ELECTRONIC RESEARCH ANNOUNCEMENTS
OF THE AMERICAN MATHEMATICAL SOCIETY
Volume 1, Issue 1, 1995

CROOKED PLANES

TODD A. DRUMM AND WILLIAM M. GOLDMAN

(Communicated by Gregory Margulis)

ABSTRACT. Crooked planes are polyhedra used to construct fundamental polyhedra for discrete groups of Lorentz isometries acting properly on Minkowski (2+1)-space. These fundamental polyhedra are regions bounded by disjoint crooked planes. We develop criteria for the intersection of crooked planes and apply these criteria to proper discontinuity of discrete isometry groups.

1. Introduction

Let Γ be a group of isometries of Minkowski (2+1)-space $\mathbb{E}$ acting freely and properly discontinuously on $\mathbb{E}$. The quotient space $\mathbb{E}/\Gamma$ is complete flat Lorentz space-time. In 1983, Margulis constructed nonamenable subgroups of the Lorentz isometry group which act properly discontinuously on $\mathbb{E}$ ([12],[13]), realizing a suggestion of Milnor [15]. That is, there exist complete flat Lorentz 3-manifolds M whose fundamental groups are nonamenable (that is, not virtually solvable) and these manifolds will be called *Margulis space-times.* By [7] the fundamental group of a Margulis space-time is a free group.

In [3], complete flat Lorentz manifolds with nonabelian free fundamental group were directly constructed using Poincaré's fundamental polyhedron technique ([3], [4]). These examples are diffeomorphic to solid handlebodies. Their fundamental groups are "affine Schottky groups" with fundamental domains derived from fundamental domains of Schottky subgroups of $\mathbf{SO^0}(2,1)$ acting on the hyperbolic plane $\mathbf{H}^2$. These fundamental polyhedra are bounded by certain disjoint polyhedral hypersurfaces in $\mathbb{R}^3$, called *crooked planes.*

Using crooked planes, it was shown in [4] that the underlying linear group of a free group of Lorentz isometries acting freely and properly discontinuously on Minkowski (2+1)-space can be any free discrete subgroup of $\mathbf{SO}(2,1)$.

Conjecture 1. Every Margulis space-time has a fundamental polyhedron whose faces are crooked planes.

The classification of intersections of crooked planes is the beginning of our program to analyze the moduli space of Margulis space-times.

Received by the editors March 11, 1995.
1991 *Mathematics Subject Classification.* 51.
Key words and phrases. Space-times, affine, fundamental polyhedra, crooked planes.
Goldman was partially supported by NSF grant DMS-8902619 and the University of Maryland Institute for Advanced Computer Studies. Drumm was partially supported by an NSF Postdoctoral fellowship, and thanks Swarthmore College's KIVA project for their hospitality.

2. Minkowski $2+1$-space

Define $\mathbb{R}^{2,1}$ to be the 3-dimensional real vector space with the indefinite inner product

$$\mathbb{B}(\mathsf{u},\mathsf{v}) = \mathsf{u}^T \cdot I_L \cdot \mathsf{v},$$

where u and v are written in terms of the usual basis $\{\mathsf{e}_1,\mathsf{e}_2,\mathsf{e}_3\}$ and I_L is the diagonal matrix with entries $+1,+1,-1$. Define the *Lorentzian cross-product* as

$$\mathsf{u} \boxtimes \mathsf{v} = I_L \cdot (\mathsf{u} \times \mathsf{v}),$$

where $\times$ is the usual cross product. Then $\mathbb{B}(\mathsf{x},\mathsf{x}\boxtimes\mathsf{y}) = 0$. Let $\mathbf{O}(2,1)$ be the full group of linear automorphisms of $\mathbb{R}^{2,1}$, with unimodular subgroup $\mathbf{SO}(2,1)$, and identity component $\mathbf{SO}^{\mathbf{0}}(2,1)$.

For a vector v denote its Euclidean norm by $|\mathsf{v}|$, the line containing v by $\mathbb{R}\mathsf{v}$, and the ray containing v by $\mathbb{R}_+\mathsf{v}$. A nonzero vector $\mathsf{v}\in\mathbb{R}^{2,1}$ is called

- *spacelike* if $\mathbb{B}(\mathsf{v},\mathsf{v}) > 0$ (and *unit-spacelike* if $\mathbb{B}(\mathsf{v},\mathsf{v}) = 1$);
- *null* (or *lightlike*) if $\mathbb{B}(\mathsf{v},\mathsf{v}) = 0$;
- *timelike* if $\mathbb{B}(\mathsf{v},\mathsf{v}) < 0$;

The *light cone* $\mathbb{W}$ is the set of all *null* vectors and its upper nappe $\mathbb{W}^+$ is called the *positive time-orientation*. Similarly $\mathbb{U}$ is the set of *timelike* vectors with its upper connected component $\mathbb{U}^+$ defining the positive time orientation.

For any nonzero vector $\mathsf{v}\in\mathbb{R}^{2,1}$, its *Lorentz-perpendicular plane* is

$$\mathcal{P}(\mathsf{v}) = \{\mathsf{u}\in\mathbb{R}^{2,1} \mid \mathbb{B}(\mathsf{u},\mathsf{v}) = 0\}$$

If v is a spacelike vector, $\mathcal{P}(\mathsf{v})\cap\mathbb{W}$ is the union of 2 null lines. There exists a unique pair

$$\mathbf{x}^-(\mathsf{v}),\mathbf{x}^+(\mathsf{v}) \in \mathcal{P}(\mathsf{v})\cap\mathbb{W}^+$$

such that both have Euclidean norm 1 and the ordered triple $(\mathsf{v},\mathbf{x}^-(\mathsf{v}),\mathbf{x}^+(\mathsf{v}))$ is a positively-oriented basis of $\mathbb{R}^{2,1}$.

For a lightlike $\mathsf{w}\in\mathbb{W}^+$ with Euclidean norm 1, the plane $\mathcal{P}(\mathsf{w})$ is the plane tangent to the light cone containing the line $\mathbb{R}\mathsf{w}$. The null line $\mathbb{R}\mathsf{w}$ divides $\mathcal{P}(\mathsf{w})$ into two half-planes with closures denoted $\mathcal{P}^+(\mathsf{w})$ and $\mathcal{P}^-(\mathsf{w})$. Choose $\mathcal{P}^+(\mathsf{w})$ so that $\mathsf{v}\in\mathcal{P}^+(\mathsf{w})$ when $\mathsf{w} = \mathbf{x}^+(\mathsf{v})$. Choose $\mathcal{P}^-(\mathsf{w})$ so that $\mathsf{v}\in\mathcal{P}^-(\mathsf{w})$ when $\mathsf{w} = \mathbf{x}^-(\mathsf{v})$.

Let $\mathbb{E}$ denote *Minkowski (2+1)-space,* the affine space corresponding to $\mathbb{R}^{2,1}$. Minkowski (2+1)-space is a simply connected complete flat Lorentzian manifold of dimension 2+1. For $p,q\in\mathbb{E}$ the vector $\mathsf{v} = q-p$ corresponds to the translation carrying p to q. We also write $q = p+\mathsf{v}$. Identify the vector space of translations of $\mathbb{E}$ with the original inner product space $\mathbb{R}^{2,1}$. Denote the group of isometries of $\mathbb{E}$ by

$$\mathbf{K} = \mathbf{O}(2,1)\ltimes\mathbb{R}^{2,1},$$

and the homomorphism assigning to $h\in\mathbf{K}$ its linear part by:

$$\mathbb{L} : \mathbf{H} \to \mathbf{O}(2,1).$$

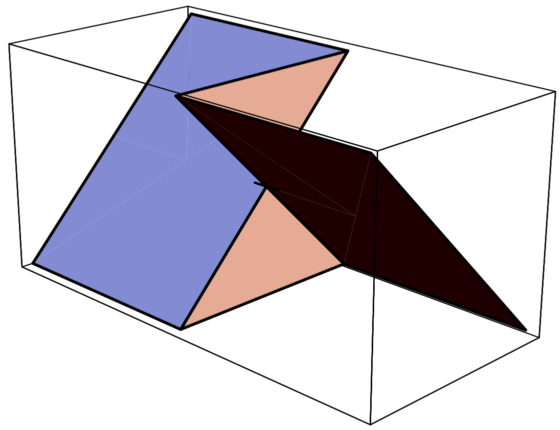

FIGURE 1. A crooked plane

Let $\mathbf{H}$ and $\mathbf{H^0}$ be the subgroups consisting of all $h \in \mathbf{K}$ whose linear parts are $\mathbb{L}(h) \in \mathbf{SO}(2,1)$ and $\mathbb{L}(h) \in \mathbf{SO^0}(2,1)$, respectively. For isometries $h \in \mathbf{H}$ of $\mathbb{E}$, write $h(\mathsf{x}) = g(\mathsf{x}) + \mathsf{v}$ where $g = \mathbb{L}(h) \in \mathbf{SO}(2,1)$. The isometries g and h are called

- *hyperbolic* if g has 3 distinct real eigenvalues;
- *parabolic* if g has 1 as its only eigenvalue;
- *elliptic* if g has imaginary eigenvalues;

If v is spacelike or timelike and $l = p + \mathbb{R}\mathsf{v}$ is a line parallel to v, then ι_l denotes the inversion in line l. If v is spacelike, then $\iota_l \notin \mathbf{H}$.

3. Crooked planes

Our strategy is to derive fundamental polyhedra for actions on Minkowski space from fundamental polygons for the corresponding actions on $\mathbf{H}^2$. A convenient model for $\mathbf{H}^2$ is the hyperboloid defined by $\mathbb{B}\,(\mathsf{v},\mathsf{v}) = -1$, which has an induced Riemannian metric of constant curvature -1. Let $\mathbf{H}^2$ be the component of the hyperboloid lying in $\mathbb{U}^+$. Points of $\mathbf{H}^2$ correspond to timelike lines and geodesics in $\mathbf{H}^2$ to spacelike planes, that is to planes that are Lorentz perpendicular to spacelike vectors. Suppose that $\mathsf{v} \in \mathbb{R}^{2,1}$ is spacelike. Then the plane $\mathcal{P}(\mathsf{v}) \subset \mathbb{R}^{2,1}$ meets $\mathbb{U}$ in timelike lines determining a geodesic in $\mathbf{H}^2$.

Two spacelike planes always intersect in a line. In particular, two ultraparallel geodesics in $\mathbf{H}^2$ do not intersect but the spacelike planes corresponding to the geodesics intersect in a spacelike line. Any translations of these spacelike planes also intersect. We now introduce crooked planes as objects in $\mathbb{E}$ that correspond to geodesics in $\mathbf{H}^2$.

Let $p \in \mathbb{E}$ be a point and $\mathsf{v} \in \mathbb{R}^{2,1}$ a spacelike vector. Define the *positively oriented crooked plane* $\mathcal{C}(\mathsf{v},p) \subset \mathbb{E}$ with *vertex* p and *direction vector* v to be the union of two *wings*

$$\mathcal{W}^+(\mathsf{v},p) = p + \mathcal{P}^+(\mathbf{x}^+(\mathsf{v})), \qquad \mathcal{W}^-(\mathsf{v},p) = p + \mathcal{P}^+(\mathbf{x}^-(\mathsf{v}))$$

and the *stem*

$$\mathcal{S}(\mathsf{v},p) = p + \{\mathsf{x} \in \mathbb{R}^{2,1} \mid \mathbb{B}\,(\mathsf{v},\mathsf{x}) = 0,\ \mathbb{B}\,(\mathsf{x},\mathsf{x}) \le 0\}.$$

Define the *negatively oriented crooked plane* $\mathcal{K}(\mathsf{v},p) \subset \mathbb{E}$ with *vertex* p and *direction vector* v to be the union of two *wings*

$$\mathcal{M}^+(\mathsf{v},p) = p + \mathcal{P}^-(\mathbf{x}^+(\mathsf{v})), \qquad \mathcal{M}^-(\mathsf{v},p) = p + \mathcal{P}^-(\mathbf{x}^-(\mathsf{v}))$$

and the stem $\mathcal{S}(\mathsf{v},p)$ defined above.

For the crooked plane $\mathcal{C}(\mathsf{v},p)$ (or $\mathcal{K}(\mathsf{v},p)$) there exists a unique spacelike line $p + \mathbb{R}\mathsf{v}$ lying on it which will be called the *spine* of the crooked plane. Given a spacelike line ℓ and a point $p \in \ell$, there is a unique positively (or negatively) oriented crooked plane with spine ℓ and vertex p.

We also define the *positively oriented crooked half-space* $\mathcal{H}(\mathsf{v},p)$ and the *negatively oriented crooked half-space* $\mathcal{N}(\mathsf{v},p)$ to be the components of $\mathbb{E} - \mathcal{C}(\mathsf{v},p)$ and $\mathbb{E} - \mathcal{K}(\mathsf{v},p)$, respectively, which contain

$$p + \{\mathsf{u} \in \mathbb{W}^+ \mid \mathbb{B}\,(\mathsf{v},\mathsf{u}) > 0\}.$$

4. Intersections of crooked planes

Since two nonparallel half planes of differing orientation meet in a ray, two oppositely oriented crooked planes always intersect:

Theorem 2 ([9])**.** *Let v_1 and v_2 be linearly independent spacelike vectors and let $p_1, p_2 \in \mathbb{E}$. Then $\mathcal{C}(\mathsf{v}_1,p_1) \cap \mathcal{K}(\mathsf{v}_2,p_2)$ is nonempty.*

To examine the intersection of two crooked planes of the same orientation we introduce the following terminology: Vectors $\mathsf{v}_1, \mathsf{v}_2$ are said to be *ultraparallel*, *asymptotic*, or *crossing* if $\mathcal{P}(\mathsf{v}_1)$ and $\mathcal{P}(\mathsf{v}_2)$ correspond to geodesics in $\mathbf{H}^2$ which are ultraparallel, asymptotic, or crossing, respectively (compare [9]).

For crossing spacelike vectors v_1 and v_2, the intersection of the stems $\mathcal{S}(\mathsf{v}_1,p_1)$ and $\mathcal{S}(\mathsf{v}_2,p_2)$ contains two rays. Since $\mathcal{C}(\mathsf{v}_2,p_2)$ and $\mathcal{K}(\mathsf{v}_2,p_2)$ share a stem,

Theorem 3 ([9])**.** *Let v_1 and v_2 be crossing spacelike vectors. Then the intersection of any two crooked planes with spines parallel to v_1 and v_2 is nonempty.*

Now consider crooked planes of the same orientation whose spines are either ultraparallel or asymptotic. The following terminology will be useful since $\mathcal{C}(\mathsf{v},p) = \mathcal{C}(-\mathsf{v},p)$ (and $\mathcal{K}(\mathsf{v},p) = \mathcal{K}(-\mathsf{v},p)$). Spacelike vectors $\mathsf{v}_1, \mathsf{v}_2, \dots, \mathsf{v}_n$ are said to be *consistently oriented* if $\mathbb{B}\,(\mathsf{v}_i,\mathsf{v}_j) \le 0$ and $\mathbb{B}\,(\mathsf{v}_i,\mathbf{x}^\pm(\mathsf{v}_j)) \le 0$ for $i \ne j$. This means that the half-planes $H(\mathsf{v}_i), H(\mathsf{v}_j) \in \mathbf{H}^2$ corresponding to $\mathcal{H}(\mathsf{v}_i,p)$ and $\mathcal{H}(\mathsf{v}_j,p)$ do not intersect.

Theorem 4 ([9])**.** *Let v_1 and v_2 be consistently oriented, ultraparallel, unit-spacelike vectors and $p_1, p_2 \in \mathbb{E}$. The positively oriented crooked planes $\mathcal{C}(\mathsf{v}_1,p_1)$ and $\mathcal{C}(\mathsf{v}_2,p_2)$ are disjoint if and only if*

$$\mathbb{B}\,(p_2 - p_1, \mathsf{v}_1 \boxtimes \mathsf{v}_2) > |\mathbb{B}\,(p_2 - p_1, \mathsf{v}_2)| + |\mathbb{B}\,(p_2 - p_1, \mathsf{v}_1)|. \tag{4.0.1}$$

The negatively oriented crooked planes $\mathcal{K}(\mathsf{v}_1,p_1)$ and $\mathcal{K}(\mathsf{v}_2,p_2)$ are disjoint if and only if

$$\mathbb{B}\,(p_2 - p_1, \mathsf{v}_1 \boxtimes \mathsf{v}_2) < -(|\mathbb{B}\,(p_2 - p_1, \mathsf{v}_2)| + |\mathbb{B}\,(p_2 - p_1, \mathsf{v}_1)|). \tag{4.0.2}$$

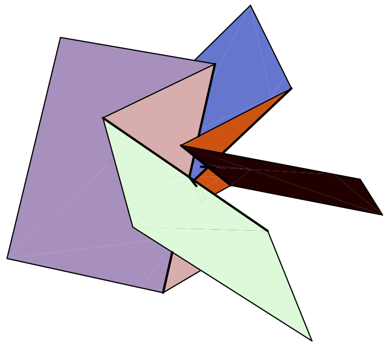

FIGURE 2. Disjoint ultraparallel crooked planes

These conditions follow from explicit description of the intersections of the components of crooked planes. We rephrase them in terms of the displacement vector $\mathsf{w} = p_2 - p_1$ between their vertices. Fix spacelike vectors v_1 and v_2 and a point $p \in \mathbb{E}$. Consider crooked planes with vertex $p_1 = p$ and $p_2 = p + \mathsf{w}$. For any $\mathsf{w} \in \mathbb{R}^{2,1}$, the stems $\mathcal{S}(\mathsf{v}_1, p)$ and and $\mathcal{S}(\mathsf{v}_2, p + \mathsf{w})$ are disjoint if and only if

$$|\mathbb{B}\,(p_2 - p_1, \mathsf{v}_1 \boxtimes \mathsf{v}_2)| > |\mathbb{B}\,(p_2 - p_1, \mathsf{v}_2)| + |\mathbb{B}\,(p_2 - p_1, \mathsf{v}_1)|.$$

The set

$$A = \left\{\mathsf{w} \in \mathbb{R}^{2,1} \mid \mathcal{S}(\mathsf{v}_1, p) \cap \mathcal{S}(\mathsf{v}_2, p + \mathsf{w}) = \emptyset\right\}$$

is a disjoint union of the two open pyramids

$$A_+ = \left\{\mathsf{w} \in \mathbb{R}^{2,1} \mid \mathcal{C}(\mathsf{v}_1, p) \cap \mathcal{C}(\mathsf{v}_2, p + \mathsf{w}) = \emptyset\right\}$$

and

$$A_- = \left\{\mathsf{w} \in \mathbb{R}^{2,1} \mid \mathcal{K}(\mathsf{v}_1, p) \cap \mathcal{K}(\mathsf{v}_2, p + \mathsf{w}) = \emptyset\right\}$$

Theorem 5 ([9]). *Let v_1 and v_2 be consistently oriented, asymptotic, unit-spacelike vectors. Let $\mathcal{C}_i = \mathcal{C}(\mathsf{v}_i, p_i)$ and $\mathcal{K}_i = \mathcal{K}(\mathsf{v}_i, p_i)$ for $i = 1, 2$. The positively oriented crooked planes $\mathcal{C}_1$ and $\mathcal{C}_2$ are disjoint if and only if*

$$\left\{\begin{array}{l} \mathbf{x}^-(\mathsf{v}_1) = \mathbf{x}^+(\mathsf{v}_2), \\ \mathbb{B}\,(p_2 - p_1, \mathsf{v}_1) < 0, \\ \mathbb{B}\,(p_2 - p_1, \mathsf{v}_2) < 0, \\ \mathbb{B}\,(p_2 - p_1, \mathbf{x}^+(\mathsf{v}_1) \boxtimes \mathbf{x}^-(\mathsf{v}_2)) > 0; \end{array}\right\} \ \mathit{or} \ \left\{\begin{array}{l} \mathbf{x}^+(\mathsf{v}_1) = \mathbf{x}^-(\mathsf{v}_2), \\ \mathbb{B}\,(p_2 - p_1, \mathsf{v}_1) > 0, \\ \mathbb{B}\,(p_2 - p_1, \mathsf{v}_2) > 0, \\ \mathbb{B}\,(p_2 - p_1, \mathbf{x}^-(\mathsf{v}_1) \boxtimes \mathbf{x}^+(\mathsf{v}_2)) > 0. \end{array}\right\}$$

The negatively oriented crooked planes $\mathcal{K}_1$ and $\mathcal{K}_2$ are disjoint if and only if

$$\left\{\begin{array}{l} \mathbf{x}^-(\mathsf{v}_1) = \mathbf{x}^+(\mathsf{v}_2), \\ \mathbb{B}\,(p_2 - p_1, \mathsf{v}_1) > 0, \\ \mathbb{B}\,(p_2 - p_1, \mathsf{v}_2) > 0, \\ \mathbb{B}\,(p_2 - p_1, \mathbf{x}^+(\mathsf{v}_1) \boxtimes \mathbf{x}^-(\mathsf{v}_2)) < 0; \end{array}\right\} \ \mathit{or} \ \left\{\begin{array}{l} \mathbf{x}^+(\mathsf{v}_1) = \mathbf{x}^-(\mathsf{v}_2), \\ \mathbb{B}\,(p_2 - p_1, \mathsf{v}_1) < 0, \\ \mathbb{B}\,(p_2 - p_1, \mathsf{v}_2) < 0, \\ \mathbb{B}\,(p_2 - p_1, \mathbf{x}^-(\mathsf{v}_1) \boxtimes \mathbf{x}^+(\mathsf{v}_2)) < 0. \end{array}\right\}$$

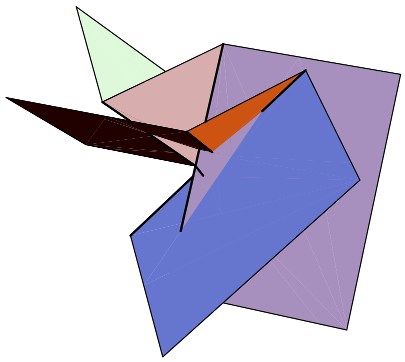

FIGURE 3. Intersecting ultraparallel crooked planes

A pair of positively oriented ultraparallel crooked planes which are disjoint are shown in Figure 2. A pair of negatively oriented ultraparallel crooked planes with the same stems and translation vectors as in Figure 2 are shown in Figure 3 and have a nonempty intersection. These two figures illustrate the following more general fact.

Corollary 6 ([9]). *Let* v_1 *and* v_2 *be spacelike vectors. Either* $\mathcal{C}(\mathsf{v}_1, p_1) \cap \mathcal{C}(\mathsf{v}_2, p_2)$ *or* $\mathcal{K}(\mathsf{v}_1, p_1) \cap \mathcal{K}(\mathsf{v}_2, p_2)$ *is nonempty.*

5. Applications

5.1. **Groups generated by inversions.** For any spacelike line l and $p \in l$, the inversion ι_l preserves $\mathcal{C}(\mathsf{v}, p)$ leaving each wing invariant, permuting the two components of $\mathcal{W}^+(\mathsf{v}, p)$ and $\mathcal{W}^-(\mathsf{v}, p)$ and the two components of $\mathcal{S}(\mathsf{v}, p) - \{p\}$. Further, $\iota_l(\mathcal{H}(\mathsf{v}, p)) = \mathcal{H}(-\mathsf{v}, p)$ so that $E - \mathcal{H}(\pm\mathsf{v}, p)$ are fundamental domains for the action of the group generated by the inversion in l.

Alternatively, ι_l also preserves $\mathcal{K}(\mathsf{v}, p)$ and $E - \mathcal{N}(\pm\mathsf{v}, p)$ are fundamental domains for the action of the group generated by the inversion in l.

Identify $\mathbb{R}^{2,1}$ with $\mathbb{E}$ via a choice of origin $o \in \mathbb{E}$. For $l = p + \mathbb{R}\mathsf{v}$, let $\ell = \mathbb{R}\mathsf{v} \subset \mathbb{R}^{2,1}$ be the line parallel to l through the origin. The inversion ι_ℓ of $\mathbb{R}^{2,1}$ through ℓ equals the reflection of $\mathbf{H}^2$ through the geodesic

$$S(l) = \mathcal{P}(\mathsf{v}) \cap \mathbf{H}^2 = \mathcal{C}(\mathsf{v}, o) \cap \mathbf{H}^2 = \mathcal{K}(\mathsf{v}, o) \cap \mathbf{H}^2.$$

Let $\Delta \subset \mathbf{H}^2$ be a convex n-gon whose pairs of adjacent sides $S_1, \dots, S_n$ are ultraparallel. ι_{ℓ_i} is the reflection in S_i and let $\Gamma \subset \mathbf{SO}(2,1)$ be the group generated by $\iota_{\ell_1}, \dots, \iota_{\ell_n}$. Then Γ acts properly and discretely on $\mathbf{H}^2$ with fundamental domain Δ. Choose $\mathsf{v}_1, \dots, \mathsf{v}_n$ to be consistently oriented so that

$$\Delta = \mathbf{H}^2 - (\cup_{i=1}^n H(\mathsf{v}_i))$$

An *affine deformation* of Γ is a collection $\mathbf{l} = (\mathbf{l_1}, \dots, \mathbf{l_n})$ of oriented spacelike lines such that each l_i has direction vector v_i. Such an affine deformation determines an affine representation of Γ whose image $\Gamma_{\mathbf{l}}$ is generated by inversions ι_j in l_j. Lines $l \subset \mathbb{E}$ with direction vector v are parametrized by a point $p \in l$ which is determined up to translation by $\mathbb{R}\mathsf{v}$. Affine deformations of Γ comprise the product

$$\mathbb{E}/\mathbb{R}\mathsf{v}_1 \times \cdots \times \mathbb{E}/\mathbb{R}\mathsf{v}_n$$

of the quotient affine spaces $\mathbb{E}/\mathbb{R}\mathsf{v}_j$.

A fundamental domain for the action of $\Gamma_{\mathbf{l}}$ is built from disjoint crooked planes $\mathcal{C}_i = \mathcal{C}(\mathsf{v}_i, q_i)$ with spine l_i which bound a common region. Since a crooked plane with given spine l is determined by its vertex, the family $(\mathcal{C}_1, \ldots, \mathcal{C}_n)$ is determined by the n-tuple $\mathbf{q} = (\mathbf{q_1}, \ldots, \mathbf{q_n}) \in \mathbf{l_1} \times \cdots \times \mathbf{l_n}$.

Theorem 7 ([9]). *Let $\Gamma \subset \mathbf{SO}(2,1)$ be as above with consistently oriented v_i's parallel to the ℓ_i's. An affine deformation $\Gamma_{\mathbf{l}}$ of Γ is properly discontinuous if there exist $\mathbf{q} \in \mathbf{l_1} \times \cdots \times \mathbf{l_n}$ such that either (but not both) of the following are true.*

- *$\mathcal{C}(q_1, \mathsf{v}_1), \ldots, \mathcal{C}(q_n, \mathsf{v}_n)$ are disjoint and bound a common region*
$$\mathbb{E} - \left(\cup_{i=1}^{n} \mathcal{H}(q_i, \mathsf{v}_i)\right),$$
which is a fundamental domain for the action.
- *$\mathcal{K}(q_1, \mathsf{v}_1), \ldots, \mathcal{K}(q_n, \mathsf{v}_n)$ are disjoint and bound a common region*
$$\mathbb{E} - \left(\cup_{i=1}^{n} \mathcal{N}(q_i, \mathsf{v}_i)\right),$$
which is a fundamental domain for the action.

These results follow directly from Theorem 3.5 of [4]. Conversely:

Conjecture 8. Let $\Gamma \subset \mathbf{SO}(2,1)$ be as above. An affine deformation $\Gamma_{\mathbf{l}}$ of Γ is properly discontinuous if and only if there exist $\mathbf{q} \in \mathbf{l_1} \times \cdots \times \mathbf{l_n}$ such that either (but not both) of the following are true.

1. $\mathcal{C}(q_1, \mathsf{v}_1), \ldots, \mathcal{C}(q_n, \mathsf{v}_n)$ are disjoint and bound a common region.
2. $\mathcal{K}(q_1, \mathsf{v}_1), \ldots, \mathcal{K}(q_n, \mathsf{v}_n)$ are disjoint and bound a common region.

5.2. **Groups generated by hyperbolic transformations.** All $g \in \mathbf{SO}(2,1)$ have 1 as an eigenvalue. For hyperbolic g, we define $\mathsf{x}^0(g)$ to be the unique fixed unit-spacelike vector such that $\mathbf{x}^-(\mathsf{x}^0(g))$ is an eigenvector with a corresponding eigenvalue < 1, and $\mathbf{x}^+(\mathsf{x}^0(g))$ is an eigenvector with a corresponding eigenvalue > 1.

For hyperbolic $h(\mathsf{x}) = g(\mathsf{x}) + \mathsf{v}$, Margulis ([12],[13]) defined an invariant $\alpha(h) = \mathbb{B}\left(\mathsf{v}, \mathsf{x}^0(g)\right)$. h admits a fixed point if and only if $\alpha(h) = 0$. Further, Margulis has shown ([12],[13], see also [7]) that

Theorem 9 (Margulis). *If h_1 and h_2 are hyperbolic Lorentz transformations such that $\alpha(h_1)$ and $\alpha(h_2)$ have opposite signs then any group containing h_1 and h_2 does not act properly on $\mathbb{R}^{2,1}$.*

If there exist crooked planes $\mathcal{C}(\mathsf{v}_1, p_1)$ and $\mathcal{C}(\mathsf{v}_2, p_2)$ such that $h(\mathcal{C}(\mathsf{v}_1, p_1)) = \mathcal{C}(\mathsf{v}_2, p_2)$ and v_1 and $\mathsf{x}^0(g)$ cross (equivalently, v_2 and $\mathsf{x}^0(g)$ cross) then the region bounded by $\mathcal{C}(\mathsf{v}_1, p_1)$ and $\mathcal{C}(\mathsf{v}_2, p_2)$ is a fundamental domain for the action of $\langle h \rangle$ on $\mathbb{R}^{2,1}$. The orientation of the crooked planes is related to $\alpha(h)$ by the following corollary of Theorems 4 and 5:

Corollary 10. *Let $h \in \mathbf{H}$ be hyperbolic. If $\alpha(h) < 0$ ($\alpha(h) > 0$) then there is no fundamental domain of the action of the group generated by h on $\mathbb{E}$ bounded by disjoint positively (negatively) oriented crooked planes.*

References

1. Auslander,L. and Markus, L., *Holonomy of flat affinely connected manifolds,* American J. Math. **62** (1955), 139–159
2. ——— *Flat Lorentz 3-manifolds,* Memoirs Amer. Math. Soc. **30** (1959)
3. Drumm, T., *Fundamental polyhedra for Margulis space-times,* Topology **31** (4) (1992), 677-683
4. ———, *Linear holonomy of Margulis space-times,* J.Diff.Geo. **38** (1993), 679–691
5. ———, *Examples of nonproper affine actions,* Mich. Math. J. **39** (1992), 435–442
6. ———, *Margulis space-times,* Proc. Symp. Pure Math. **54** (1993), Part 2, 191–195
7. ———, *Translations and the holonomy of complete affine flat manifolds* (preprint)
8. Drumm, T. and Goldman, W., *Complete flat Lorentz 3-manifolds with free fundamental group,* Int. J. Math.**1** (1990), 149–161
9. ———, *The geometry of crooked planes,* preprint, 30 pp.
10. Fried, D. and Goldman, W., *Three-dimensional affine crystallographic groups,* Adv. Math. **47** (1983), 1–49
11. Goldman, W., "Complex hyperbolic geometry," Oxford Mathematical Monographs (to appear)
12. Margulis, G., *Free properly discontinuous groups of affine transformations,* Dokl. Akad. Nauk SSSR **272** (1983), 937–940;
13. ———, *Complete affine locally flat manifolds with a free fundamental group,* J. Soviet Math. **134** (1987), 129–134
14. Mess, G., *Lorentz spacetimes of constant curvature,* (preprint)
15. Milnor, J., *On fundamental groups of complete affinely flat manifolds,* Adv. Math. **25** (1977), 178–187
16. Tits, J., *Free subgroups in linear groups,* J. Algebra **20** (1972), 250–270

Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104 *(Drumm)*, Department of Mathematics, University of Maryland, College Park, MD 20742 *(Goldman)*

E-mail address: `tad@math.upenn.edu` *(Drumm),* `wmg@math.umd.edu` *(Goldman)*