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# THE INTRINSIC INVARIANT OF AN APPROXIMATELY FINITE DIMENSIONAL FACTOR AND THE COCYCLE CONJUGACY OF DISCRETE AMENABLE GROUP ACTIONS

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ABSTRACT. We announce in this article that i) to each approximately finite dimensional factor $\mathcal{R}$ of any type there corresponds canonically a group cohomological invariant, to be called the **intrinsic invariant** of $\mathcal{R}$ and denoted $\Theta(\mathcal{R})$, on which $\mathrm{Aut}(\mathcal{R})$ acts canonically; ii) when a group $G$ acts on $\mathcal{R}$ via $\alpha : G \mapsto \mathrm{Aut}(\mathcal{R})$, the pull back of $\mathrm{Orb}(\Theta(\mathcal{R}))$, the orbit of $\Theta(\mathcal{R})$ under $\mathrm{Aut}(\mathcal{R})$,by $\alpha$ is a cocycle conjugacy invariant of $\alpha$; iii) if $G$ is a discrete countable amenable group, then the pair of the module, $\mathrm{mod}(\alpha)$, and the above pull back is a complete invariant for the cocycle conjugacy class of $\alpha$. This result settles the open problem of the general cocycle conjugacy classification of discrete amenable group actions on an AFD factor of type $\mathrm{III}_1$, and unifies known results for other types.

## Introduction

The celebrated work of Connes, [1,3], surveyed in [2], Ocneanu's analysis, [12], and the previous work of Kawahigashi, and the second and third authors, [11], reveal the beatiful structure of the automorphism group of an approximately finite dimensional, or AFD, factor $\mathcal{R}$. In this note, we announce that it is possible to describe the structure of $\mathrm{Aut}(\mathcal{R})$ independently of the type of $\mathcal{R}$. This structure of $\mathrm{Aut}(\mathcal{R})$ enables us to define a cohomological invariant which we call the **intrinsic invariant** of $\mathcal{R}$. If a group G acts on $\mathcal{R}$ via $\alpha$, then the pull back of the orbit of the intrinsic invariant of $\mathcal{R}$ under the natural action of the group $\mathrm{Aut}(\mathcal{R})$ gives a cocyle conjugacy invariant which is complete if G is a countable discrete amenable group. This completes the cocylce conjugacy classification of discrete amenable group actions on an AFD factor $\mathcal{R}$ including the type $\mathrm{III}_1$ case.

## Intrinsic Invariant and Main Theorem

Let $\mathcal{R}$ be an AFD factor, and let $\mathrm{Aut}(\mathcal{R})$ and $\mathrm{Int}(\mathcal{R})$ be the group of automorphisms and the group of inner automorphisms respectively. Let

$$\varepsilon : \alpha \in \mathrm{Aut}(\mathcal{R}) \longmapsto \dot{\alpha} \in \mathrm{Out}(\mathcal{R}) = \mathrm{Aut}(\mathcal{R})/\mathrm{Int}(\mathcal{R})$$

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be the canonical quotient map and set

$$\mathrm{Cnt}(\mathcal{R}) = \varepsilon^{-1}(\text{Center of } \mathrm{Out}(\mathcal{R})).$$

As $\mathrm{Int}(\mathcal{R})$ is not closed in $\mathrm{Aut}(\mathcal{R})$, we need to consider its closure $\overline{\mathrm{Int}}(\mathcal{R})$ and the quotient group $\mathrm{Mod}(\mathcal{R}) = \mathrm{Aut}(\mathcal{R})/\overline{\mathrm{Int}}(\mathcal{R})$. We denote the canonical quotient map by $\mathrm{mod} : \alpha \in \mathrm{Aut}(\mathcal{R}) \longmapsto \mathrm{mod}(\alpha) \in \mathrm{Mod}(\mathcal{R})$. The map mod will be called the *module* and the image $\mathrm{mod}(\alpha)$ of $\alpha \in \mathrm{Aut}(\mathcal{R})$ in $\mathrm{Mod}(\mathcal{R})$ the *module* of $\alpha$. Two more groups and a map are evidently associated with $\mathcal{R}$: the unitary group $\mathcal{U}(\mathcal{R})$, its center $\mathbf{T}$ which is the one dimensional torus group of complex numbers of modulus one, and the adjoint map:

$$\mathrm{Ad} : u \in \mathcal{U}(\mathcal{R}) \longmapsto \mathrm{Ad}(u) \in \mathrm{Int}(\mathcal{R}).$$

Of course, $\mathbf{T}$ is the kernel of the map Ad. Finally, we need to consider the flow of weights $F(\mathcal{R})$ and its cohomology groups: $B^1(F(\mathcal{R}))$, $Z^1(F(\mathcal{R}))$ and $H^1(F(\mathcal{R}))$. It is known,[5], that $\mathrm{Mod}(\mathcal{R}) = \mathrm{Aut}(\mathcal{R})/\overline{\mathrm{Int}}(\mathcal{R})$ is canonically identified with $\mathrm{Aut}(F(\mathcal{R}))$. By the work of Wong, [15], the short exact sequence:

$$1 \longrightarrow \overline{\mathrm{Int}}(\mathcal{R}) \longrightarrow \mathrm{Aut}(\mathcal{R}) \longrightarrow \mathrm{Mod}(\mathcal{R}) \longrightarrow 1$$

splits, but not canonically. By [11; Theorem 1] and [5], there exists a canonical isomorphism from $H^1(F(\mathcal{R}))$ onto the Center of $\mathrm{Out}(\mathcal{R})$, which will be denoted by $\delta$. These groups and maps are related as described in the following commutative diagrams of exact sequences:

$$\begin{array}{ccccccccc}
 & & & & \mathrm{Cnt}(\mathcal{R}) & \longrightarrow & \mathrm{H}^1(F(\mathcal{R})) & & \\
 & & & \nearrow & & & & \searrow & \\
1 & \longrightarrow & \mathrm{Int}(\mathcal{R}) & & \downarrow & & \downarrow \delta & & 1. \\
 & & & \searrow & & & & \nearrow & \\
 & & & & \mathrm{Aut}(\mathcal{R}) & \xrightarrow{\varepsilon} & \mathrm{Out}(\mathcal{R}) & &
\end{array}$$

The sequence involving $\mathrm{Cnt}(\mathcal{R})$ forms part of an exact square, as follows:

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbf{T} & \longrightarrow & \mathcal{U}(F(\mathcal{R})) & \xrightarrow{\partial} & \mathrm{B}^1(F(\mathcal{R})) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathcal{U}(\mathcal{R}) & \longrightarrow & \widetilde{\mathcal{U}}(\mathcal{R}) & \longrightarrow & \mathrm{Z}^1(F(\mathcal{R}))) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Int}(\mathcal{R}) & \longrightarrow & \mathrm{Cnt}(\mathcal{R}) & \underset{\delta^{-1}\circ\varepsilon}{\longrightarrow} & \mathrm{H}^1(F(\mathcal{R})) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

Here, $\widetilde{\mathcal{U}}(\mathcal{R})$ is the semi direct product of $\mathcal{U}(\mathcal{R})$ by the extended modular action of $\mathrm{Z}^1(F(\mathcal{R}))$ as in [14]. Except for the lower right corner $\mathrm{H}^1(F(\mathcal{R}))$, all groups are Polish and all maps are continuous. As the above square of exact sequences is canonical, $\mathrm{Aut}(\mathcal{R})$ acts on the square, i.e. the above sequare is an equivariant square under the action of $\mathrm{Aut}(\mathcal{R})$. Let $\nu$ denote the map $\delta^{-1} \circ \varepsilon$, called the *modular invariant*. The middle vertical $\mathrm{Aut}(\mathcal{R})$ equivariant exact sequence of the above exact square:

$$1 \longrightarrow \mathcal{U}(F(\mathcal{R})) \longrightarrow \widetilde{\mathcal{U}}(\mathcal{R}) \longrightarrow \mathrm{Cnt}(\mathcal{R}) \longrightarrow 1$$

gives rise to a cohomological invariant, called the *characteristic invariant* $\chi \in \Lambda(\mathrm{Aut}(\mathcal{R}), \mathrm{Cnt}(\mathcal{R}), \mathcal{U}(F(\mathcal{R})))$. Thus we have the triplet:

$$\begin{aligned}(\mathrm{mod}, \chi, \nu) \in \mathrm{Hom}(\mathrm{Aut}(\mathcal{R}), \mathrm{Aut}(F(\mathcal{R})) \times \Lambda(\mathrm{Aut}(\mathcal{R}), \mathrm{Cnt}(\mathcal{R}), \mathcal{U}(F(\mathcal{R}))) \\ \times\ \mathrm{Hom}_{\mathrm{Aut}(\mathcal{R})}(\mathrm{Cnt}(\mathcal{R}), \mathrm{H}^1(F(\mathcal{R}))),\end{aligned}$$

consiting of the action mod of $\mathrm{Aut}(\mathcal{R})$ on $F(\mathcal{R})$, the characteristic invariant and the $\mathrm{Aut}(\mathcal{R})$-equivariant homomorphism $\nu$, which will be called the *intrinsic invariant* of the AFD factor $\mathcal{R}$ and denoted by $\Theta(\mathcal{R})$. Naturally, $\mathrm{Aut}(\mathcal{R})$ acts on $\mathrm{Hom}(\mathrm{Aut}(\mathcal{R}), \mathrm{Aut}(F(\mathcal{R})))$, $\Lambda(\mathrm{Aut}(\mathcal{R}), \mathrm{Cnt}(\mathcal{R}), \mathcal{U}(F(\mathcal{R}))))$ and $\mathrm{Hom}_{\mathrm{Aut}(\mathcal{R})}(\mathrm{Cnt}(\mathcal{R})), \mathrm{H}^1(F(\mathcal{R})))$. Let $\mathrm{Orb}(\Theta(\mathcal{R}))$ denote the orbit of $\Theta(\mathcal{R})$ under the actionof $\mathrm{Aut}(\mathcal{R})$.

We are now at the position to state the main result:

**Theorem 1.** *Let $\mathcal{R}$ be an approximately finite dimensional separable factor and $G$ be a countable discrete amenable group. If $\alpha$ is an action of $G$ on $\mathcal{R}$, then the pull back $\alpha^*(\mathrm{Orb}(\Theta(\mathcal{R}))$ of the orbit of the intrinsic invariant is a complete invariant of the cocylce conjugacy class of $\alpha$. More precisely, the inverse image, $N(\alpha) = \alpha^{-1}(\mathrm{Cnt}(\mathcal{R}))$, of $\mathrm{Cnt}(\mathcal{R})$ under $\alpha$, the action $\mathrm{mod} \circ \alpha$ of $G$ on $F(\mathcal{R})$, the characteristic invariant $\chi(\alpha) \in \Lambda(G, N(\alpha), \mathcal{U}(F(\mathcal{R})))$ of $\alpha$ which is obtained as the pull back of $\Theta(\mathcal{R})$ and $\nu_\alpha \in \mathrm{Hom}_G(N(\alpha), \mathcal{U}(F(\mathcal{R})))$ determine the cocycle conjugacy class of $\alpha$.*

It should be noted that this one theorem applies to all AFD factors of **any type**. Of course, the type of the carrier factor $\mathcal{R}$ affects on the nature of these invariants. We list the the invariants in each type as follows:
Type $\mathrm{I}_n, n \in \mathbf{N}$:

$$\mathrm{Int}(\mathcal{R}) = \mathrm{Cnt}(\mathcal{R}) = \mathrm{Aut}(\mathcal{R}) \quad \textit{is compact}$$

$$\mathrm{Mod}(\mathcal{R}) = 1$$

$$G = N(\alpha),\ \chi_\alpha \in \mathrm{H}^2(G, \mathbf{T}), \nu_\alpha = 1.$$

Type $\mathrm{I}_\infty$:

$$\mathrm{Aut}(\mathcal{R}) = \mathrm{Cnt}(\mathcal{R}) = \mathrm{Int}(\mathcal{R}) \quad \textit{is not compact.}$$

$$\mathrm{Mod}(\mathcal{R}) = 1$$

$$G = N(\alpha),\ \chi_\alpha \in \mathrm{H}^2(G, \mathbf{T}), \nu_\alpha = 1.$$

Type $\mathrm{II}_1$:

$$\mathrm{Int}(\mathcal{R}) = \mathrm{Cnt}(\mathcal{R}),\ \overline{\mathrm{Int}}(\mathcal{R}) = \mathrm{Aut}(\mathcal{R}),\ \mathrm{Mod}(\mathcal{R}) = 1.$$

Therefore the characteristic invariant $\chi_\alpha \in \Lambda(G, N(\alpha), \mathbf{T})$ alone determines the cocylcle conjugacy of the action $\alpha$ of $G$.
Type $\mathrm{II}_\infty$:

$$\mathrm{Int}(\mathcal{R}) = \mathrm{Cnt}(\mathcal{R}), \qquad F(\mathcal{R}) = \{L^\infty(\mathbf{R}), \textit{Translation}\}, \qquad \mathrm{H}^1(F(\mathcal{R})) = 1,$$

$$\mathrm{Mod}(\mathcal{R}) = \mathbf{R}_+^*, \qquad \mathrm{Aut}(\mathcal{R}) = \overline{\mathrm{Int}}(\mathcal{R}) \rtimes \mathbf{R}_+^*$$

$$\widetilde{\mathcal{U}}(\mathcal{R}) = \mathcal{U}(\mathcal{R}) \times \mathcal{U}(L^\infty(\mathbf{R}))/\mathbf{T}.$$

Type $\mathrm{III}_0$: All invariants are non-trivial in general.

$$\mathrm{Aut}(\mathcal{R}) = \overline{\mathrm{Int}}(\mathcal{R}) \rtimes \mathrm{Aut}(F(\mathcal{R})) \quad \text{by} \quad \text{Wong, [23];}$$

$$\widetilde{\mathcal{U}}(\mathcal{R}) = \mathcal{U}(\mathcal{R}) \rtimes \mathbf{Z}^1(F(\mathcal{R})) \quad \text{by} \quad [22].$$

Type $\mathrm{III}_\lambda, 0 < \lambda < 1$:

$$\mathcal{F}(\mathcal{R}) = \{L^\infty(\mathbf{R}/(-\log(\lambda)\mathbf{Z}), \textit{Translation}\},$$
$$\mathrm{H}^1(F(\mathcal{R})) = \mathbf{R}/T\mathbf{Z} \quad \textit{with} \quad T = -2\pi/\log\lambda;$$

$$\mathrm{Cnt}(\mathcal{R}) = \mathrm{Int}(\mathcal{R})\sigma(\mathbf{R}) \text{ where } \sigma(\mathbf{R}) = \text{ Modular Automorphism Group;}$$

$$\widetilde{\mathcal{U}}(\mathcal{R}) = (\mathcal{U}(\mathcal{R}) \rtimes Z^1(F(\mathcal{R})), \quad \mathrm{Aut}(\mathcal{R}) = \overline{\mathrm{Int}}(\mathcal{R}) \rtimes \mathbf{R}/(\log\lambda)\mathbf{Z};$$

and

$$Z^1(F(\mathcal{R})) = (\mathcal{U}(L^\infty(\mathbf{R}/(-\log(\lambda)\mathbf{Z})/\mathbf{T}) \rtimes \mathbf{T}.$$

Type $\mathrm{III}_1$:

$$\mathcal{F}(\mathcal{R}) = \{\mathbf{C}, \textit{Trivial action of } \mathbf{R}\}, \overline{\mathrm{Int}}(\mathcal{R}) = \mathrm{Aut}(\mathcal{R}), \ \mathrm{Mod}(\mathcal{R}) = 1;$$

$$\widetilde{\mathcal{U}}(\mathcal{R}) = \mathcal{U}(\mathcal{R}) \rtimes \mathbf{R}, \qquad \mathrm{H}^1(F(\mathcal{R})) = \mathbf{R}, \qquad \mathrm{Cnt}(\mathcal{R}) = \mathrm{Int}(\mathcal{R}) \rtimes \mathbf{R}.$$

It is interesting to note that the structure of the invariants in the type $\mathrm{III}_1$case is simplest among type III cases yet the proof is the hardest. Special cases of the result in the $\mathrm{III}_1$case have been established in [11]. The general case will appear in [10]. We state it here as an independent result:

**Corollary 2.** *If $\mathcal{R}$ is an AFD factor of type $III_1$, then with $N = \alpha^{-1}(\mathrm{Cnt}(\mathcal{R}))$ the pair*

$$(\chi_\alpha, \nu_\alpha) \in \Lambda(G, N, \mathbf{T}) \times \mathrm{Hom}_G(N, \mathbf{R})$$

*is a complete invariant for the cocycle conjugacy class of the action $\alpha$ of a countable discrete amenable group $G$ on $\mathcal{R}$. Every element $(\chi, \nu) \in \Lambda(G, N.\mathbf{T}) \times Hom_G(N, \mathbf{R})$ arises as the invariant of an action of $G$ on $\mathcal{R}$.*

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