Conjugacy Classes of the Hyperelliptic Mapping Class Group of Genus 2 and 3

Kazushi Ahara and Mitsuhiko Takasawa

CONTENTS

We present tables of conjugacy classes of the hyperelliptic mapping class group of genus 2 and 3, and some theorems on the Sp representation, the Jones representation, and Meyer's function.

1. INTRODUCTION

Let Σ_g be a closed Riemann surface of genus g. Let $\mathcal{M}_g$ be the mapping class group of Σ_g, a finitely generated and finitely represented infinite group [Lickorish 1964; Humphries 1979; Wajnryb 1983; Matsumoto 2000]. Let $\sigma \in \mathcal{M}_g$ be the hyperelliptic involution (see Section 2 for definitions), and Δ_g the hyperelliptic mapping class group, that is, the centralizer of σ.

There are several important representations and 1-cocycles of Δ_g. One is the fundamental linear representation or Sp-*representation* of $\mathcal{M}_g$, whose target is the Siegel modular group $\mathrm{Sp}(2g, \mathbb{Z})$. Its kernel is the Torelli group $\mathcal{I}_g$. The second is Meyer's function $\varphi_g : \Delta_g \to \frac{1}{2g+1}\mathbb{Z}$. This map is not a homomorphism, but the coboundary of φ_g is called Meyer's signature cocycle $\tau_g \in Z^2(\mathcal{M}_g, \mathbb{Z})$ [Meyer 1973]. The third is the Jones representation on Δ_g, which arises from a representation of the Hecke algebra corresponding to a rectangular Young diagram [Jones 1987]. It is known how to get an explicit formula of the Jones representation [Kazhdan and Lusztig 1979; Wenzl 1988]. In the case $g = 2$ Jones himself gave an explicit formula. We shall calculate an explicit representation in the case $g = 3$ in the same way as Jones'. In fact, the Jones representation ρ_g is given by maps $\rho_g : \Delta_g \to \widetilde{\mathrm{GL}}_a(\mathbb{Z}) := \mathrm{GL}(a, \mathbb{Z}[q^{1/a}, q^{-1/a}])$, where a is an integer determined by g and $\widetilde{\mathrm{GL}}_a(\mathbb{Z})$ is the set of $a \times a$ matrices with coefficients in the Laurent polynomial ring $\mathbb{Z}[q^{1/a}, q^{-1/a}]$. If $g = 2, 3, 4$ we get $a = 5, 14, 42$

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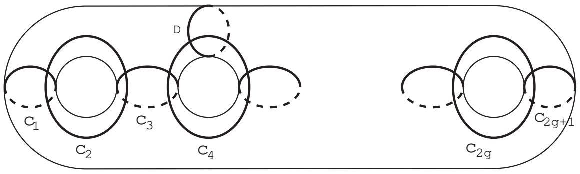
respectively. Finally, any element of $\mathcal{M}_g$ is either periodic, reducible, or pseudo-Anosov. We call this classification the *Thurston type* [Thurston 1988].

This paper gives, for the cases $g = 2, 3$, a table of conjugacy classes of Δ_g up to word length 4. The table contains the Sp-representation, Meyer's function, the Jones representation, and the Thurston type. We also calculate the table of some conjugacy classes of Torelli group $\mathcal{I}_g$.

This paper is organized as follows. Section 2 introduces notations on the Sp-representation and the Meyer's function. Section 3 discusses the Jones representation. Section 4 presents some theorems obtained from the tables. Section 5 lists the conjugacy classes.

2. PRELIMINARIES

Let $\mathcal{M}_g$ be the mapping class group of Σ_g, that is, the group of isotopy classes of orientation preserving diffeomorphisms of Σ_g. Let $C_1, C_2, \ldots, C_{2g+1}, D$ be the simple closed curves on Σ_g as follows:



Let $\zeta_1, \zeta_2, \ldots, \zeta_{2g+1}, \eta$ be the Dehn twists along $C_1, C_2, \ldots, C_{2g+1}, D$; they generate $\mathcal{M}_g$ [Lickorish 1964; Humphries 1979]. The map

$$\sigma := \zeta_1\zeta_2\cdots\zeta_{2g}\zeta_{2g+1}^2\zeta_{2g}\cdots\zeta_2\zeta_1$$

satisfies $\sigma^2 = 1$ and is called the hyperelliptic involution. The centralizer

$$\Delta_g := \{\zeta \in \mathcal{M}_g \mid \zeta\sigma = \sigma\zeta\}$$

of σ is generated by $\zeta_1, \zeta_2, \ldots, \zeta_{2g+1}$: by [Birman and Hilden 1973], we have

$$\Delta_g = \langle \zeta_1, \zeta_2, \ldots, \zeta_{2g+1} \mid \zeta_i\zeta_j = \zeta_j\zeta_i \text{ for } |i-j| \geq 2, \\ \zeta_i\zeta_{i+1}\zeta_i = \zeta_{i+1}\zeta_i\zeta_{i+1},\ \xi^{2g+2} = 1,\ \sigma^2 = 1,\ \sigma\zeta_i = \zeta_i\sigma\rangle,$$

where $\xi = \zeta_1\zeta_2\cdots\zeta_{2g+1}$. For $g = 2$, $\eta = \zeta_5$ and hence $\Delta_2 = \mathcal{M}_2$.

For any $\kappa \in \mathcal{M}_g$, define $C(\kappa) : \mathcal{M}_g \to \mathcal{M}_g$ by

$$C(\kappa)(\zeta) = \kappa\zeta\kappa^{-1}.$$

Proposition 2.1. *Let $\zeta_{a,b} := \prod_{i=a}^{b} \zeta_i$ for $1 \leq a \leq b \leq 2g+1$.*

(1) *For any a, b such that $1 \leq a < b \leq 2g+1$,*

$$C(\zeta_{a,b})(\zeta_i) = \begin{cases} \zeta_i & \text{if } i < a-1, \\ \zeta_{i+1} & \text{if } a \leq i < b, \\ \zeta_i & \text{if } b+1 < i. \end{cases}$$

(2) $C(\zeta_{a,b}^2)(\zeta_b) = \zeta_a$.

(3) *For any a, b such that $1 \leq a < b \leq 2g+1$, let $\eta_{a,b} := \prod_{c=0}^{b-a} \zeta_{a,b-c}$. Then*

$$C(\eta_{a,b})(\zeta_i) = \begin{cases} \zeta_i & \text{if } i < a-1, \\ \zeta_{a+b-i} & \text{if } a \leq i \leq b, \\ \zeta_i & \text{if } b+1 < i. \end{cases}$$

(4) *For any $\kappa, \zeta \in \mathcal{M}_g$, we have $C(\kappa)\circ C(\zeta) = C(\kappa\zeta)$, $C(\kappa)^{-1} = C(\kappa^{-1})$, and $C(\kappa)(\zeta^{-1}) = (C(\kappa)(\zeta))^{-1}$.*

(5) *For any $t_1, t_2, \ldots, t_{2g+1} \in \mathbb{Z}$ and $s \in S_{2g+1}$ (the symmetric group of degree $2g+1$), we have*

$$\zeta_1^{t_1}\zeta_2^{t_2}\cdots\zeta_{2g+1}^{t_{2g+1}} \sim \zeta_{s(1)}^{t_{s(1)}}\zeta_{s(2)}^{t_{s(2)}}\cdots\zeta_{s(2g+1)}^{t_{s(2g+1)}},$$

where $\sim$ means conjugacy in $\mathcal{M}_g$.

Proof. Parts (1), (2), (3) and (5) follow via straightforward computations from the relations of Birman and Hilden. Part (4) is trivial. □

Because $\zeta \in \mathcal{M}_g$ is an isotopy class of homeomorphisms on Σ_g, ζ naturally induces a homomorphism

$$\zeta_* : H_1(\Sigma_g; \mathbb{Z}) \to H_1(\Sigma_g; \mathbb{Z}).$$

Because $\operatorname{rank} H_1 = 2g$, we have $\zeta_* \in M_{2g}(\mathbb{Z})$. It is known that ζ_* is contained in the Siegel modular group

$$\mathrm{Sp}(2g; \mathbb{Z}) := \{X \in M_{2g}(\mathbb{Z}) \mid {}^tXJX = J\}.$$

Here

$$J = \begin{pmatrix} 0 & E_g \\ -E_g & 0 \end{pmatrix}$$

for E_g the $g \times g$ identity matrix. The map $\zeta \mapsto \zeta_*$ is a homomorphism

$$\mathrm{Sp} : \mathcal{M}_g \to \mathrm{Sp}(2g; \mathbb{Z}) : \zeta \mapsto \zeta_*,$$

called the Sp-*representation*.

For $\alpha, \beta \in \mathcal{M}_g$, let $A = \mathrm{Sp}(\alpha)$, $B = \mathrm{Sp}(\beta)$. Define a real vector space

$$V_{A,B} = \{(x, y) \in \mathbb{R}^{2g} \times \mathbb{R}^{2g} \mid \\ (E_{2g} - A^{-1})x + (E_{2g} - B)y = 0\}.$$

Let $\psi_{A,B}$ is a quadratic form on $V_{A,B}$ defined by

$$\psi_{A,B}((x_1, y_1), (x_2, y_2)) := {}^t(x_1 + y_1)J(E_{2g} - B)y_2$$

Then $\psi_{A,B}$ is a symmetric form; we define the *signature cocycle* τ_g by

$$\tau_g(\alpha,\beta) := \operatorname{sgn}\psi_{A,B}.$$

Lemma 2.2. (1) $\tau_g \in Z^2(\mathcal{M}_g;\mathbb{Z})$. *That is, for all* $\alpha,\beta,\gamma \in \mathcal{M}_g$, *we have*

$$\tau_g(\beta,\gamma) - \tau_g(\alpha\beta,\gamma) + \tau_g(\alpha,\beta\gamma) - \tau_g(\alpha,\beta) = 0.$$

(2) $\tau_g(\alpha,1) = \tau_g(\alpha,\alpha^{-1}) = 0$.
(3) $\tau_g(\alpha,\beta) = \tau_g(\beta,\alpha)$.
(4) $\tau_g(\alpha^{-1},\beta^{-1}) = -\tau_g(\alpha,\beta)$.
(5) $\tau_g(\gamma\alpha\gamma^{-1},\gamma\beta\gamma^{-1}) = \tau_g(\alpha,\beta)$.

This follows easily from the definition of τ_g.

We have $(2g+1)\tau_g \in B^2(\Delta_g;\mathbb{Z})$ [Endo 2000]. Therefore there is a function $\varphi_g : \Delta_g \to \frac{1}{2g+1}\mathbb{Z}$ satisfying

$$\delta\varphi_g = \tau_g|\Delta_g.$$

That is, for any $A, B \in \Delta_g$,

$$\varphi_g(B) - \varphi_g(AB) + \varphi_g(A) = \tau_g(A,B).$$

This φ_g is called *Meyer's function* of genus g.

Lemma 2.3. *Let* $\alpha,\beta \in \Delta_g$.

(1) $\varphi_g(1) = 0$.
(2) $\varphi_g(\alpha^{-1}) = -\varphi_g(\alpha)$.
(3) $\varphi_g(\beta\alpha\beta^{-1}) = \varphi_g(\alpha)$

The proof is straightforward from Lemma 2.2.

The next lemma gives the explicit value of Meyer's function.

Lemma 2.4. (1) $\varphi_g(\zeta_i) = (g+1)/(2g+1)$.

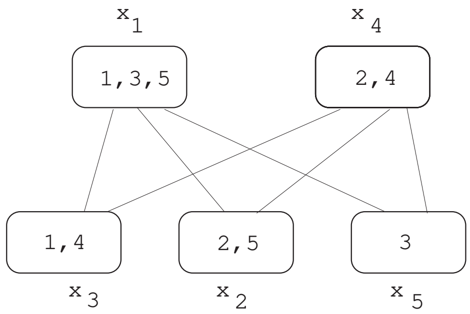
(2) $$\varphi_g(\zeta_{i_1}\cdots\zeta_{i_r}) = \frac{r(g+1)}{2g+1} - \sum_{j=1}^{r-1}\tau_g(\zeta_{i_1}\cdots\zeta_{i_j},\zeta_{i_{j+1}}).$$

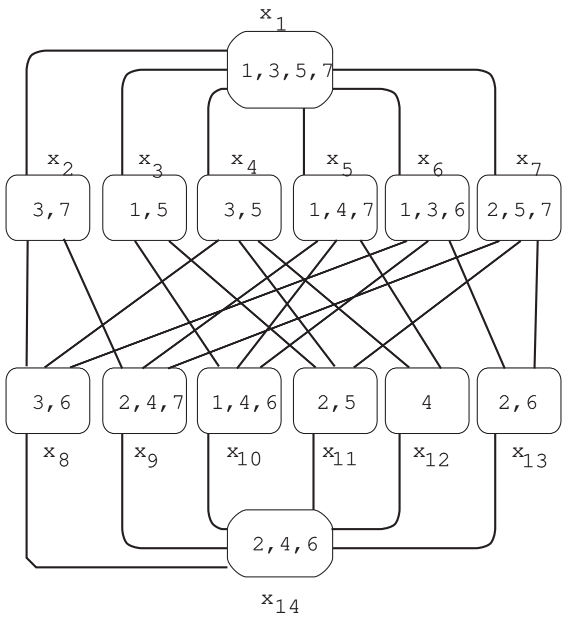
For the proof of (1), see [Endo 2000]. Part (2) follows easily from (1).

3. THE JONES REPRESENTATION

Let $H(q,n)$ be the Hecke algebra of type A_{n-1}. That is, $H(q,n)$ is an algebra over $\mathbb{Z}[q,q^{-1}]$ generated by $g_1, g_2, \ldots, g_{n-1}$ with relations $(g_i - q)(g_i + 1) = 0$ for $i = 1,\ldots,n-1$, $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$ for $i = 1,\ldots,n-2$, and $g_i g_j = g_j g_i$ for $|i-j| \geq 2$. Jones [1987] showed that, if we regard q as a complex number close to 1, the irreducible representations of $H(q,n)$ are in one-to-one correspondence with Young diagrams. He also showed that if we adjust the representation so that $\rho(g_1 g_2 \cdots g_{n-1})^n = 1$, then ρ is a representation of Δ_g if and only if $n = 2g+2$ and the Young diagram is rectangular. Therefore such a representation of Δ_g is called Jones representation and is denoted by ρ_g. Jones representations are in one-to-one correspondence with rectangular Young diagrams of size $2g+2$.

In this section we shall obtain Jones representation explicitly using the Kazhdan–Lusztig formalism of W-graphs in case $g = 2, 3$. Kazhdan and Lusztig [1979] introduced W-graphs that allow us to get a representation of $H(q,n)$. But, as Jones points out, a simple way to go from a Young diagram to a W-graph seems to be lacking in [Kazhdan and Lusztig 1979]. Ochiai and Kako [1995] wrote software to list up all of irreducible representations of $H(q,n)$. Using this software one can get the complete correspondence between Young diagrams and W-graphs, and in particular the W-graphs corresponding to 2×3 and 2×4 rectangular Young diagrams:





Let $\{x_1, \dots, x_a\}$ be vertices of the W-graphs. We assign a subset of $\{1, 2, \dots, 2g+1\}$ to each vertex. Let X_i be the set assigned to x_i.

If two vertices x_j and x_k are connected by an edge, we assign two Laurent polynomials $\mu(j,k)$ and $\mu(k,j)$ to the edge.

Let V be a vector space over $\mathbb{Z}[q, q^{-1}]$ spanned by $\{x_1, \dots, x_a\}$, and define nondegenerate matrices $\rho_g(\zeta_i) : V \to V$ by

$$\rho_g(\zeta_i)(x_j) = \begin{cases} qx_j & \text{if } i \in X_j, \\ -x_j + \sum_{k:(*)} \mu(j,k)x_k & \text{if } i \notin X_j, \end{cases}$$

where $k : (*)$ means that there is an edge $\overline{x_j x_k}$ and that $i \in X_k$.

Kazhdan and Lusztig [1979] give the values of $\mu(j,k)$, but we shall recompute them. This is not hard to do if we use the relations among ζ_i's. First we obtain

$$\mu(j,k)\mu(k,j) = q. \tag{3–1}$$

To illustrate this, take $g = 2$, $j = 2$, $k = 5$. Then

$$\begin{aligned} \rho_2(\zeta_1)\rho_2(\zeta_2)\rho_2(\zeta_1)(x_5) &= \big(1 - \mu(2,5)\mu(5,2)\big)x_5 \\ &\quad + \mu(5,2)\big(-2q + \mu(5,2)\mu(2,5)\big)x_2, \\ \rho_2(\zeta_2)\rho_2(\zeta_1)\rho_2(\zeta_2)(x_5) &= \big(-q^2 + q\mu(5,2)\mu(2,5)\big)x_5 \\ &\quad - q\mu(5,2)x_2. \end{aligned}$$

Thus $\mu(2,5)\mu(5,2) = q$.

Next, if $\{x_i, x_j, x_k, x_l\}$ is a simple closed path of length 4 on the W-graph, then

$$\mu(i,j)\mu(j,k) = \mu(i,l)\mu(l,k). \tag{3–2}$$

Again as an illustration, take $g = 2$ and $(i,j,k,l) = (1,2,5,4)$. Then

$$\begin{aligned} \rho_2(\zeta_2)\rho_2(\zeta_4)(x_1) &= x_1 - \mu(1,4)x_4 - \mu(1,2)x_2 \\ &\quad + \mu(1,2)\mu(2,5)x_5, \\ \rho_2(\zeta_4)\rho_2(\zeta_2)(x_1) &= x_1 - \mu(1,4)x_4 - \mu(1,2)x_2 \\ &\quad + \mu(1,4)\mu(4,5)x_5. \end{aligned}$$

Hence $\mu(1,2)\mu(2,5) = \mu(1,4)\mu(4,5)$.

It is easy to check that for any $\mu(j,k)$ we have

$$(\rho_g(\zeta_i) - qE_a)(\rho_g(\zeta_i) + E_a) = 0$$

for any i. We cannot determine $\mu(j,k)$ uniquely. But any choice must allow us to get the same representation. For example, it is easy to understand the following assignment. In the case $g = 2$, let $A = \{x_1, x_3\}$, and $B = \{x_2, x_4, x_5\}$. We observe that the W-graph is a (complete) dipartite graph. Using this feature, define $\mu(j,k)$ by

$$\mu(j,k) = \begin{cases} q & \text{if } x_j \in A,\ x_k \in B, \\ 1 & \text{if } x_j \in B,\ x_k \in A. \end{cases}$$

Jones [1987] obtains matrices of $\rho_2(\zeta_i)$ in this way. In the same fashion, in the case $g = 3$, let

$$\begin{aligned} A &= \{x_1, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}\}, \\ B &= \{x_2, x_3, x_4, x_5, x_6, x_7, x_{14}\}. \end{aligned}$$

Define $\mu(j,k)$ as above. See formulas for ρ_3 in Section 5C.

4. THEOREMS

Theorem 4.1. *For any $\zeta \in \mathcal{M}_g$,*

$$\det(yE_{2g} - \mathrm{Sp}(\zeta)) = \det(yE_{2g} - \mathrm{Sp}(\zeta^{-1})).$$

Proof. For $\zeta \in \mathcal{M}_g$, we have $J\,\mathrm{Sp}(\zeta)J^{-1} = {}^t\,\mathrm{Sp}(\zeta^{-1})$. Therefore

$$\begin{aligned} \det(yE_{2g} - \mathrm{Sp}(\zeta)) &= \det(yE_{2g} - {}^t\,\mathrm{Sp}(\zeta^{-1})) \\ &= \det(yE_{2g} - \mathrm{Sp}(\zeta^{-1})). \qquad \square \end{aligned}$$

The following corollary is equivalent to this theorem.

Corollary 4.2. *For $\zeta \in \mathcal{M}_g$, suppose that the characteristic function of $\mathrm{Sp}(\zeta)$ is given by*

$$\det(yE_{2g} - \mathrm{Sp}(\zeta)) = \sum_{i=0}^{2g} s_i y^i.$$

Then $s_i = s_{2g-i}$.

We now show that the characteristic function of the Jones representation also has a symmetry as that of the Sp-representation.

Theorem 4.3. *Let $g = 2$, 3 or 4. For any $\zeta \in \Delta_g$, if $J(\zeta)$ is given by $J(\zeta)(y,q) := \det(yE_a - \rho_g(\zeta))$, then*

$$J(\zeta)(y,q) = J(\zeta^{-1})(y, q^{-1}),$$

where $a = a(g)$ is the size of the Jones representation of genus g.

Proof. Define $\bar\rho_g : \Delta_g \to \widetilde{\mathrm{GL}}_a(\mathbb{Z})$ by

$$\bar\rho_g(\zeta) := {}^t(\overline{\rho_g(\zeta)})^{-1}.$$

Here $X \mapsto \bar X : \widetilde{\mathrm{GL}}_a(\mathbb{Z}) \to \widetilde{\mathrm{GL}}_a(\mathbb{Z})$ is an automorphism induced from a map $q \mapsto q^{-1}$. Clearly $\bar\rho_g$ is an irreducible representation of Δ_g. In fact, $\bar\rho_g$ is a representation because

$$\begin{aligned} \bar\rho_g(\zeta\eta) &= {}^t(\overline{\rho_g(\zeta\eta)})^{-1} \\ &= {}^t(\overline{\rho_g(\zeta)})^{-1}\,{}^t(\overline{\rho_g(\eta)})^{-1} = \bar\rho_g(\zeta)\bar\rho_g(\eta). \end{aligned}$$

Irreducibility can be shown easily. Jones [1987] has shown that the simple $H(q,n)$ modules are in one-to-one correspondence with Young diagrams (if q is close to 1), and that it defines a representation of Δ_g if and only if the Young diagram is rectangular. It follows that ρ_g and $\bar{\rho}_g$ are the same representation. That is, there is a nonsingular matrix $P \in \widetilde{\mathrm{GL}}_a(\mathbb{Z})$ such that P has an inverse matrix P^{-1} with $\det(P)P^{-1} \in \widetilde{\mathrm{GL}}_a(\mathbb{Z})$, and $P\rho_g(\zeta)P^{-1} = \bar{\rho}_g(\zeta)$ for any ζ. Therefore

$$\begin{aligned} J(\zeta)(y,q) &= \det(yE_a - \rho_g(\zeta)) = \det(yE_a - \bar{\rho}(\zeta)) \\ &= \det(yE_a - \overline{\rho_g(\zeta)}^{-1}) = J(\zeta^{-1})(y,q^{-1}). \quad \square \end{aligned}$$

In the case $g = 2$, we can get P explicitly. It is easily shown that such P is determined uniquely up to constant multiplication and equals

$$\begin{pmatrix} (q+1)^2 & -q(q+1) & 2q & -q(q+1) & -q(q+1) \\ -q-1 & q^2+q+1 & -q-1 & q & q \\ 2q & -q(q+1) & (q+1)^2 & -q(q+1) & -q(q+1) \\ -q-1 & q & -q-1 & q^2+q+1 & q \\ -q-1 & q & -q-1 & q & q^2+q+1 \end{pmatrix}.$$

Remark. The target space of the Sp-representation is the Siegel modular group $\mathrm{Sp}(2g,\mathbb{Z})$. This allows us to show that the target of the Jones representation is contained in an extension of the Siegel modular group. That is, let

$$\mathrm{Sp}(q,g) := \{A \in \widetilde{\mathrm{GL}}_a(\mathbb{Z}) \mid {}^t\overline{A}PA = P\}.$$

It is easy to show that Theorem 4.3 and the following corollary are equivalent. Remark that $\det(\rho_g(\zeta_i)) = -1$ for $g = 2,3$. See the formulas of ρ_g in [Jones 1987] and Section 5C.

Corollary 4.4. *Let $g = 2,3$. For $\zeta \in \Delta_g$, suppose that the characteristic function of $\rho_g(\zeta)$ is given by*

$$\det(yE_a - \rho_g(\zeta)) = \sum_{i=0}^{a} J_i(q)y^i.$$

Here $a = a(g)$ is the size of the Jones representation and $J_i(q)$ is a Laurent polynomial of $q^{1/a}$. Then

$$J_i(q) = (-1)^{a\varepsilon(\zeta)} J_{a-i}(q^{-1}),$$

where $\varepsilon(\zeta)$ is the parity of the word length of ζ.

There is a relationship between the Jones representation and Meyer's function:

Theorem 4.5. *If $g = 2$,*

$$q^{-\varphi_g(\zeta)}\rho_g(\zeta) \in \mathrm{GL}(a, \mathbb{Z}[q,q^{-1}]).$$

Proof. For any $A, B \in \Delta_g$, we have $\varphi_g(B) - \varphi_g(AB) + \varphi_g(A) = \tau_g(A,B)$ by definition. Therefore φ_g mod. $\mathbb{Z}$ is a homomorphism. It follows that it is sufficient to show Theorem 4.5 for generators ζ_i. We do it by a straightforward calculation in the case $g = 2$. $\square$

Remark. For higher genus we don't have a similar theorem. This theorem implies that the abelianization of Δ_2 contains $\mathbb{Z}/5\mathbb{Z}$. (If fact, $\Delta_2^{\text{abel}} = \mathbb{Z}/10\mathbb{Z}$.) Therefore Theorem 4.5 asserts nothing new. But it remains an open problem whether or not Meyer's function is wholly contained in the Jones representation.

We now consider a modified characteristic function of the Jones representation. Suppose $g = 2$ and set

$$\tilde{J}(\zeta)(y,q) := \det(yq^{\varphi_g(\zeta)}E_a - \rho_g(\zeta)) \in \mathbb{Z}[y,q,q^{-1}].$$

Corollary 4.6. $\tilde{J}(\zeta)(y,q) = \tilde{J}(\zeta^{-1})(y,q^{-1})$.

This follows from Lemma 4.4 and Lemma 2.3(2).

Finally, let $\mathrm{inv} : \Delta_g \to \Delta_g$ be a map defined by

$$\mathrm{inv}(\zeta_{i_1}\zeta_{i_2}\cdots\zeta_{i_n}) := \zeta_{i_n}\cdots\zeta_{i_2}\zeta_{i_1}.$$

This map is well defined because Birman–Hilden relations of Δ_g are invariant under inv. (Note that $\xi = \zeta_1\cdots\zeta_{2g+1}$ is conjugate to $\xi' = \mathrm{inv}(\xi)$ via $C(\eta_{1,2g+1})$. See Proposition 2.1(3).)

Theorem 4.7. *If $g = 2,3,4$, then*

$$J(\zeta)(y,q) = J(\mathrm{inv}(\zeta))(y,q)$$

for any $\zeta \in \Delta_g$.

Proof. Define $\hat{\rho}_g : \Delta_g \to \widetilde{M}_a(\mathbb{Z})$ by

$$\hat{\rho}_g(\zeta) := {}^t(\rho_g(\mathrm{inv}(\zeta))).$$

It is easy to show that $\hat{\rho}_g$ is also a representation of Δ_g and $H(q,2g+1)$, and it is irreducible. We can conclude that $J(\zeta) = J(\mathrm{inv}(\zeta))$ in the same way as the proof of Theorem 4.3. $\square$

Remark. It is an open problem whether the kernel of the Jones representation is generated by only the hyperelliptic involution σ. Using the preceding theorem, we cannot answer this question straightforward but we can have an evidence such that there are at least two conjugacy classes of Δ_g such that they cannot distinguish by $J(\zeta)$. For example, $g = 2$ and let

$$\zeta_6 = (\zeta_1\zeta_2)^6,$$
$$\eta_1 = \zeta_6\zeta_3\zeta_2^{-1}\zeta_3^2\zeta_6^{-1}\zeta_3^{-2}\zeta_2\zeta_3^{-1},$$
$$\eta_2 = \zeta_6\zeta_3^2\zeta_2^{-1}\zeta_3\zeta_6^{-1}\zeta_3^{-1}\zeta_2\zeta_3^{-2}.$$

Then we can check that $J(\eta_1) = J(\eta_2)$ but η_1 and η_2 are not conjugate in Δ_g, because

$$\{A \in \widetilde{\mathrm{GL}}_5(\mathbb{Z}) \mid A\eta_1 = \eta_2 A\} \cap \mathrm{Sp}(q,2) = \varnothing.$$

5. TABLES

5A. Conjugacy Classes for $g = 2$

Table 1 gives the word expression, Meyer's function, Sp-representation, and modified Jones representation for all conjugacy classes up to word length 4. In the first column we have elements of Δ_2. As an example of how to read the word expression, $\{1, -2, 3\}$ means $\zeta_1\zeta_2^{-1}\zeta_3$.

The characteristic function $I(\zeta)$ of the Sp-representation is defined by

$$I(\zeta) = \det(zE_4 - \mathrm{Sp}(\zeta))$$

for $\zeta \in \Delta_2 = \mathcal{M}_2$. It is easy to show that

$$I(\zeta) = y^4 + i_1(\zeta)y^3 + i_2(\zeta)y^2 + i_1(\zeta)y + 1.$$

We have $i(\zeta)$ in our table.

The characteristic function $J(\zeta)$ of the Jones representation ρ_2 is defined by

$$J(\zeta) = \det(yE_5 - \rho_2(\zeta))$$

for $\zeta \in \Delta_2 = \mathcal{M}_2$. Here ρ_2 was already given by Jones [1987]. From Corollary 4.4,

$$J(\zeta) = y^5 + \varepsilon j_1(q)y^4 + j_2(q)y^3 + \varepsilon j_2(q^{-1})y^2 + j_1(q^{-1})y + \varepsilon,$$

where $\varepsilon = 1$ if the length of the word is even and $\varepsilon = -1$ if the length of the word is odd. In the table, $n(b_0, b_1, b_2, \ldots)$ means $q^n(b_0+b_1q+b_2q^2+\cdots)$. In the fourth column, we have the Thurston type of each conjugacy classes. We can get the invariant train track (and hence Thurston type) for all of elements of $\mathcal{M}_g$ using the algorithm from [1992].

In the table we omit the data for inverses. For example, if we have the data of ζ_1^3 we omit those of ζ_1^{-3}. The data for inverse elements are obtained via the formulas

$$\varphi_2(\zeta^{-1}) = -\varphi_2(\zeta),$$
$$i_1(\zeta^{-1}) = i_1(\zeta),$$
$$i_2(\zeta^{-1}) = i_2(\zeta),$$
$$j_1(\zeta^{-1})(q) = j_1(\zeta)(q^{-1}),$$
$$j_2(\zeta^{-1})(q) = j_2(\zeta)(q^{-1}).$$

ζ	φ_2	i_1	i_2	type	$j_1(q)$	$j_2(q)$
$\{1\}$	$3/5$	-4	6	reducible	$-2/5(-3,2)$	$-4/5(3,-6,1)$
$\{1,1\}$	$1/5$	-4	6	reducible	$-4/5(-3,0,-2)$	$-8/5(3,0,6,0,1)$
$\{1,2\}$	$6/5$	-3	4	reducible	$-4/5(-1,2)$	$-3/5(-2,3)$
$\{1,-2\}$	0	-5	8	reducible	$-1(2,-3,2)$	$-2(1,-4,7,-4,1)$
$\{1,3\}$	$6/5$	-4	6	reducible	$-4/5(-2,2,-1)$	$-8/5(1,-4,3,-2)$
$\{1,-3\}$	0	-4	6	reducible	$-1(1,-3,1)$	$-1(-3,4,-3)$
$\{1,1,1\}$	$-1/5$	-4	6	reducible	$-6/5(-3,0,0,2)$	$-12/5(3,0,0,-6,0,0,1)$
$\{1,1,2\}$	$4/5$	-2	2	reducible	$-6/5(-1)$	$3/5(-2)$
$\{1,1,-2\}$	$-2/5$	-6	10	reducible	$-7/5(2,-3,2,-2)$	$-14/5(1,-4,5,-8,5,-2,1)$
$\{1,1,3\}$	$4/5$	-4	6	reducible	$-6/5(-2,1,-1,1)$	$-12/5(1,-2,2,-3,1,-1)$
$\{1,1,-3\}$	$-2/5$	-4	6	reducible	$-7/5(1,-2,1,-1)$	$-9/5(-2,2,-3,2,-1)$
$\{1,2,3\}$	$9/5$	-2	2	reducible	$-1/5(1)$	$-2/5(1,-1)$
$\{1,2,-3\}$	$3/5$	-4	6	reducible	$-7/5(1,-2,2)$	$-9/5(-1,3,-4,2)$
$\{1,2,4\}$	$9/5$	-3	4	reducible	$-6/5(-1,1,-1)$	$-7/5(-1,2,-1,1)$
$\{1,2,-4\}$	$3/5$	-3	4	reducible	$-2/5(-2,1)$	$-4/5(2,-2,1)$
$\{1,-2,3\}$	$3/5$	-6	10	reducible	$-7/5(2,-3,3,-1)$	$-14/5(1,-3,7,-8,5,-2)$

TABLE 1. Word expression, Meyer's function, Sp-representation, and modified Jones representation for conjugacy classes up to word length 4, when $g = 2$. See Section 5A.

ζ	φ_2	i_1	i_2	type	$j_1(q)$	$j_2(q)$
$\{1,-2,4\}$	$3/5$	-5	8	reducible	$-7/5(1,-3,2,-1)$	$-9/5(-2,5,-5,4,-1)$
$\{1,3,5\}$	$4/5$	-4	6	reducible	$-6/5(-1,3,0,1)$	$-7/5(-3,3,-1,3)$
$\{1,3,-5\}$	$-2/5$	-4	6	reducible	$-7/5(1,-1,3)$	$-9/5(-1,3,-3,3)$
$\{1,1,1,1\}$	$-3/5$	-4	6	reducible	$-8/5(-3,0,0,0,-2)$	$-16/5(3,0,0,0,6,0,0,0,1)$
$\{1,1,1,2\}$	$2/5$	-1	0	reducible	$-8/5(-1,0,2)$	$-6/5(-2,0,3)$
$\{1,1,1,-2\}$	$-4/5$	-7	12	reducible	$-9/5(2,-3,2,-2,2)$	$-18/5(1,-4,5,-6,9,-6,3,-2,1)$
$\{1,1,1,3\}$	$2/5$	-4	6	reducible	$-8/5(-2,1,0,1,-1)$	$-16/5(1,-2,0,-2,3,-1,0,-1)$
$\{1,1,1,-3\}$	$-4/5$	-4	6	reducible	$-9/5(1,-2,0,-1,1)$	$-13/5(-2,1,-1,3,-2,0,-1)$
$\{1,1,2,2\}$	$2/5$	0	-2	reducible	$-8/5(-1,-2,0,-2)$	$-11/5(2,1,2,4,0,1)$
$\{1,1,2,3\}$	$7/5$	-1	0	reducible	$2/5(1)$	$4/5(1)$
$\{1,1,2,-3\}$	$1/5$	-3	4	reducible	$-9/5(1,-1,1)$	$-8/5(1,-1,2,-1)$
$\{1,1,2,4\}$	$7/5$	-2	2	reducible	$-8/5(-1)$	$-1/5(-1,0,-1)$
$\{1,1,2,-4\}$	$1/5$	-2	2	reducible	$-4/5(-1)$	$-3/5(-1,0,-1)$
$\{1,1,-2,-2\}$	0	-8	14	reducible	$-2(-2,2,-5,2,-2)$	$-4(1,-2,7,-8,14,-8,7,-2,1)$
$\{1,1,-2,3\}$	$1/5$	-7	12	reducible	$-9/5(2,-3,3,-2,1)$	$-18/5(1,-3,6,-8,9,-6,3,-1)$
$\{1,1,-2,-3\}$	-1	-5	8	reducible	$-1(2,-2,2,-1)$	$-2(2,-3,5,-4,2,-1)$
$\{1,1,-2,4\}$	$1/5$	-6	10	reducible	$-9/5(1,-3,2,-2,1)$	$-13/5(-2,4,-6,6,-5,2,-1)$
$\{1,1,-2,-4\}$	-1	-6	10	reducible	$-2(-1,2,-3,2,-1)$	$-3(-1,3,-6,6,-6,3,-1)$
$\{1,1,3,3\}$	$2/5$	-4	6	reducible	$-8/5(-2,0,-2,0,-1)$	$-16/5(1,0,4,0,3,0,2)$
$\{1,1,3,4\}$	$7/5$	-3	4	reducible	$-8/5(-1,1,0,1)$	$-11/5(-1,1,-1,1,0,1)$
$\{1,1,3,-4\}$	$1/5$	-5	8	reducible	$-9/5(1,-2,2,-1,1)$	$-13/5(-1,3,-4,4,-3,2)$
$\{1,1,3,5\}$	$2/5$	-4	6	reducible	$-8/5(-1,2,-1,0,-1)$	$-11/5(-2,2,-2,1,-2,1)$
$\{1,1,3,-5\}$	$-4/5$	-4	6	reducible	$-9/5(1,-1,1,-2)$	$-13/5(-1,1,-3,2,-2,1)$
$\{1,1,-3,-3\}$	0	-4	6	reducible	$-2(-1,0,-3,0,-1)$	$-2(3,0,4,0,3)$
$\{1,1,-3,-4\}$	-1	-3	4	reducible	$-1(1,-1,1)$	$-2(1,-1,1,-1,1)$
$\{1,1,-3,-5\}$	0	-4	6	reducible	$-1(2,-2,0,-1)$	$-2(1,-4,1,-2,2)$
$\{1,2,2,3\}$	$7/5$	0	-2	reducible	$-3/5(-1,1,-1)$	$-6/5(-1,-1,1,-1)$
$\{1,2,2,-3\}$	$1/5$	-4	6	reducible	$-9/5(1,-2,1,-1)$	$-13/5(-1,1,-3,3,-2)$
$\{1,2,3,4\}$	$12/5$	-1	1	period 5	$0(0)$	$0(0)$
$\{1,2,3,-4\}$	$6/5$	-3	3	pseudo-An.	$-4/5(-1,1)$	$-8/5(1,-2,2,-1)$
$\{1,2,3,5\}$	$7/5$	-2	2	reducible	$-3/5(1)$	$-6/5(1,0,0,-1)$
$\{1,2,3,-5\}$	$1/5$	-2	2	reducible	$1/5(1)$	$-3/5(-1,1)$
$\{1,2,-3,4\}$	$6/5$	-5	7	pseudo-An.	$-9/5(1,-2,2,-1)$	$-13/5(-1,3,-5,5,-3,1)$
$\{1,2,-3,-4\}$	0	-3	5	pseudo-An.	$-1(1,-2,1)$	$-1(-1,2,-1)$
$\{1,2,-3,5\}$	$1/5$	-4	6	reducible	$-9/5(1,-1,2,-1)$	$-8/5(2,-3,3,-2)$
$\{1,2,-3,-5\}$	1	-4	6	reducible	$-1(2,-2,1)$	$-2(1,-3,3,-2,1)$
$\{1,2,4,5\}$	$12/5$	-2	3	reducible	$-8/5(-1,0,-1)$	$-6/5(1)$
$\{1,2,4,-5\}$	$6/5$	-4	5	reducible	$-4/5(-2,1,-1)$	$-8/5(2,-3,3,-2,1)$
$\{1,2,-4,-5\}$	0	-2	3	period 3	$0(-2)$	$0(1)$
$\{1,-2,1,-2\}$	0	-9	16	reducible	$-2(-2,4,-3,4,-2)$	$-4(1,-4,8,-12,15,-12,8,-4,1)$
$\{1,-2,-2,3\}$	1	-8	14	reducible	$-2(-2,3,-4,3,-1)$	$-4(1,-3,6,-11,12,-10,5,-2)$
$\{1,-2,3,-2\}$	1	-9	16	reducible	$-2(-2,4,-4,4,-1)$	$-4(1,-4,8,-14,16,-12,8,-2)$
$\{1,-2,3,-4\}$	0	-7	13	pseudo-An.	$-2(-1,3,-4,3,-1)$	$-3(-1,4,-8,10,-8,4,-1)$
$\{1,-2,3,5\}$	$1/5$	-6	10	reducible	$-9/5(1,-3,3,-1,1)$	$-13/5(-1,5,-7,6,-5,2)$
$\{1,-2,3,-5\}$	-1	-6	10	reducible	$-2(-1,2,-3,3)$	$-3(-1,3,-6,7,-6,3)$
$\{1,-2,-3,4\}$	0	-5	9	pseudo-An.	$-1(2,-3,2,-1)$	$-2(1,-4,5,-4,2)$
$\{1,-2,4,-5\}$	0	-6	11	reducible	$-2(-1,2,-4,2,-1)$	$-2(3,-6,7,-6,3)$

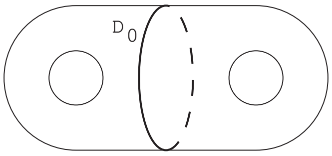
TABLE 1 (continued)

5B. The Torelli Group

The Torelli group $\mathcal{I}_g \subset \mathcal{M}_g$ is the kernel of the Sp-homomorphism. When $g = 2$, it is generated by Dehn twists of all separating simple closed curves:

$$\mathcal{I}_2 = \langle C(\kappa)\zeta_6 \mid \kappa \in \mathcal{M}_g \rangle,$$

where $\zeta_6 = (\zeta_1\zeta_2)^6$ is the Dehn twist of a standard separating simple closed curve D_0 as in the figure.



It is known that $\mathcal{I}_2$ is infinitely generated. The following relations on ζ_6 hold:

Lemma 5.1. (1) $\zeta_6 = (\zeta_4\zeta_5)^6$.
(2) $C(\eta_{1,5})(\zeta_6) = \zeta_6$, $C(\xi^3)(\zeta_6) = \zeta_6$.
(3) $\zeta_i\zeta_6 = \zeta_6\zeta_i$ *for* $i = 1, 2, 4, 5$.

Proof. From the Birman–Hilden relations,

$$\sigma\zeta_1^{-1}\zeta_2^{-1}\zeta_3^{-1}\zeta_4^{-1}\zeta_5^{-1} = \zeta_1\zeta_2\zeta_3\zeta_4\zeta_5,$$
$$\sigma^3(\zeta_1^{-1}\zeta_2^{-1}\zeta_3^{-1}\zeta_4^{-1}\zeta_5^{-1})^3(\zeta_1\zeta_2\zeta_3\zeta_4\zeta_5)^3 = 1,$$
$$(\zeta_1\zeta_2)^3(\zeta_4^{-1}\zeta_5^{-1})^3 = \sigma,$$
$$(\zeta_1\zeta_2)^6 = (\zeta_5\zeta_4)^6 = (\zeta_4\zeta_5)^6.$$

Next, from Lemma 2.1(1),(3),

$$C(\eta_{1,5})(\zeta_1) = \zeta_5, \quad C(\xi^3)(\zeta_1) = \zeta_4,$$
$$C(\eta_{1,5})(\zeta_2) = \zeta_4, \quad C(\xi^3)(\zeta_2) = \zeta_5.$$

The conclusions follow. □

There is exactly one conjugacy class of word length 1 in the Torelli group $\mathcal{I}_2$, namely ζ_6. Conjugacy classes of word length 2 in $\mathcal{I}_2$ are given by

$$D_\pm(\kappa) := \kappa\zeta_6\kappa^{-1}\zeta_6^{\pm 1}$$

for any $\kappa \in \mathcal{M}_2$. Table 2 shows ζ_6 and $D_\pm(\kappa)$ for κ with at most word length 4. From this table we see that the characteristic function of the Jones representation cannot distinguish all conjugacy classes of $\mathcal{M}_2$. We also give some lemmas on $D_\pm(\kappa)$, which follow easily from Lemma 5.1. As before, $\sim$ means conjugacy in $\mathcal{M}_g$.

Lemma 5.2. (1) *For any* κ, $D_\pm(\kappa\eta_{1,5}) \sim D_\pm(\eta_{1,5}\kappa) \sim D_\pm(\kappa)$.
(2) *For any* κ, $D_\pm(\kappa\xi^3) \sim D_\pm(\xi^3\kappa) \sim D_\pm(\kappa)$.
(3) *For any* κ *and* $i = 1, 2, 4, 5$,

$$D_\pm(\kappa\zeta_i) \sim D_\pm(\zeta_i\kappa) \sim D_\pm(\kappa).$$

(4) *For any* κ, $D_+(\kappa^{-1}) \sim D_+(\kappa)$ *and* $D_-(\kappa^{-1}) \sim (D_-(\kappa))^{-1}$.

We give some nontrivial relations. Let $D_\pm(b_1, b_2, \ldots)$ denote $D_\pm(\{b_1, b_2, \ldots\})$.

Lemma 5.3. (1) $D_\pm(3,2,4,3) \sim D_\pm(-3,-2,4,3) \sim D_\pm(-3)$.
(2) $D_\pm(3,2,2,3) \sim D_\pm(-3,-3)$.
(3) $D_\pm(3,2,4,-3) \sim D_\pm(3,2,-4,3) \sim D_\pm(3,-2,4,3) \sim D_\pm(-3,2,4,3)$.

Proof. (1) If $\xi' = \zeta_5\zeta_4\zeta_3\zeta_2\zeta_1$ then $D_\pm(\kappa) \sim D_\pm(\xi'^3\kappa)$. Using Lemma 5.1,

$$\begin{aligned} &D_\pm(3,2,4,3)\\ &\quad\sim D_\pm(\xi'^{-3}\zeta_3\zeta_2\zeta_4\zeta_3)\\ &\quad= D_\pm(-1,-2,-3,-4,-5,-1,-2,-3,\\ &\qquad -4,-5,-1,-2,-3,-4,-5,3,2,4,3)\\ &\quad= D_\pm(-1,-2,-1,-2,-1,-2,-3,-2,-1,-4,-5)\\ &\quad\sim D_\pm(-3). \end{aligned}$$

Because $\xi\xi'^{-2} = \xi^3$,

$$\begin{aligned} &D_\pm(-3,-2,4,3)\\ &\quad\sim D_\pm(\xi\xi'^{-2}\zeta_3\zeta_2\zeta_4\zeta_3)\\ &\quad= D_\pm(1,2,3,4,5,-1,-2,-3,-4,-5,\\ &\qquad -1,-2,-3,-4,-5,-3,-2,4,3)\\ &\quad= D_\pm(1,2,-1,-2,-1,-2,-3,-2,1,-4,-5)\\ &\quad\sim D_\pm(-3). \end{aligned}$$

(2) We have

$$\begin{aligned} C(\xi'^{-3})(\zeta_3) &= C(\xi'^{-1}\xi^2)(\zeta_3)\\ &= \{-1,-2,-3,-4,-5,-4,-3,-2,-1\}. \end{aligned}$$

Therefore

$$\begin{aligned} &D_\pm(3,2,2,3)\\ &\quad\sim D_\pm(C(\xi'^{-3})(\zeta_3\zeta_2^2\zeta_3))\\ &\quad= D_\pm(-1,-2,-3,-4,-5,-4,-3,-2,-1,5,\\ &\qquad 5,-1,-2,-3,-4,-5,-4,-3,-2,-1)\\ &\quad= D_\pm(-1,-2,-3,-3,-2,-1)\\ &\quad\sim D_\pm(-3,-3). \end{aligned}$$

ζ	φ_2	i	type	$j_1(q)$ $j_2(q)$
$\{6\}$	$-4/5$	-2	reducible	$-24/5(-1,0,0,0,0,0,-4)$ $-18/5(4,0,0,0,0,0,6)$
$D_+(3)$	$-8/5$	-2	reducible	$-38/5(-1,2,-1,-2,2,-2,-1,2,-1,0,-3)$ $-36/5(1,0,3,-6,3,6,-6,6,3,-6,3,0,3)$
$D_+(3,3)$	$-8/5$	-2	reducible	$-53/5(-1,1,0,-4,6,-2,-5,8,-5,-2,6,-4,0,-2,-1)$ $-41/5(3,-2,0,12,-18,6,15,-24,15,6,-18,12,0,0,3)$
$D_+(3,-2,3)$	$-8/5$	-2	reducible	$-58/5(1,-3,3,2,-11,14,-4,-13,20,-13,-4,14,-11,2,0,-3,1)$ $-46/5(-3,9,-8,-6,33,-42,12,39,-60,39,12,-42,33,-6,-6,9,-3)$
$D_+(3,3,3)$	$-8/5$	-2	reducible	$-58/5(1,-2,1,2,-7,8,-2,-8,12,-8,-2,8,-7,2,-2,-2,1)$ $-46/5(-3,6,-2,-6,21,-24,6,24,-36,24,6,-24,21,-6,0,6,-3)$
$D_+(3,-2,-2,3)$	$-8/5$	-2	reducible	$-63/5(-1,3,-5,2,7,-18,19,-3,-20,30,-20,-3,19,-18,7,-1,-5,3,-1)$ $-51/5(3,-9,15,-5,-21,54,-57,9,60,-90,60,9,-57,54,-21,-3,15,-9,3)$
$D_+(3,-2,3,3)$	$-8/5$	-2	reducible	$-63/5(-1,4,-7,4,9,-25,27,-5,-28,42,-28,-5,27,-25,9,1,-7,4,-1)$ $-51/5(3,-12,21,-11,-27,75,-81,15,84,-126,84,15,-81,75,-27,-9,21,-12,3)$
$D_+(3,-2,4,3)$	$-8/5$	-2	pseudo-An.	$-48/5(-1,4,-8,8,0,-12,16,-12,0,8,-8,4,-4)$ $-36/5(4,-12,24,-24,0,36,-48,36,0,-24,24,-12,6)$
$D_+(3,-2,-4,3)$	$-8/5$	-2	pseudo-An.	$-63/5(-1,4,-7,4,9,-25,27,-5,-28,42,-28,-5,27,-25,9,1,-7,4,-1)$ $-51/5(3,-12,21,-11,-27,75,-81,15,84,-126,84,15,-81,75,-27,-9,21,-12,3)$
$D_+(3,3,-2,3)$	$-8/5$	-2	reducible	$-63/5(-1,4,-7,4,9,-25,27,-5,-28,42,-28,-5,27,-25,9,1,-7,4,-1)$ $-51/5(3,-12,21,-11,-27,75,-81,15,84,-126,84,15,-81,75,-27,-9,21,-12,3)$
$D_+(3,3,3,3)$	$-8/5$	-2	reducible	$-63/5(-1,2,-2,-1,5,-8,7,0,-9,12,-9,0,7,-8,5,-4,-2,2,-1)$ $-51/5(3,-6,6,4,-15,24,-21,0,27,-36,27,0,-21,24,-15,6,6,-6,3)$
$D_-(3)$	0	-2	reducible	$-6(-1,0,1,-2,1,2,-7,2,1,-2,1,0,-1)$ $-6(3,0,-3,6,-3,-6,16,-6,-3,6,-3,0,3)$
$D_-(3,3)$	0	-2	reducible	$-7(1,-2,0,4,-6,2,5,-13,5,2,-6,4,0,-2,1)$ $-7(-3,6,0,-12,18,-6,-15,34,-15,-6,18,-12,0,6,-3)$
$D_-(3,-2,3)$	0	-2	reducible	$-8(-1,3,-4,-2,11,-14,4,13,-25,13,4,-14,11,-2,-4,3,-1)$ $-8(3,-9,12,6,-33,42,-12,-39,70,-39,-12,42,-33,6,12,-9,3)$
$D_-(3,3,3)$	0	-2	reducible	$-8(-1,2,-2,-2,7,-8,2,8,-17,8,2,-8,7,-2,-2,2,-1)$ $-8(3,-6,6,6,-21,24,-6,-24,46,-24,-6,24,-21,6,6,-6,3)$
$D_-(3,-2,-2,3)$	0	-2	reducible	$-9(1,-3,5,-3,-7,18,-19,3,20,-35,20,3,-19,18,-7,-3,5,-3,1)$ $-9(-3,9,-15,9,21,-54,57,-9,-60,100,-60,-9,57,-54,21,9,-15,9,-3)$
$D_-(3,-2,3,3)$	0	-2	reducible	$-9(1,-4,7,-5,-9,25,-27,5,28,-47,28,5,-27,25,-9,-5,7,-4,1)$ $-9(-3,12,-21,15,27,-75,81,-15,-84,136,-84,-15,81,-75,27,15,-21,12,-3)$
$D_-(3,-2,4,3)$	0	-2	pseudo-An.	$-5(-4,8,-8,0,12,-21,12,0,-8,8,-4)$ $-5(12,-24,24,0,-36,58,-36,0,24,-24,12)$
$D_-(3,-2,-4,3)$	0	-2	pseudo-An.	$-9(1,-4,7,-5,-9,25,-27,5,28,-47,28,5,-27,25,-9,-5,7,-4,1)$ $-9(-3,12,-21,15,27,-75,81,-15,-84,136,-84,-15,81,-75,27,15,-21,12,-3)$
$D_-(3,3,-2,3)$	0	-2	reducible	$-9(1,-4,7,-5,-9,25,-27,5,28,-47,28,5,-27,25,-9,-5,7,-4,1)$ $-9(-3,12,-21,15,27,-75,81,-15,-84,136,-84,-15,81,-75,27,15,-21,12,-3)$
$D_-(3,3,3,3)$	0	-2	reducible	$-9(1,-2,2,0,-5,8,-7,0,9,-17,9,0,-7,8,-5,0,2,-2,1)$ $-9(-3,6,-6,0,15,-24,21,0,-27,46,-27,0,21,-24,15,0,-6,6,-3)$

TABLE 2. Data for ζ_6 and $D_\pm(\kappa)$, for κ of word length at most 4.

(3) We have

$$\begin{aligned}
D_\pm(3,2,4,-3) &= D_\pm(3,2,-3,3,4,-3)\\
&= D_\pm(-2,3,2,-4,3,4) \sim D_\pm(3,2,-4,3),\\
D_\pm(3,2,-4,3) &= D_\pm(3,-4,2,3)\\
&\sim D_\pm(C(\eta_{1,5})(\{3,-4,2,3\})) = D_\pm(3,-2,4,3),\\
D_\pm(3,-2,4,3) &\sim D_\pm(2,3,-2,4,3,-4)\\
&= D_\pm(-3,2,3,-3,4,3) = D_\pm(-3,2,4,3). \qquad \square
\end{aligned}$$

Lemma 5.4. (1) $D_\pm(3,-2,3,3) \not\sim D_\pm(3,3,-2,3)$.
(2) $D_\pm(3,-2,3,3) \not\sim D_\pm(3,-2,-4,3)$.
(3) $D_\pm(3,3,-2,3) \not\sim D_\pm(3,-2,-4,3)$.

Proof. (1) Let

$$\eta_1 = D_+(3,-2,3,3), \quad \eta_2 = D_+(3,3,-2,3).$$

A short calculation shows that the set

$$\{A \in \widetilde{\mathrm{GL}}_5(\mathbb{Z}) \mid A\eta_1 = \eta_2 A\} \cap \mathrm{Sp}(q,2)$$

is empty. Thus η_1 and η_2 are not conjugate.
(2), (3) Using the second author's software (see section on Electronic Availability at the end of this paper), we check that the stretch factors of the invariant train tracks of $D_+(3,-2,3,3)$, $D_+(3,-2,-4,3)$, $D_-(3,-2,3,3)$, and $D_-(3,-2,-4,3)$ are approximately 398, 254, 402, and 258, respectively. Since

the stretch factor is an invariant of conjugation, none of these D's can be conjugate. □

5C. Conjugacy Classes for $g = 3$

Finally we discuss the Jones representation in the case $g = 3$. We consider a 2×4 rectangular Young diagram, obtaining the W-graph shown on page 385, bottom right. Moreover $\rho_3(\zeta_1)$ equals $q^{-5/14}$ times

$$\begin{pmatrix}
q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
q & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
q & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
q & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q & 0 & 0 & 0 & -1
\end{pmatrix}$$

and $\rho_3(\zeta_1\zeta_2\zeta_3\zeta_4\zeta_5\zeta_6\zeta_7)$ equals

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & q_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & q_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & q_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & q_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & q_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & q_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & q_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
q_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

where we have put $q_1 = -q^{1/2}$ and $q_2 = -q^{-1/2}$.

Table 3 shows the results obtained.

ELECTRONIC AVAILABILITY

The software used to compute the tables given in this paper is available at http://www.is.titech.ac.jp/~takasawa/MCG/index.html. Software implementing the train track algorithm can be found at the same address.

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ζ	$\varphi_3(\zeta)$	$I(\zeta)$	$J(\zeta)(q)$
$\{1\}$	$2/7$	$(-1+y)^6$	$-(t^9-y)^5(1+t^5y)^9t^{-45}$
$\{1,1\}$	$1/7$	$(-1+y)^6$	$-(t^{18}-y)^5(-1+t^{10}y)^9t^{-90}$
$\{1,2\}$	$8/7$	$(-1+y)^4(1-y+y^2)$	$(-1+t^{10}y)^4(t^8+t^4y+y^2)^5t^{-40}$
$\{1,-2\}$	0	$(-1+y)^4(1-y+y^2)$	$-(1+y)^4(-t^{14}+y-t^{14}y+t^{28}y-t^{14}y^2)^5t^{-70}$
$\{1,3\}$	$8/7$	$(-1+y)^6$	$(t^{18}-y)^2(t^4+y)^6(-1+t^{10}y)^6t^{-60}$
$\{1,-3\}$	0	$(-1+y)^6$	$(t^{14}-y)^3(1+y)^8(-1+t^{14}y)^3t^{-42}$
$\{1,1,1\}$	$-2/7$	$(-1+y)^6$	$-(t^{27}-y)^5(1+t^{15}y)^9t^{-135}$
$\{1,1,2\}$	$5/7$	$(-1+y)^4(1+y^2)$	$-(t^6-y)^5(t^6+y)^5(1+t^{15}y)^4t^{-60}$
$\{1,1,-2\}$	$-3/7$	$(-1+y)^4(1-4y+y^2)$	$-(-1+t^5y)^4(t^{23}-y+t^{14}y-t^{28}y+t^{42}y-t^{19}y^2)^5t^{-115}$
$\{1,1,3\}$	$5/7$	$(-1+y)^6$	$(t^{27}-y)^2(t^{13}+y)^3(-1+ty)^3(1+t^{15}y)^6t^{-93}$
$\{1,1,-3\}$	$-3/7$	$(-1+y)^6$	$-(t^{23}-y)^3(t^9+y)^2(-1+t^5y)^6(1+t^{19}y)^3t^{-87}$
$\{1,2,3\}$	$12/7$	$(-1+y)^4(1+y^2)$	$(t^6-y)^2(t^6+y)^2(-1+ty)^3(1+t^{15}y)(1+t^2y^2)^3t^{-24}$
$\{1,2,-3\}$	$4/7$	$(-1+y)^6$	$(t^9+y)^2(-1+t^5y)^3 \times (t^{18}+t^9y-t^{23}y+t^{37}y+y^2-t^{14}y^2+t^{28}y^2+t^{19}y^3)^3t^{-72}$
$\{1,2,4\}$	$12/7$	$(-1+y)^4(1-y+y^2)$	$(-1+ty)(1+t^{15}y)^3(t^{26}+t^{13}y+y^2)^2(1-ty+t^2y^2)^3t^{-52}$
$\{1,2,-4\}$	$4/7$	$(-1+y)^4(1-y+y^2)$	$(1+t^5y)^3(-1+t^{19}y)(t^{18}-t^9y+y^2)^3(1+t^5y+t^{10}y^2)^2t^{-54}$
$\{1,-2,3\}$	$4/7$	$(-1+y)^4(1-4y+y^2)$	$(t^9+y)^3(-1+t^5y)(t^{23}-y+t^{14}y-t^{28}y+t^{42}y-t^{19}y^2)^2 \times (t^9+y-2t^{14}y+t^{28}y+t^{19}y^2)^3t^{-100}$
$\{1,-2,4\}$	$4/7$	$(-1+y)^4(1-3y+y^2)$	$(t^9+y)(-1+t^5y)^3(-t^{23}+y-t^{14}y+t^{28}y-t^5y^2)^2 \times (t^9+y-t^{14}y+t^{28}y+t^{19}y^2)^3t^{-82}$
$\{1,3,5\}$	$12/7$	$(-1+y)^6$	$(t^{27}-y)(t^{13}+y)^3(-1+ty)^6(1+t^{15}y)^4t^{-66}$
$\{1,3,-5\}$	$4/7$	$(-1+y)^6$	$-(t^{23}-y)(t^9+y)^5(-1+t^5y)^6(1+t^{19}y)^2t^{-68}$
$\{1,1,1,1\}$	$-5/7$	$(-1+y)^6$	$-(t^{36}-y)^5(-1+t^{20}y)^9t^{-180}$
$\{1,1,1,2\}$	$2/7$	$(-1+y)^4(1+y+y^2)$	$(-1+t^{20}y)^4(t^{16}+t^8y+y^2)^5t^{-80}$
$\{1,1,1,-2\}$	$-6/7$	$(-1+y)^4(1-5y+y^2)$	$-(1+t^{10}y)^4(-t^{32}+y-t^{14}y+t^{28}y-t^{42}y+t^{56}y-t^{24}y^2)^5t^{-160}$
$\{1,1,1,3\}$	$2/7$	$(-1+y)^6$	$(t^{36}-y)^2(t^{22}+y)^3(1+t^6y)^3(-1+t^{20}y)^6t^{-138}$
$\{1,1,1,-3\}$	$-6/7$	$(-1+y)^6$	$-(t^{32}-y)^3(t^{18}+y)^2(1+t^{10}y)^6(-1+t^{24}y)^3t^{-132}$
$\{1,1,2,2\}$	$2/7$	$(-1+y)^4(1+y)^2$	$-(t^{22}-y)^5(-1+t^6y)^5(-1+t^{20}y)^4t^{-110}$
$\{1,1,2,3\}$	$9/7$	$(-1+y)^4(1+y+y^2)$	$(-1+t^{20}y)(t^{16}+t^8y+y^2)^2(-1+t^4y^3)^3t^{-32}$
$\{1,1,2,-3\}$	$1/7$	$(-1+y)^4(1-y+y^2)$	$-(1+t^{10}y)(t^8-t^4y+y^2)^2(-t^{22}-t^{32}y+t^{46}y+y^2-t^{14}y^2-t^{24}y^3)^3t^{-82}$
$\{1,1,2,4\}$	$9/7$	$(-1+y)^4(1+y^2)$	$-(t-y)^3(t^{15}-y)^2(t+y)^3(t^{15}+y)^2(1+t^6y)(-1+t^{20}y)^3t^{-66}$
$\{1,1,2,-4\}$	$1/7$	$(-1+y)^4(1+y^2)$	$-(t^{11}-y)^3(t^{11}+y)^3(-1+t^3y)^2(1+t^3y)^2(-1+t^{10}y)^3(1+t^{24}y)t^{-66}$
$\{1,1,-2,-2\}$	0	$(-1+y)^4(1-6y+y^2)$	$-(-1+y)^4(-t^{28}+y-t^{14}y+2t^{28}y-t^{42}y+t^{56}y-t^{28}y^2)^5t^{-140}$
$\{1,1,-2,3\}$	$1/7$	$(-1+y)^4(1-5y+y^2)$	$-(1+t^{10}y)(-t^{32}+y-t^{14}y+t^{28}y-t^{42}y+t^{56}y-t^{24}y^2)^2 \times (-t^{22}-t^4y+2t^{18}y-2t^{32}y+t^{46}y+y^2-2t^{14}y^2+2t^{28}y^2 -t^{42}y^2-t^{24}y^3)^3t^{-130}$

TABLE 3. Data for the case $g=3$: Meyer's function φ_3 and the polynomials $I(\zeta)=\det(yE_6-\mathrm{Sp}(\zeta))$ and $J(\zeta)=\det(yE_{14}-\rho_3(\zeta))$. We use $t:=q^{1/14}$ for simplicity. The Thurston type is reducible in every case.

ζ	$\varphi_3(\zeta)$	$I(\zeta)$	$J(\zeta)(q)$
$\{1,1,-2,-3\}$	-1	$(-1+y)^4(1-3y+y^2)$	$-(-1+y)$ $\times(t^{14}+y-t^{14}y+t^{28}y+t^{14}y^2)^2-(t^{28}-y+t^{14}y-t^{28}y+t^{42}y-t^{14}y^2$ $+t^{28}y^2-t^{42}y^2+t^{56}y^2-t^{28}y^3)^3t^{-112}$
$\{1,1,-2,4\}$	$1/7$	$(-1+y)^4(1-4y+y^2)$	$-(t^4-y)(1+t^{10}y)^3(t^{32}-y+t^{14}y-t^{28}y+t^{42}y-t^{10}y^2)^2$ $\times(-t^{18}-y+t^{14}y-t^{28}y+t^{42}y+t^{24}y^2)^3t^{-122}$
$\{1,1,-2,-4\}$	-1	$(-1+y)^4(1-4y+y^2)$	$(1+y)^3(-1+t^{14}y)(-t^{28}-y+t^{14}y-t^{28}y+t^{42}y+t^{14}y^2)^3$ $\times(t^{14}-y+t^{14}y-t^{28}y+t^{42}y-t^{28}y^2)^2t^{-112}$
$\{1,1,3,3\}$	$2/7$	$(-1+y)^6$	$(t^8-y)^6(t^{36}-y)^2(-1+t^{20}y)^6t^{-120}$
$\{1,1,3,4\}$	$9/7$	$(-1+y)^4(1-y+y^2)$	$-(t^8-y)(-1+t^{20}y)^3(t^{44}+t^{22}y+y^2)^2(1+t^6y+t^{12}y^2)^3t^{-96}$
$\{1,1,3,-4\}$	$1/7$	$(-1+y)^4(1-3y+y^2)$	$-(t^{18}-y)(-1+t^{10}y)^3(t^{36}+t^4y-t^{18}y+t^{32}y+y^2)^2$ $\times(t^4+y-t^{14}y+t^{28}y+t^{24}y^2)^3t^{-102}$
$\{1,1,3,5\}$	$9/7$	$(-1+y)^6$	$(t^8-y)^3(t^{36}-y)(t^{22}+y)^2(1+t^6y)^4(-1+t^{20}y)^4t^{-104}$
$\{1,1,3,-5\}$	$1/7$	$(-1+y)^6$	$(t^4-y)^3(t^{32}-y)(t^{18}+y)^3(1+t^{10}y)^5(-1+t^{24}y)^2t^{-98}$
$\{1,1,-3,-3\}$	0	$(-1+y)^6$	$-(t^{28}-y)^3(-1+y)^8(-1+t^{28}y)^3t^{-84}$
$\{1,1,-3,-4\}$	-1	$(-1+y)^4(1-y+y^2)$	$(1+y)^3(t^{28}+y)(t^{28}-t^{14}y+y^2)^2(1-t^{14}y+t^{28}y^2)^3t^{-84}$
$\{1,1,-3,-5\}$	-1	$(-1+y)^6$	$(t^{28}-y)^2(-1+y)^5(t^{14}+y)^2(1+t^{14}y)^4(-1+t^{28}y)t^{-84}$
$\{1,2,2,3\}$	$9/7$	$(-1+y)^4(1+y)^2$	$(t^{22}-y)^2(t^8+y)^3(-1+t^6y)^5(1+t^6y)^3(-1+t^{20}y)t^{-68}$
$\{1,2,2,-3\}$	$1/7$	$(-1+y)^6$	$(t^{18}+y)^2(1+t^{10}y)^3$ $\times(-t^{22}+t^{18}y-t^{32}y+t^{46}y+y^2-t^{14}y^2+t^{28}y^2-t^{24}y^3)^3t^{-102}$
$\{1,2,3,4\}$	$16/7$	$(-1+y)^2$ $\times(1-y+y^2-y^3+y^4)$	$(1+t^6y+t^{12}y^2+t^{18}y^3+t^{24}y^4)(t^{12}-y^5)^2t^{-24}$
$\{1,2,3,-4\}$	$8/7$	$(-1+y)^2$ $\times(1-3y+3y^2-3y^3+y^4)$	$(t^{12}+t^8y-t^{22}y+t^{36}y+t^4y^2-t^{18}y^2+t^{32}y^2+y^3-t^{14}y^3+t^{28}y^3+t^{24}y^4)$ $\times(t^{26}+t^{22}y-t^{36}y-t^4y^2+2t^{18}y^2-2t^{32}y^2+t^{46}y^2-y^3+2t^{14}y^3$ $-2t^{28}y^3+t^{42}y^3+t^{10}y^4-t^{24}y^4-t^{20}y^5)^2t^{-64}$
$\{1,2,3,5\}$	$16/7$	$(-1+y)^4(1+y^2)$	$-(t-y)(t^8-y)(t^{15}-y)(t+y)(t^{15}+y)(1+t^6y)^2(-1+t^{20}y)(t^{16}+y^2)$ $\times(1+t^{12}y^2)^2t^{-56}$
$\{1,2,3,-5\}$	$8/7$	$(-1+y)^4(1+y^2)$	$-(t^4-y)^2(t^{11}-y)(t^{11}+y)(-1+t^3y)(1+t^3y)(1+t^{10}y)^2(t^8+y^2)^2$ $\times(1+t^{20}y^2)t^{-46}$
$\{1,2,-3,4\}$	$8/7$	$(-1+y)^2$ $\times(1-5y+7y^2-5y^3+y^4)$	$-(-t^{12}+2t^8y-2t^{22}y+t^{36}y-2t^4y^2+3t^{18}y^2-2t^{32}y^2+y^3-2t^{14}y^3$ $+2t^{28}y^3-t^{24}y^4)$ $\times(-t^{40}-t^{22}y+2t^{36}y-2t^{50}y+t^{64}y-t^4y^2+3t^{18}y^2-5t^{32}y^2+5t^{46}y^2$ $-3t^{60}y^2+t^{74}y^2+y^3-3t^{14}y^3+5t^{28}y^3-5t^{42}y^3+3t^{56}y^3-t^{70}y^3$ $+t^{10}y^4-2t^{24}y^4+2t^{38}y^4-t^{52}y^4-t^{34}y^5)^2t^{-92}$
$\{1,2,-3,-4\}$	0	$(-1+y)^2$ $\times(1-3y+5y^2-3y^3+y^4)$	$(1+y)^2(-t^{14}+y-t^{14}y+t^{28}y-t^{14}y^2+y^3-t^{14}y^3+t^{28}y^3-t^{14}y^4)^2$ $\times(t^{28}-t^{14}y+t^{28}y-t^{42}y+y^2-t^{14}y^2+t^{28}y^2-t^{42}y^2+t^{56}y^2$ $-t^{14}y^3+t^{28}y^3-t^{42}y^3+t^{28}y^4)t^{-56}$
$\{1,2,-3,5\}$	$8/7$	$(-1+y)^6$	$(t^4-y)^2(t^{18}+y)(1+t^{10}y)^2$ $\times(t^{36}+t^{18}y-t^{32}y+t^{46}y+y^2-t^{14}y^2+t^{28}y^2+t^{10}y^3)$ $\times(-t^8+t^4y-t^{18}y+t^{32}y-y^2+t^{14}y^2-t^{28}y^2+t^{24}y^3)^2t^{-78}$
$\{1,2,-3,-5\}$	0	$(-1+y)^6$	$(-1+y)^3(t^{14}+y)(1+t^{14}y)$ $\times(t^{28}+t^{14}y-t^{28}y+t^{42}y+y^2-t^{14}y^2+t^{28}y^2+t^{14}y^3)^2$ $\times(-1+y-t^{14}y+t^{28}y-y^2+t^{14}y^2-t^{28}y^2+t^{28}y^3)t^{-70}$

TABLE 3 (continued)

ζ	$\varphi_3(\zeta)$	$I(\zeta)$	$J(\zeta)(q)$
$\{1,2,4,5\}$	$16/7$	$(-1+y)^2(1-y+y^2)^2$	$(t^8-y)^4(-1+t^{20}y)^2(t^{16}+t^8y+y^2)^2(1+t^6y+t^{12}y^2)^2t^{-64}$
$\{1,2,4,-5\}$	$8/7$	$(-1+y)^2 \times(1-3y+y^2)(1-y+y^2)$	$-(1+t^{10}y)^2(t^8-t^4y+y^2)(-t^4+y-t^{14}y+t^{28}y-t^{24}y^2) \times(t^{36}+t^{18}y-t^{32}y+t^{46}y+y^2-2t^{14}y^2+2t^{28}y^2-2t^{42}y^2+t^{56}y^2 +t^{10}y^3-t^{24}y^3+t^{38}y^3+t^{20}y^4)^2t^{-84}$
$\{1,2,4,6\}$	$16/7$	$(-1+y)^4(1-y+y^2)$	$(1+t^6y)^2(-1+t^{20}y)^2(t^{16}-t^8y+y^2)^2(t^{44}+t^{22}y+y^2) \times(1+t^6y+t^{12}y^2)^2t^{-76}$
$\{1,2,4,-6\}$	$8/7$	$(-1+y)^4(1-y+y^2)$	$-(t^4-y)(1+t^{10}y)^2(-1+t^{24}y)(t^8-t^4y+y^2)^3(t^{36}+t^{18}y+y^2) \times(1+t^{10}y+t^{20}y^2)t^{-64}$
$\{1,2,-4,-5\}$	0	$(-1+y)^2(1-y+y^2)^2$	$(1+y)^6(1-y+y^2)^2(t^{28}-t^{14}y+y^2)(1-t^{14}y+t^{28}y^2)t^{-28}$
$\{1,2,-4,-6\}$	0	$(-1+y)^4(1-y+y^2)$	$(1+y)^2(-1+t^{14}y)^2(1+y+y^2)^2(t^{28}-t^{14}y+y^2)^2 \times(1-t^{14}y+t^{28}y^2)t^{-56}$
$\{1,-2,1,-2\}$	0	$(-1+y)^4(1-7y+y^2)$	$-(-1+y)^4(-t^{28}+y-2t^{14}y+t^{28}y-2t^{42}y+t^{56}y-t^{28}y^2)^5t^{-140}$
$\{1,-2,-2,3\}$	1	$(-1+y)^4(1-6y+y^2)$	$(-1+y)(t^{14}+y)^3(-t^{28}+y-t^{14}y+2t^{28}y-t^{42}y+t^{56}y-t^{28}y^2)^2 \times(-t^{14}-y+2t^{14}y-2t^{28}y+t^{42}y+t^{28}y^2)^3t^{-140}$
$\{1,-2,3,-2\}$	1	$(-1+y)^4(1-7y+y^2)$	$(-1+y)(-t^{28}+y-2t^{14}y+t^{28}y-2t^{42}y+t^{56}y-t^{28}y^2)^2 \times(-t^{28}-2t^{14}y+3t^{28}y-2t^{42}y+t^{56}y-y^2+2t^{14}y^2-3t^{28}y^2 +2t^{42}y^2+t^{28}y^3)^3t^{-140}$
$\{1,-2,3,-4\}$	0	$(-1+y)^2 \times(1-7y+13y^2-7y^3+y^4)$	$(1+y)^2(t^{28}-2t^{14}y+3t^{28}y-2t^{42}y+y^2-3t^{14}y^2+5t^{28}y^2-3t^{42}y^2 +t^{56}y^2-2t^{14}y^3+3t^{28}y^3-2t^{42}y^3+t^{28}y^4) \times(-t^{42}-t^{14}y+3t^{28}y-3t^{42}y+3t^{56}y-t^{70}y+y^2-3t^{14}y^2 +5t^{28}y^2-7t^{42}y^2+5t^{56}y^2-3t^{70}y^2+t^{84}y^2-t^{14}y^3 +3t^{28}y^3-3t^{42}y^3+3t^{56}y^3-t^{70}y^3-t^{42}y^4)^2t^{-112}$
$\{1,-2,3,5\}$	$8/7$	$(-1+y)^4(1-4y+y^2)$	$-(t^4-y)^2(t^{18}+y)(1+t^{10}y)(t^{32}-y+t^{14}y-t^{28}y+t^{42}y-t^{10}y^2) \times(t^{18}+y-2t^{14}y+t^{28}y+t^{10}y^2)(-t^4+y-2t^{14}y+t^{28}y-t^{24}y^2)^2 \times(-t^{18}-y+t^{14}y-t^{28}y+t^{42}y+t^{24}y^2)t^{-102}$
$\{1,-2,3,-5\}$	0	$(-1+y)^4(1-4y+y^2)$	$-(-1+y)^2(t^{14}+y)^2(t^{28}-y+t^{14}y-t^{28}y+t^{42}y-t^{14}y^2) \times(t^{14}+y-2t^{14}y+t^{28}y+t^{14}y^2)^2(1-y+2t^{14}y-t^{28}y+t^{28}y^2) \times(-t^{14}-y+t^{14}y-t^{28}y+t^{42}y+t^{28}y^2)t^{-98}$
$\{1,-2,-3,4\}$	0	$(-1+y)^2 \times(1-5y+9y^2-5y^3+y^4)$	$(-1+y)^2(-t^{28}+y-2t^{14}y+2t^{28}y-2t^{42}y-y^2+2t^{14}y^2-3t^{28}y^2 +2t^{42}y^2-t^{56}y^2-2t^{14}y^3+2t^{28}y^3-2t^{42}y^3+t^{56}y^3-t^{28}y^4)^2 \times(t^{28}+2t^{14}y-2t^{28}y+t^{42}y+y^2-2t^{14}y^2+3t^{28}y^2-2t^{42}y^2 +t^{56}y^2+t^{14}y^3-2t^{28}y^3+2t^{42}y^3+t^{28}y^4)t^{-84}$
$\{1,-2,4,-5\}$	0	$(-1+y)^2(1-3y+y^2)^2$	$(-1+y)^6(t^{14}+y-t^{14}y+t^{28}y+t^{14}y^2)^2 \times(-t^{28}+y-2t^{14}y+t^{28}y-2t^{42}y+t^{56}y-t^{28}y^2)^2t^{-84}$
$\{1,-2,4,6\}$	$8/7$	$(-1+y)^4(1-3y+y^2)$	$(t^4-y)^2(1+t^{10}y)^2(t^{36}-t^4y+t^{18}y-t^{32}y+y^2) \times(t^{18}+y-t^{14}y+t^{28}y+t^{10}y^2)^2(-t^4+y-t^{14}y+t^{28}y-t^{24}y^2)^2 \times t^{-88}$
$\{1,-2,4,-6\}$	0	$(-1+y)^4(1-3y+y^2)$	$-(-1+y)^2(t^{14}+y)(1+t^{14}y)(-t^{28}+y-t^{14}y+t^{28}y-y^2) \times(t^{14}+y-t^{14}y+t^{28}y+t^{14}y^2)^3(1-y+t^{14}y-t^{28}y+t^{28}y^2)t^{-84}$
$\{1,3,5,7\}$	$9/7$	$(-1+y)^6$	$(t^8-y)^6(t^{36}-y)(1+t^6y)^4(-1+t^{20}y)^3t^{-84}$
$\{1,3,5,-7\}$	$1/7$	$(-1+y)^6$	$-(t^4-y)^3(t^{18}+y)^4(1+t^{10}y)^6(-1+t^{24}y)t^{-84}$
$\{1,3,-5,-7\}$	0	$(-1+y)^6$	$-(t^{28}-y)(-1+y)^8(t^{14}+y)^2(1+t^{14}y)^2(-1+t^{28}y)t^{-56}$

TABLE 3 (continued)

Kazushi Ahara, Department of Mathematics, Meiji University, Higashimita 1-1-1, Tama, Kawasaki, Kanagawa, Japan 214-8571 (ahara@math.meiji.ac.jp)

Mitsuhiko Takasawa, Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, 2-12-1 Ookayama, Meguro-ku, Tokyo, Japan 152-8551 (takasawa@is.titech.ac.jp)

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