972448089536737	1.02265	C_1
14	0.1275163863998600644	1.06348	C_1	32	0.0788506226168764215	1.02643	C_1
15	0.1154700538379251529	1	D_3	33	0.0776371221483728244	1.02728	C_1
16	0.1137125784440782042	1.02002	C_1	34	0.0763874538343494465	1.02688	C_1
17	0.1113943099632405880	1.03269	D_1	35	0.0751604548962267707	1.02603	D_1
18	0.1091089451179961906	1.04333	C_3	36	0.0721687836487032206	1	D_3

TABLE 1. Properties of the coverings. For the (conjecturally) optimal cover by n circles we give the radius of the circles, the normalized radius, and the symmetry group. Normalization means multiplying the radius by the factor $\sqrt{3}\,k$, where $n = k(k+1)/2$; thus the normalized radius is 1 whenever n is a triangular number, and the smaller the normalized radius, the more efficient the cover. The symmetry group G is either C_n, the cyclic group of order n, or D_n, the dihedral group of order $2n$.

numbers $n > 10$. Other infinite families of possibly optimal coverings are not known, although in view of the results in this paper it seems that an infinite family of possibly optimal coverings could also be constructed for

$$n = \frac{k(k+1)}{2} - 1$$

circles when $k = 6, 8, 10, \ldots$ and maybe also when $k = 5, 7, 9, \ldots$. The coverings of these two families would have one loose circle (a circle that is not part of the rigid structure of the covering), which would give infinitely many (possibly) optimal coverings with at least one loose circle. This contrasts with coverings of a square, where not a single conjecturally optimal covering with loose circles is known so far [Nurmela and Östergård 2000]. Note also in Figures 1–4 the orderly arrangement of the circles in the coverings for $n = k(k+1)/2 + 1$, for $k = 2, 3, 4, \ldots$; this is yet another candidate for an infinite family of optimal coverings.

The radius can be calculated symbolically for some of the best known coverings [Melissen 1997b]. However, in addition to the smallest few coverings, the symbolic values can usually be calculated only for very regular coverings.

In the coverings for $n = 2$, 4, 5, and 9 the circle centers that appear to lie on the triangle boundary do indeed coincide with the boundary. However, for $n = 7$, 8, 11, 16, 22, and 29 this does not hold; the centers that seem to lie on the boundary are actually slightly off the boundary line (for example in the covering with 8 circles the distance between the boundary and the center of the leftmost circle is about $4.1 \cdot 10^{-5}$).

In view of our results it seems probable that for optimal coverings of an equilateral triangle the radius is a strictly decreasing function of the number of circles. This is in contrast to the corresponding packing problem; for example, an optimal packing of five equal circles in an equilateral triangle is obtained by removing one of the circles in the optimal packing of six circles [Melissen 1993].

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