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# INTERSECTION PAIRINGS IN MODULI SPACES OF HOLOMORPHIC BUNDLES ON A RIEMANN SURFACE

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Abstract. We outline a proof of formulas (found by Witten in 1992 using physical methods) for intersection pairings in the cohomology of the moduli space $M(n,d)$ of stable holomorphic vector bundles of rank $n$ and degree $d$ (assumed coprime) and fixed determinant on a Riemann surface of genus $g \geq 2$.

## 1. Introduction

The moduli space $M(n,d)$ of semistable rank $n$ degree $d$ holomorphic vector bundles with fixed determinant on a compact Riemann surface $\Sigma$ is a smooth Kähler manifold when $n$ and $d$ are coprime. This space had long been studied by algebraic geometers (see for instance Narasimhan and Seshadri 1965 [22]), but a new viewpoint on it was revealed by the seminal 1982 paper [1] of Atiyah and Bott on the Yang-Mills equations on Riemann surfaces. In this paper a set of generators for the (rational) cohomology ring of $M(n,d)$ was given and formulas for the Poincaré polynomial were proved. Given the specification of a set of generators, knowledge of the intersection pairings between products of these generators (or equivalently knowledge of the evaluation on the fundamental class of products of the generators) completely determines the structure of the cohomology ring.

Little progress was made on the problem of determining these intersection pairings until 1991, when Donaldson [7] and Thaddeus [24] gave formulas for the intersection pairings in $H^*(M(2,1))$ (in terms of Bernoulli numbers). Then, using physical methods, Witten [26] found formulas for the evaluation of any product of generators on the fundamental class of $M(n,d)$ for arbitrary rank $n$. These generalized his (rigorously proved) formulas [25] for the symplectic volume of $M(n,d)$: for instance, the symplectic volume of $M(2,1)$ is given by

$$\mathrm{vol}(M(2,1)) = 2\frac{(-1)^{g-1}}{(2\pi^2)^{g-1}}\Big(1 - \frac{1}{2^{2g-3}}\Big)\zeta(2g-2)$$

where $g$ is the genus of the Riemann surface and $\zeta$ is the Riemann zeta function.

In this article we outline a mathematically rigorous proof of Witten's result. Full details, in particular for the case of rank $n$ at least three, will appear later [18]. The key idea is to use a description of $M(n,d)$ as a symplectic quotient and to apply the *nonabelian localization theorem* (Witten [26]; Jeffrey-Kirwan [17]), which is a generalization of the Duistermaat-Heckman theorem [8]. A different approach to the nonabelian localization principle has been given recently by Guillemin-Kalkman

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[11] and independently by Martin [21]; their ideas are crucial in the technicalities of our proof since they can be adapted to certain noncompact situations where the approach of [17] is not valid.

This paper is organized as follows. In Section 2 we describe the generators for the cohomology ring $H^*(\mathcal{M}(n,d))$. In Section 3 we outline tools from the Cartan model of equivariant cohomology and some different versions of localization. In Section 4 we recall properties of the *extended moduli space*, a finite dimensional symplectic space equipped with a Hamiltonian action of $SU(n)$ for which the symplectic quotient is $\mathcal{M}(n,d)$. Finally Section 5 gives an outline of the proof of Witten's formulas.

## 2. Generators for the cohomology ring

Let us assume throughout, in order to avoid exceptional cases, that the Riemann surface $\Sigma$ has genus $g \geq 2$.

In [1], Atiyah and Bott gave a set of generators for $H^*(\mathcal{M}(n,d))$; their procedure may be described as follows[1]. We may form a normalized universal rank $n$ vector bundle

$$\mathbb{U} \to \Sigma \times \mathcal{M}(n,d)$$

(see [1], p. 582). Then there is the following description [1] of a set of generators of $H^*(\mathcal{M}(n,d))$:

$$f_r = ([\Sigma], c_r(\mathbb{U})), \tag{1}$$

$$b_r^j = (\alpha_j, c_r(\mathbb{U})), \tag{2}$$

$$a_r = (1, c_r(\mathbb{U})). \tag{3}$$

Here, $[\Sigma] \in H_2(\Sigma)$ and $\alpha_j \in H_1(\Sigma)$ $(j = 1, \ldots, 2g)$ form standard bases of $H_2(\Sigma, \mathbb{Z})$ and $H_1(\Sigma, \mathbb{Z})$. The bracket represents the slant product $H^N(\Sigma \times \mathcal{M}(n,d)) \otimes H_j(\Sigma) \to H^{N-j}(\mathcal{M}(n,d))$. More generally if $K = SU(n)$ and $Q$ is an invariant polynomial of degree $s$ on its Lie algebra $\mathbf{k} = su(n)$ then there is an associated element of $H^*(BSU(n))$ and hence an associated element of $H^*(\Sigma \times \mathcal{M}(n,d))$ which is a characteristic class $Q(\mathbb{U})$ of the universal bundle $\mathbb{U}$. Hence the slant product gives rise to classes

$$Q([\Sigma]) \in H^{2s-2}(\mathcal{M}(n,d)), \tag{4}$$

$$Q(\alpha_j) \in H^{2s-1}(\mathcal{M}(n,d)), \tag{5}$$

$$Q(1) \in H^{2s}(\mathcal{M}(n,d)). \tag{6}$$

In particular, letting $\tau_r \in S^r(\mathbf{k}^*)^K$ denote the invariant polynomial associated to the $r$-th Chern class, we recover our generators of $H^*(\mathcal{M}(n,d))$:

$$\begin{aligned} f_r &= \tau_r([\Sigma]), \qquad (7)\\ b_r^j &= \tau_r(\alpha_j), \\ a_r &= \tau_r(1). \end{aligned}$$

[1] In this paper, all cohomology groups are assumed to be with complex coefficients.

A special role is played by the invariant polynomial $\tau_2 = \frac{1}{2}\langle\cdot,\cdot\rangle$ on $\mathbf{k}$ given by an invariant inner product proportional to the Killing form. We normalize the inner product as follows for $K = SU(n)$:

$$\langle X, X\rangle = -\mathrm{Trace}(X^2)/(4\pi^2).$$

The associated class $f_2$ is the cohomology class of the symplectic form on $\mathcal{M}(n,d)$.

In Sections 4 and 5 of [26], Witten obtained formulas for generating functionals from which one may extract all intersection pairings

$$\prod_{r=2}^{n} a_r^{m_r} f_r^{n_r} \prod_{k_r=1}^{2g} (b_r^{k_r})^{p_{r,k_r}}[\mathcal{M}(n,d)].$$

Let us begin with pairings of the form

$$\prod_{r=2}^{n} a_r^{m_r} \exp f_2[\mathcal{M}(n,d)]. \tag{8}$$

For $m_r$ sufficiently small to ensure convergence of the sum, Witten obtains ([26], (4.74))

$$\prod_{r=2}^{n} a_r^{m_r} \exp f_2[\mathcal{M}(n,d)] = \Gamma\left(\sum_{\lambda} \frac{c^{-\lambda}\prod_{r=2}^{n}\tau_r(2\pi i\lambda)^{m_r}}{\mathcal{D}^{2g-2}(2\pi i\lambda)}\right), \tag{9}$$

where $\Gamma$ is a universal constant depending on $n$, $d$ and $g$, and the Weyl odd polynomial $\mathcal{D}$ on $\mathbf{t}^*$ is defined by

$$\mathcal{D}(X) = \prod_{\gamma>0}\gamma(X)$$

where $\gamma$ runs over the positive roots. The sum over $\lambda$ in (9) runs over those elements of the weight lattice $\Lambda^w$ that are in the interior of the fundamental Weyl chamber.[2] The element

$$c = e^{2\pi i d/n}\mathrm{diag}(1,\ldots,1) \tag{10}$$

is a generator of the centre $Z(K)$, so since $\lambda \in \mathbf{t}^*$ is in $\mathrm{Hom}(T, U(1))$, we may evaluate $\lambda$ on $c$ as in (9): $c^\lambda$ is defined as $\exp\lambda(\tilde{c})$ where $\tilde{c}$ is any element of the Lie algebra of $T$ such that $\exp\tilde{c} = c$.

There are similar formulas ([26], (5.15) and (5.18)) for pairings involving the $f_r$ for $r > 2$ and the $b_r^j$ as well as $f_2$ and the $a_r$.

For concreteness it is worth examining the special case $n = 2, d = 1$. Here the dominant weights $\lambda$ are just the positive integers. The relevant generators of $H^*(\mathcal{M}(2,1))$ are

$$f_2 \in H^2(\mathcal{M}(2,1)) \tag{11}$$

(which is the cohomology class of the symplectic form on $\mathcal{M}(2,1)$) and

$$a_2 \in H^4(\mathcal{M}(2,1)): \tag{12}$$

[2]The weight lattice $\Lambda^w \subset \mathbf{t}$ is the dual lattice of the integer lattice $\Lambda^I = \mathrm{Ker}(\exp)$ in $\mathbf{t}$ under the inner product $\langle\cdot,\cdot\rangle$.

these arise from the invariant polynomial $\tau_2 = \langle \cdot, \cdot \rangle$ by $a_2 = \tau_2(1)$, $f_2 = \tau_2([\Sigma])$ (see (7)). We find then that the formula (9) reduces for $m \le g - 2$ to[3] ([26], (4.44))

$$a_2^j \exp(f_2)[\mathcal{M}(2,1)] = \frac{2^{2g}(-1)^{g-1-j}}{2(8\pi^2)^{g-1}} \left( \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\pi^{2j}}{n^{2g-2-2j}} \right). \tag{13}$$

Thus one obtains the formulas found in [24] for the intersection pairings $a_2^m f_2^n [\mathcal{M}(2,1)]$; these intersection pairings are given by Bernoulli numbers, or equivalently are given in terms of the Riemann zeta function $\zeta(s) = \sum_{n \ge 1} 1/n^s$.

## 3. Equivariant cohomology and localization

The methods we shall use involve the application of the *nonabelian localization theorem* [17, 26]. In [17] we considered a compact symplectic manifold $M$ equipped with the Hamiltonian action of a compact group $K$. We expressed the cohomology ring $H^*(M_{\rm red})$ of the reduced space or symplectic quotient $M_{\rm red} = \mu^{-1}(0)/K$ of $M$ by $K$ in terms of the equivariant cohomology of $M$. (This was under the assumption that 0 is a regular value of $\mu$, which implies that $M_{\rm red}$ is a symplectic orbifold.) There is a surjective ring homomorphism $\Phi$ from the equivariant cohomology $H_K^*(M)$ of $M$ to the cohomology $H^*(M_{\rm red})$ of the reduced space. In terms of this map, we derived a formula for the evaluation $\eta_0[M_{\rm red}]$ of a cohomology class $\eta_0 \in H^*(M_{\rm red})$ on the fundamental class of $M_{\rm red}$. This formula involves data that enter the Duistermaat-Heckman formula, and its generalization the abelian localization formula [2, 4, 5] for the action of a maximal torus $T$ of $K$ on $M$: that is, the components $F$ of the fixed point set $M^T$ of $T$ on $M$, and the equivariant Euler classes $e_F$ of the normal bundles to $F$ in $M$. In the following paragraphs we sketch this construction.

The $K$-equivariant cohomology of $M$ is the cohomology of a certain chain complex $\Omega_K^*(M)$, which can be expressed as

$$\Omega_K^*(M) = (S(\mathbf{k}^*) \otimes \Omega^*(M))^K \tag{14}$$

(where $\Omega^*(M)$ denotes the space of differential forms on $M$, and $S(\mathbf{k}^*)$ denotes the ring of polynomial functions on the Lie algebra $\mathbf{k}$ of $K$). An element $f \in \Omega_K^*(M)$ may be thought of as a $K$-equivariant polynomial function from $\mathbf{k}$ to $\Omega^*(M)$, or alternatively as a family of differential forms on $M$ parametrized by $X \in \mathbf{k}$. The differential $D$ on the complex $\Omega_K^*(M)$ is then defined by

$$(D\alpha)(X) = d(\alpha(X)) - \iota_{X^\#}(\alpha(X)) \tag{15}$$

where $X^\#$ is the vector field on $M$ generated by the action of $X$.

In terms of this notation, the map $\Omega_K^*(M) \to \Omega_K^*(\mathrm{pt}) = S(\mathbf{k}^*)^K$ given by integration over $M$ passes to $H_K^*(M)$. Thus for any $D$-closed element $\eta \in \Omega_K^*(M)$ representing a cohomology class $[\eta]$, there is a corresponding element $\int_M \eta \in \Omega_K^*(\mathrm{pt})$ which depends only on $[\eta]$. The same is true for any $D$-closed element $\eta = \sum_j \eta_j$ which is a formal series of elements $\eta_j$ in $\Omega_K^j(M)$ without polynomial dependence on $X$: we shall in particular consider terms of the form $\eta(X) e^{i(\omega + \mu(X))}$ (where $\eta \in \Omega_K^*(M)$), since the element

$$\bar{\omega}(X) = \omega + \mu(X) \in \Omega_K^2(M)$$

satisfies $D\bar{\omega} = 0$.

[3] Here, we have identified $a_2$ with Witten's class $\Theta$ and $f_2$ with Witten's class $\omega$.

If $X$ lies in $\mathbf{t}$, the Lie algebra of a chosen maximal torus $T$ of $K$, then there is a formula for $\int_M \eta(X)$ (the *abelian localization formula* [1, 3, 4, 5]) which depends only on the fixed point set of $T$ in $M$. It tells us that

$$\int_M \eta(X) = \sum_{F\in\mathcal{F}} \int \frac{i_F^*\eta(X)}{e_F(X)} \tag{16}$$

where $\mathcal{F}$ indexes the components $F$ of the fixed point set of $T$ in $M$, the inclusion of $F$ in $M$ is denoted by $i_F$ and $e_F \in H_T^*(M)$ is the equivariant Euler class of the normal bundle to $F$ in $M$. In particular, applying (16) with $\eta$ replaced by the formal sum $\eta e^{i\bar{\omega}}$ we have

$$h^\eta(X) \stackrel{\text{def}}{=} \int_M \eta(X)e^{i\bar{\omega}(X)} = \sum_{F\in\mathcal{F}} h_F^\eta(X), \tag{17}$$

where

$$h_F^\eta(X) = e^{i\mu(F)(X)} \int_F \frac{i_F^*\eta(X)e^{i\omega}}{e_F(X)}. \tag{18}$$

Note that the moment map $\mu$ takes a constant value $\mu(F) \in \mathbf{t}^*$ on each $F \in \mathcal{F}$, and that the integral in (18) is a rational function of $X$.

We assume that $\eta_0 \in H^*(M_{\rm red})$ comes via the surjective homomorphism $\Phi$ from a class in $H_K^*(M)$ which is represented as an equivariant differential form by $\eta \in \Omega_K^*(M)$, and denote the symplectic form on $M_{\rm red}$ by $\omega_0$. Let us recall the main result (the residue formula, Theorem 8.1) of [17]:

**Theorem 3.1.** [17] *Let $\eta \in H_K^*(M)$ induce $\eta_0 \in H^*(M_{\rm red})$. Then we have*

$$\eta_0 e^{i\omega_0}[M_{\rm red}] = n_0 C^K \operatorname{Res}\left(\mathcal{D}^2(X) \sum_{F\in\mathcal{F}} h_F^\eta(X)[dX]\right), \tag{19}$$

*where $n_0$ is the order of the subgroup of $K$ that acts trivially on*[4] *$M$, and the constant $C^K$ is defined by*

$$C^K = \frac{(-1)^{n_+}}{|W| \operatorname{vol}(T)}. \tag{20}$$

*We have introduced $s = \dim K$ and $l = \dim T$; here $n_+ = (s-l)/2$ is the number of positive roots*[5]*. Also, $\mathcal{F}$ denotes the set of components of the fixed point set of $T$, and if $F$ is one of these components then the meromorphic function $h_F^\eta$ on $\mathbf{t}\otimes\mathbb{C}$ is defined by (18) and the polynomial $\mathcal{D} : \mathbf{t} \to \mathbb{R}$ is defined by $\mathcal{D}(X) = \prod_{\gamma>0}\gamma(X)$, where $\gamma$ runs over the positive roots of $K$.*

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[4]The subgroup of $K$ which acts trivially on $M$ is the same as the subgroup, $N$ say, of $K$ which acts trivially on $\mu^{-1}(0)$, and is a finite central subgroup of $K$. For since 0 is a regular value of $\mu$ it follows that $N$ is a finite normal subgroup of $K$, and because $K$ is connected $N$ is therefore contained in the center of $K$. Thus the coadjoint action of $N$ on $\mathbf{k}^*$ is trivial, so $N$ acts trivially on the normal bundle to $\mu^{-1}(0)$ in $M$, and as $M$ is connected this means $N$ acts trivially on $M$.

[5]Here, the roots of $K$ are the nonzero weights of its complexified adjoint action. We fix the convention that weights $\beta \in \mathbf{t}^*$ satisfy $\beta \in \operatorname{Hom}(\Lambda^I, \mathbb{Z})$ rather than $\beta \in \operatorname{Hom}(\Lambda^I, 2\pi\mathbb{Z})$ (where $\Lambda^I = \operatorname{Ker}(\exp : \mathbf{t} \to T)$ is the integer lattice). This definition of roots differs by a factor of $2\pi$ from the definition used in [17]: there, the roots $\gamma$ satisfy $\gamma(\Lambda^I) \subset 2\pi\mathbb{Z}$. This is why the constant $C^K$ differs from that of [17] by a factor of $(2\pi)^{s-l}$.

The above formula was called a residue formula in [17] because the quantity Res (whose general definition was given in Section 8 of [17]) can be expressed as a multivariable residue (or alternatively in terms of iterated 1-variable residues).

Here we shall make particular use of the case where $K$ has rank 1, for which the results are as follows.

**Corollary 3.2.** *In the situation of Theorem 3.1, let $K = U(1)$. Then*

$$\eta_0 e^{i\omega_0}[M_{\rm red}] = n_0 \mathrm{Res}_{X=0}\Big( \sum_{F \in \mathcal{F}_+} h_F^\eta(X) \Big).$$

*Here, the meromorphic function $h_F^\eta$ on $\mathbb{C}$ was defined by (18), and $\mathrm{Res}_{X=0}$ denotes the coefficient of $1/X$, where $X \in \mathbb{R}$ has been identified with $2\pi i X \in \mathbf{k}$. The set $\mathcal{F}_+$ is defined by $\mathcal{F}_+ = \{F \in \mathcal{F} : \mu_T(F) > 0\}$. The integer $n_0$ is as in Theorem 3.1.*

**Corollary 3.3. (cf. [17], Corollary 8.2)** *In the situation of Theorem 3.1, let $K = SU(2)$. Then*

$$\eta_0 e^{i\omega_0}[M_{\rm red}] = -\frac{n_0}{2} \mathrm{Res}_{X=0}\Big( (2X)^2 \sum_{F \in \mathcal{F}_+} h_F^\eta(X) \Big).$$

*Here, $\mathrm{Res}_{X=0}$, $h_F^\eta$ and $\mathcal{F}_+$ are as in Corollary 3.2, and $X \in \mathbb{R}$ has been identified with $\mathrm{diag}(2\pi i, -2\pi i)X \in \mathbf{t}$. The integer $n_0$ is as in Theorem 3.1.*

*Remark:* Notice that we shall not only be considering the reduced space $M_{\rm red} = \mu^{-1}(0)/K$ with respect to the action of the nonabelian group $K$, but also $\mu^{-1}(0)/T$ and $M_{\rm red}^T(t) = \mu_T^{-1}(t)/T$ for regular values $t$ of the $T$-moment map $\mu_T$ which is the composition of $\mu$ with restriction from $\mathbf{k}^*$ to $\mathbf{t}^*$. We shall use the same notation $\eta_0$ for the image of $\eta$ under the surjective homomorphism $\Phi$ for whichever of the spaces $\mu^{-1}(0)/K$, $\mu^{-1}(0)/T$ or $\mu_T^{-1}(0)/T$ we are working with, and the notation $\eta_t$ if we are working with $\mu_T^{-1}(t)/T$. It should be clear from the context which version of the map $\Phi$ is being used.

We shall need the following very recent additional results, which are expressed in the above notation:

**Proposition 3.4.** *(Reduction to the abelian case) (*S. Martin [21]*) If $T$ is a maximal torus of $K$, then we have that*

$$\int_{\mu_K^{-1}(0)/K} (\eta e^{i\bar\omega})_0 = \frac{1}{|W|} \int_{\mu_K^{-1}(0)/T} (\mathcal{D}\eta e^{i\bar\omega})_0 = \frac{(-1)^{n+}}{|W|} \int_{\mu_T^{-1}(t)/T} (\mathcal{D}^2 \eta e^{i\bar\omega})_t$$

*for any regular value $t$ of $\mu_T$ sufficiently close to 0.*

**Proposition 3.5.** *(Dependence of symplectic quotients on parameters) (*Guillemin-Kalkman [11] ; S. Martin [21]*) Consider a symplectic manifold $M$ acted on in a Hamiltonian fashion by $T = U(1)$ and let $\mu$ denote the moment map for this action. Let $t_0 < t_1$ be two regular values of $\mu$. Then we may subtract $t_i$ from the moment map to get a modified symplectic quotient $M_{\rm red}^T(t_i) = \mu^{-1}(t_i)/T$ for $i = 0, 1$ with maps*

$$\Phi : H_T^*(M) \to H^*(M_{\rm red}^T(t_i))$$

*sending $\eta$ to $(\eta)_{t_i}$. We then have*

$$\int_{M_{\rm red}^T(t_0)} (\eta e^{i\bar\omega})_{t_0} - \int_{M_{\rm red}^T(t_1)} (\eta e^{i\bar\omega})_{t_1} = n_0 \sum_{F \in \mathcal{F} : t_0 < \mu(F) < t_1} \mathrm{Res}_{X=0} e^{i\mu(F)(X)} \int_F \frac{\eta(X) e^{i\omega}}{e_F(X)}.$$

*Here, $X \in \mathbb{C}$ has been identified with $2\pi i X \in \mathbf{t} \otimes \mathbb{C}$.*

**Corollary 3.6.** *If $t_1 \in \mathbf{t}$ is close enough to $t_0 \in \mathbf{t}$ that there is a path between $t_0$ and $t_1$ in $\mu(M)$ consisting entirely of regular values of $\mu$, then*

$$\int_{M^T_{\mathrm{red}}(t_0)} (\eta e^{i\bar{\omega}})_{t_0} = \int_{M^T_{\mathrm{red}}(t_1)} (\eta e^{i\bar{\omega}})_{t_1}.$$

*Remark:* In fact both the equality

$$\int_{\mu_K^{-1}(0)/K} (\eta e^{i\bar{\omega}})_0 = \frac{(-1)^{n_+}}{|W|} \int_{\mu_T^{-1}(t)/T} (\mathcal{D}^2 \eta e^{i\bar{\omega}})_t$$

of Proposition 3.4 and the residue formula of Proposition 3.5 can be deduced easily from the residue formula of [17] when $M$ is a compact symplectic manifold. However the proofs of Propositions 3.4 and 3.5 can be adapted to apply in certain circumstances when $M$ is not compact and the residue formula of [17] is not valid. This will be crucial later.

## 4. Extended moduli spaces

To make progress we invoke a description of a symplectic space $M(c)$ equipped with a Hamiltonian action of $K = SU(n)$ such that the symplectic quotient of $M(c)$ at 0 is $\mathcal{M}(n,d)$, and explain our general strategy for obtaining (9) and its generalizations. The moduli space $\mathcal{M}(n,d)$ was described by Atiyah and Bott [1] as the symplectic reduction of an infinite dimensional symplectic vector space $\mathcal{A}$ with respect to the action of an infinite dimensional group $\mathcal{G}$ (the *gauge group*).[6] We however shall exhibit $\mathcal{M}(n,d)$ as the symplectic quotient of a *finite dimensional* symplectic space $M(c)$ by the Hamiltonian action of a finite dimensional group $K$. In this case the group $K$ is $SU(n)$. One characterization of the space $M(c)$ is that it is the symplectic reduction of the infinite dimensional affine space $\mathcal{A}$ by the action of the *based* gauge group $\mathcal{G}_0$ (which is the kernel of the evaluation map $\mathcal{G} \to K$ at a prescribed basepoint: see [14]). Now if a compact group $G$ containing a closed normal subgroup $H$ acts in a Hamiltonian fashion on a symplectic manifold $Y$, then one may "reduce in stages": the space $Y//H = \mu_H^{-1}(0)/H$ has a residual Hamiltonian action of the quotient group $G/H$ with moment map $\mu_{G/H} : Y//H \to (\mathfrak{g}/\mathfrak{h})^*$, and $\mu_G^{-1}(0)/G$ is naturally identified as a symplectic manifold with $\mu_{G/H}^{-1}(0)/(G/H)$. Similarly $M(c)$ has a Hamiltonian action of $\mathcal{G}/\mathcal{G}_0 \cong K$, and the symplectic reduction with respect to this action is identified with the symplectic reduction of $\mathcal{A}$ with respect to the full gauge group $\mathcal{G}$.

A more concrete (and entirely finite dimensional) characterization of $M(c)$ is given in [14]. The space is defined by

$$M(c) = (\varepsilon_r \times e_c)^{-1}(\triangle) \subset \mathrm{Hom}(\mathbb{F}, K) \times \mathbf{k}. \tag{21}$$

Here, we identify $\mathrm{Hom}(\mathbb{F}, K)$ with $K^{2g}$ through a choice of generators $\{x_1, \ldots, x_{2g}\}$ for the free group $\mathbb{F}$ on $2g$ generators; then $\varepsilon_r : \mathrm{Hom}(\mathbb{F}, K) \to K$ is the evaluation map on the relator $r = \prod_{j=1}^g [x_{2j-1}, x_{2j}]$

$$\varepsilon_r(h_1, \ldots, h_{2g}) = \prod_{j=1}^g [h_{2j-1}, h_{2j}]. \tag{22}$$

[6] To obtain his generating functionals, Witten formally applied his version of nonabelian localization to the infinite dimensional space $\mathcal{A}$.

The map $e_c : \mathbf{k} \to K$ is defined by

$$e_c(Y) = c\exp(Y), \tag{23}$$

where the generator $c$ of the centre of $K$ was defined at (10) above. The diagonal in $K \times K$ is denoted $\triangle$. The space $M(c)$ then has canonical projection maps $\mathrm{pr}_1, \mathrm{pr}_2$ which make the following diagram commute:

$$\begin{array}{ccc} M(c) & \xrightarrow{\mathrm{pr}_2} & \mathbf{k} \\ \mathrm{pr}_1 \downarrow & & \downarrow e_c \\ \mathrm{Hom}(\mathbb{F}, K) & \xrightarrow{\varepsilon_r} & K \end{array} \tag{24}$$

In other words, $M(c)$ is the fibre product of $\mathrm{Hom}(\mathbb{F}, K)$ and $\mathbf{k}$ under the maps $\varepsilon_r$ and $e_c$. The action of $K$ on $M(c)$ is given by the adjoint actions on $K$ and $\mathbf{k}$. The space $M(c)$ has the following properties (see [14] and [15]):

**Proposition 4.1.** **(a)** *The space $M(c)$ is smooth near all $(h, \Lambda) \in \mathrm{Hom}(\mathbb{F}, K) \times \mathbf{k}$ for which the linear space $z(h) \cap \ker(d\exp)_\Lambda \neq \{0\}$. Here, $z(h)$ is the Lie algebra of the stabilizer $Z(h)$ of $h$.*

**(b)** *There is a $K$-invariant 2-form $\omega$ on $\mathrm{Hom}(\mathbb{F}, K) \times \mathbf{k}$ whose restriction to $M(c)$ is closed and which defines a nondegenerate bilinear form on the Zariski tangent space to $M(c)$ at every $(h, \Lambda)$ in an open dense set in $M(c)$ which includes the subset of $M(c)$ where $\Lambda = 0$. (Thus the form $\omega$ gives rise to a symplectic structure on an open dense subset of $M(c)$.)*

**(c)** *With respect to the symplectic structure given by the 2-form $\omega$, a moment map $\mu : M(c) \to \mathbf{k}^*$ for the action of $K$ on $M(c)$ is given by $\sigma \mathrm{pr}_2$, where $\mathrm{pr}_2 : M(c) \to \mathbf{k}$ is the projection map to $\mathbf{k}$ (composed with the canonical isomorphism $\mathbf{k} \to \mathbf{k}^*$ given by the invariant inner product on $\mathbf{k}$) and $\sigma = -2$.*

**(d)** *The space $M(c)$ is smooth in a neighbourhood of $\mu^{-1}(0)$.*

**(e)** *The symplectic quotient $M_{\mathrm{red}} = \mu^{-1}(0)/K$ can be naturally identified with $Y_c/K = \mathcal{M}(n, d)$.*

Using our description of $M(c)$ as a fibre product, it is easy to identify the components $F$ of the fixed point set of the action of $T$. We examine the fixed point sets of the action of $T$ on $\mathrm{Hom}(\mathbb{F}, K)$ and $\mathbf{k}$ and we find

$$\begin{array}{ccc} M(c)^T & \xrightarrow{\mathrm{pr}_2} & \mathbf{t} \\ \mathrm{pr}_1 \downarrow & & \downarrow e_c \\ \mathrm{Hom}(\mathbb{F}, T) & \xrightarrow{\varepsilon_r} & 1 \in T \end{array} \tag{25}$$

(Notice that $\varepsilon_r$ sends $\mathrm{Hom}(\mathbb{F}, T)$ to 1 because $T$ is abelian.) Thus

$$M(c)^T = \mathrm{Hom}(\mathbb{F}, T) \times e_c^{-1}(1) = T^{2g} \times \{\delta - \tilde{c} : \ \delta \in \Lambda^I \subset \mathbf{t}\} \tag{26}$$

where $\tilde{c}$ is a fixed element of $\mathbf{t}$ for which $\exp \tilde{c} = c$. (Here, $\Lambda^I$ denotes the integer lattice $\mathrm{Ker}(\exp) \subset \mathbf{t}$.) If we ignore the singularities of $M(c)$, this description also enables us to identify the equivariant Euler class $e_{F_\delta}$ of the normal bundle of each component $T^{2g} \times (\delta - \tilde{c})$ in $M(c)^T$ (indexed by $\delta \in \Lambda^I$). This should be simply the equivariant Euler class of the normal bundle to $T^{2g}$ in $K^{2g}$, implying that $e_{F_\delta}$ is in fact independent of $\delta$ and is given by

$$e_{F_\delta}(X) = (-1)^{n+g} \mathcal{D}(X)^{2g}. \tag{27}$$

The symplectic volume of the component $F_\delta$ is independent of $\delta$ (indeed these components are all identified symplectically with $T^{2g}$): we denote the volume of $F_\delta$ by $\int_F \mathrm{vol}_\omega$.

The constant value taken by the moment map $\mu_T$ on the component $F = F_\delta$ is given by $\sigma(\delta - \tilde{c})$ where $\sigma = -2$ as in Proposition 4.1(c).

We shall need also the following property ([16]):

**Proposition 4.2.** *The generating classes $a_r$, $b_r^j$ and $f_r$ $(r = 2, \ldots, n,\ j = 1, \ldots, 2g)$ extend to classes $\tilde{a}_r(X)$, $\tilde{b}_r^j(X)$ and $\tilde{f}_r(X) \in H_K^*(M(c))$.*

Indeed, the equivariant differential form $\tilde{a}_r(X) \in \Omega_K^*(M(c))$ whose restriction represents the cohomology class $a_r \in H^*(\mathcal{M}(n,d))$ is simply the invariant polynomial $\tau_r \in S^r(\mathbf{k})^K \cong H_K^*(\mathrm{pt})$ which is associated to the $r$th Chern class.

Finally we shall need to work with the symplectic subspace $M_{\mathbf{t}}(c) = \mu_K^{-1}(\mathbf{t})$ of $M(c)$, which is no longer acted on by $K$ but is acted on by $T$. The space $M_{\mathbf{t}}(c)$ has an important periodicity property:

**Lemma 4.3.** *Suppose $\Lambda_0 \in \mathrm{Ker}(\exp) \subset \mathbf{t}$. Then there is a homeomorphism $s_{\Lambda_0}$ : $M_{\mathbf{t}}(c) \to M_{\mathbf{t}}(c)$ defined by*

$$s_{\Lambda_0} : (h, \Lambda) \mapsto (h, \Lambda + \Lambda_0).$$

## 5. Proof of the residue formula

Naïve application of the residue formula (Theorem 3.1) to the space $M(c)$, ignoring the fact that it is noncompact and has singularities, would thus yield

$$\begin{aligned}\prod_{r=2}^{n} a_r^{m_r} \exp(f_2)[\mathcal{M}(n,d)] &= \\ & n_0 C^K \mathrm{Res}\left(\mathcal{D}^2(X)\Big(\int_F \mathrm{vol}_\omega\Big) \sum_{\delta \in \Lambda^I} \frac{\prod_{r=2}^{n} \tau_r(X)^{m_r} e^{i(-\tilde{c}+\delta)(X)}}{(-1)^{n+g}\mathcal{D}^{2g}(X)}\right). \qquad (28)\end{aligned}$$

The main problem with (28) (related to the noncompactness of $M(c)$, which permits the fixed point set $M(c)^T$ to consist of infinitely many components $F_\delta$) is that the sum over $\delta$ does not converge for $X \in \mathbf{t}$. In this section we shall sketch a sequence of results that enable us nonetheless to conclude that (28) is true if interpreted appropriately. We shall concentrate mainly on the case when $n$ is 2 but the argument can be generalized to higher $n$; details will be given in a subsequent paper. (See Theorem 5.10 and the remarks following it.)

**Lemma 5.1.** *For generic $\xi \in \mathbf{t}$,*

$$\begin{aligned} M(c\exp(\xi)) \stackrel{\mathrm{def}}{=} & \Big\{(h_1, \ldots, h_{2g}, \Lambda) \in K^{2g} \times \mathbf{k} \ : \\ & \prod_{j=1}^{g} h_{2j-1} h_{2j} h_{2j-1}^{-1} h_{2j}^{-1} = c\exp(\xi)\exp(\Lambda)\Big\}\end{aligned}$$

*is a smooth manifold.*

*Proof:* We see that $M(c) = F^{-1}(c)$ where $F : K^{2g} \times \mathbf{k} \to K$ is defined by

$$F\Big(h_1, \ldots, h_{2g}, \Lambda\Big) = \prod_{j=1}^{g} h_{2j-1} h_{2j} h_{2j-1}^{-1} h_{2j}^{-1} \exp(-\Lambda).$$

By Sard's theorem, for generic $\xi \in \mathbf{k}$, $c\exp(\xi)$ is a regular value of $F$ (although $c$ itself cannot be a regular value of $F$ since $M(c)$ is not smooth, see [15],[16]). Moreover, since $F$ is $K$-equivariant, we may assume without loss of generality that $\xi \in \mathbf{t}$. $\square$

*Remark:* Note that when $\exp(\xi) = 1$ we recover the definition of $M(c)$.

**Lemma 5.2.** *Define* $M_{\mathbf{t}}(c\exp(\xi))) = \mu^{-1}(\mathbf{t}) \cap M(c\exp(\xi))$, *where* $\mu : K^{2g} \times \mathbf{k} \to \mathbf{k}$ *is defined by*

$$\mu(h_1, \ldots, h_{2g}, \Lambda) = \sigma\Lambda$$

*(see Proposition 4.1(c)) and* $\mu_T$ *is the projection of* $\mu$ *onto* $\mathbf{t}$. *Then there is a* $T$*-equivariant homeomorphism* $s_\xi : M_{\mathbf{t}}(c\exp(\xi)) \to M_{\mathbf{t}}(c)$ *which sends* $\mu_T^{-1}(0) \cap M_{\mathbf{t}}(c\exp(\xi))$ *to* $\mu_T^{-1}(\sigma\xi) \cap M_{\mathbf{t}}(c)$.

*Proof:* One may define the homeomorphism as follows:

$$s_\xi : (h_1, \ldots, h_{2g}, \Lambda) \to (h_1, \ldots, h_{2g}, \Lambda + \xi). \qquad \square$$

**Lemma 5.3.** *We have*

$$\int_{M_{\mathrm{red}}} \Phi(\eta e^{i\bar{\omega}}) = \frac{1}{|W|} \int_N s_\xi^* \Phi(\mathcal{D}\eta e^{i\bar{\omega}})$$

*where*

$$N = \mu_T^{-1}(0) \cap M_{\mathbf{t}}(c\exp(\xi))/T$$

*and* $|W| = n!$ *is the order of the Weyl group of* $K = SU(n)$.

*Proof:* We first identify $\int_{M_{\mathrm{red}}} \Phi(\eta e^{i\bar{\omega}})$ with

$$\frac{1}{|W|} \int_{\mu^{-1}(0)/T} \Phi(\mathcal{D}\eta e^{i\bar{\omega}}) = \frac{1}{|W|} \int_{\mu_T|_{M_{\mathbf{t}}(c)}^{-1}(0)/T} \Phi(\mathcal{D}\eta e^{i\bar{\omega}}),$$

via Proposition 3.4, whose proof can be made to work in this situation, even though $M(c)$ is noncompact and singular, because $\mu^{-1}(0)$ is compact and $M(c)$ is nonsingular in a neighbourhood of $\mu^{-1}(0)$. Then we observe that for sufficiently small $\xi \in \mathbf{t}$,

$$\int_{\mu_T|_{M_{\mathbf{t}}(c)}^{-1}(0)/T} \Phi(\mathcal{D}\eta e^{i\bar{\omega}}) = \int_{\mu_T|_{M_{\mathbf{t}}(c)}^{-1}(\sigma\xi)/T} \Phi(\mathcal{D}\eta e^{i\bar{\omega}})$$

(by Corollary 3.6). Finally we use the homeomorphism $s_\xi$ of Lemma 5.2 to establish an identification between $\mu_T|_{M_{\mathbf{t}}(c)}^{-1}(\sigma\xi)/T$ and $N = \mu_T|_{M_{\mathbf{t}}(c\exp(\xi))}^{-1}(0)/T$.$\square$

Let us now examine the behaviour of the images in $H_T^*(M_{\mathbf{t}}(c))$ of the generating classes $\tilde{a}_r(X), \tilde{b}_r^j(X), \tilde{f}_r(X) \in H_K^*(M(c))$ under pullback under the homeomorphisms $s_{\Lambda_0} : M_{\mathbf{t}}(c) \to M_{\mathbf{t}}(c)$ defined at Lemma 4.3. By abuse of language, we shall refer to these images also as $\tilde{a}_r(X), \tilde{b}_r^j(X)$ and $\tilde{f}_r(X)$. It follows from [16] that the classes $\tilde{a}_r(X)$ are the images in $H_K^*(M(c))$ of the polynomials $\tau_r \in H_K^* = S(\mathbf{k}^*)^K$. Moreover ([16], (8.18)) the classes $\tilde{b}_r^j(X) \in H_K^*(M(c))$ are of the form $\tilde{b}_r^j(X) = \mathrm{pr}_1^*(\tilde{b}_r^j(X))_1$ where $(\tilde{b}_r^j(X))_1 \in H_K^*(K^{2g})$ and $\mathrm{pr}_1 : M(c) \to K^{2g}$ is the projection in (24). It follows that $s_{\Lambda_0}^* \tilde{b}_r^j(X) = \tilde{b}_r^j(X)$ and $s_{\Lambda_0}^* \tilde{a}_r(X) = \tilde{a}_r(X)$.

Furthermore we see from (8.30) of [16] that $\tilde{f}_2(X)$ is of the form

$$\tilde{f}_2(X) = \mathrm{pr}_1^* f_2^1 + \langle \mu, X \rangle \tag{29}$$

where $f_2^1 \in H_K^*(K^{2g})$ and $\mu : M(c) \to \mathbf{k}$ is the moment map (which is $\sigma$ times the restriction to $M(c)$ of the projection $K^{2g} \times \mathbf{k} \to \mathbf{k}$: see Proposition 4.1(c)). It follows from the definition (29) of $\tilde{f}_2$ that for any $\Lambda_0$ in the integer lattice of $\mathbf{t}$ (the kernel of the exponential map),

$$s_{\Lambda_0}^* \tilde{f}_2(X) = \tilde{f}_2(X) + \sigma\langle \Lambda_0, X\rangle. \tag{30}$$

Note also that there is a commutative diagram of homeomorphisms

$$\begin{array}{ccc} M_{\mathbf{t}}(c\exp(\xi)) & \xrightarrow{s_\xi} & M_{\mathbf{t}}(c) \\ s_{\Lambda_0}\downarrow & & \downarrow s_{\Lambda_0} \\ M_{\mathbf{t}}(c\exp(\xi)) & \xrightarrow{s_\xi} & M_{\mathbf{t}}(c) \end{array} \tag{31}$$

For simplicity, we restrict from now on to $K = SU(2)$, and let $\Lambda_0 = 2\pi i m \mathrm{diag}(1,-1)$ (for $m \in \mathbb{Z}$) and identify $X \in \mathbb{R}$ with $2\pi i X \,\mathrm{diag}(1,-1) \in \mathbf{t}$.

**Lemma 5.4.** *Suppose $\eta$ is a polynomial in the $\tilde{a}_r(X)$. Then for any positive integer $m$ we have that*

$$\int_N s_\xi^* \Phi(\eta e^{i\bar{\omega}} e^{im\sigma X}) = \int_{M_{\mathbf{t}}(c\exp(\xi))\cap\mu_T^{-1}(\sigma m)/T} s_\xi^*\Phi(\eta e^{i\bar{\omega}})$$

$$= \int_N s_\xi^*\Phi(\eta e^{i\bar{\omega}}) + 2\,\mathrm{sgn}(\sigma) \sum_{F\in\mathcal{F},0<|\mu_T(F)|<|\sigma| m} \mathrm{Res}_{X=0}\Big(\int_F \frac{\eta(X)e^{i\bar{\omega}(X)}\gamma(X)}{e_F(X)}\Big).$$

*Here, $\mathcal{F}$ is the set of connected components of the fixed point set of the action of $T$ on $M(c)$ and if $F \in \mathcal{F}$ then $e_F$ denotes the equivariant Euler class of the normal bundle of $s_\xi^{-1}(F)$ in $M(c\exp\xi)$. Also $\gamma(X) = 2X$ denotes the positive root of $SU(2)$ and $\sigma = -2$ is the normalization constant resulting from the normalization of the symplectic form (see Proposition 4.1 (c)). The overall factor of 2 multiplying the sum over $\mathcal{F}$ results from the fact that the element $-1 \in U(1)$ acts trivially on $M(c)$.*

*Proof:* We see that

$$\int_{M_{\mathbf{t}}(c\exp(\xi))\cap\mu_T^{-1}(\sigma m)/T} s_\xi^*\Phi(\eta e^{i\bar{\omega}}) = \int_N s_\xi^*\Phi(s_m^*\eta e^{i\bar{\omega}})$$

$$= \int_N s_\xi^*\Phi(\eta e^{i\bar{\omega}} e^{im\sigma X})$$

by (30). By (26) the set of components of the fixed point set of the action of $T$ on $M(c\exp\xi)$ is $\{s_\xi^{-1}(F) : F \in \mathcal{F}\}$. Using the fact that the root $-\gamma(X) = -2X$ of $SU(2)$ represents the Poincaré dual of $M_{\mathbf{t}}(c\exp\xi)$ in $M(c\exp\xi)$, Guillemin and Kalkman's proof of Proposition 3.5 can be applied to compactly supported equivariant forms in a neighbourhood of $M_{\mathbf{t}}(c\exp(\xi)) \cap \mu_T^{-1}[\sigma m, 0]$ in the smooth manifold $M(c\exp\xi)$ to show that

$$-\mathrm{sgn}(\sigma)\Big(\int_{M_{\mathbf{t}}(c\exp(\xi))\cap\mu_T^{-1}(\sigma m)/T} s_\xi^*\Phi(\eta e^{i\bar{\omega}}) - \int_N s_\xi^*\Phi(\eta e^{i\bar{\omega}})\Big)$$

$$= -2 \sum_{F\in\mathcal{F},0<|\mu_T(F)|<|\sigma| m} \mathrm{Res}_{X=0}\Big(\int_F \frac{\eta(X)e^{i\bar{\omega}(X)}\gamma(X)}{e_F(X)}\Big).\square$$

**Corollary 5.5.** *For all $j \geq 0$,*

$$-\mathrm{sgn}(\sigma)\int_N s_\xi^*\ \Phi(X^j(e^{im\sigma X} - 1)\eta e^{i\bar{\omega}}) = \frac{(-1)^{g-1}}{2^{2g-2}} \sum_{k=1}^m \mathrm{Res}_{X=0} \frac{e^{i\sigma(k-1/2)X}\int_F \eta e^{i\omega}}{X^{2g-1-j}}.$$

*Proof:* This follows from Lemma 5.4 using the fact that $e_F(X) = (-1)^g \gamma(X)^{2g}$ where $\gamma(X) = 2X$ is the positive root of $SU(2)$.□

We shall encode this information by defining a formal Laurent series $L_\eta$ in an indeterminate $x$ as follows:

$$L_\eta(x) = \sum_{j \geq 0} \frac{\int_N s_\xi^* \Phi(\eta X^j e^{i\bar{\omega}})}{x^{j+1}}. \tag{32}$$

We appeal to the following elementary observation:

**Lemma 5.6.** *Suppose $f(x) = \sum_{j \in \mathbb{Z}} f_j x^j$ is a formal Laurent series in $x$. Then*

$$\sum_{j \geq 0} \frac{\mathrm{Res}_{X=0}(X^j f(X))}{x^{j+1}} = \mathcal{P}(f(x)),$$

*where $\mathcal{P}$ denotes the principal part.*

We have also

**Proposition 5.7.** *For any formal power series $g(x) = \sum_{j \geq 0} g_j x^j$ in $x$,*

$$\mathcal{P}(L_{g(X)\eta}(x)) = \mathcal{P}(g(x) L_\eta(x)).$$

*Proof:* This follows because for any $k \geq 0$,

$$L_{X^k \eta}(x) = \sum_{j \geq 0} \frac{\int_N s_\xi^* \Phi(\eta X^{k+j}) e^{i\bar{\omega}}}{x^{k+j+1}} x^k.$$

**Proposition 5.8.**

$$-\mathrm{sgn}(\sigma) \mathcal{P}((e^{i\sigma x} - 1) L_\eta) = \frac{(-1)^{g-1}}{2^{2g-2}} \mathcal{P} \frac{(\int_{T^{2g}} \eta e^{i\omega}) e^{i\sigma x/2}}{x^{2g-1}}.$$

*Proof:*

$$\mathcal{P}\Big((e^{i\sigma x} - 1) L_\eta(x)\Big) = \mathcal{P}\Big(L_{(e^{i\sigma X}-1)\eta}(x)\Big) \quad \text{by Proposition 5.7}$$

$$= \mathcal{P}\Big(\sum_{j \geq 0} x^{-j-1} \int_N s_\xi^* \Phi\Big((X^j \eta e^{i\bar{\omega}})(e^{i\sigma X} - 1)\Big) \quad \text{by definition of } L_\eta$$

so that

$$-\mathrm{sgn}(\sigma) \mathcal{P}\Big((e^{i\sigma x} - 1) L_\eta(x)\Big) = \frac{(-1)^{g-1}}{2^{2g-2}} \mathcal{P} \sum_{j \geq 0} x^{-j-1} \mathrm{Res}_{X=0}\Big(\frac{(\int_{T^{2g}} \eta(X) e^{i\omega}) e^{i\sigma X/2}}{X^{2g-1-j}}\Big)$$

(by Corollary 5.5 when $m = 1$ )

$$= \frac{(-1)^{g-1}}{2^{2g-2}} \mathcal{P} \, \frac{(\int_{T^{2g}} \eta(x) e^{i\omega}) e^{i\sigma x/2}}{x^{2g-1}} \quad \text{by Lemma 5.6. } \square$$

**Corollary 5.9.**

$$-\mathrm{sgn}(\sigma) L_\eta(x) = \frac{(-1)^{g-1}}{2^{2g-2}} \frac{\int_{T^{2g}} \eta e^{i\omega} e^{i\sigma x/2}}{(e^{i\sigma x} - 1) x^{2g-1}} + h(x)$$

*where $h$ is a formal Laurent series with at most a simple pole at 0.*

*Proof:* This follows immediately when we divide through the result of Proposition 5.8 by $(e^{i\sigma x}-1)$. □

Unwinding the definition of $L_\eta$ we have from Corollary 5.9 (provided that $k \geq 1$) that

$$\int_N s_\xi^* \Phi(X^k \eta e^{i\bar\omega}) = \mathrm{Res}_{x=0}\Big(x^k L_\eta(x)\Big)$$

$$= \frac{(-1)^g}{2^{2g-2}} \mathrm{Res}_{x=0} \frac{1}{2ix^{2g-1-k}\sin(|\sigma|x/2)} (\int_{T^{2g}} \eta(x) e^{i\omega}). \tag{33}$$

It follows finally that

**Theorem 5.10.**

$$\int_{M(2,1)} a_2^j e^{if_2} = \frac{(-1)^g}{2^{2g-1}i} \int_{T^{2g}} e^{i\omega} \mathrm{Res}_{X=0}\Big(\frac{1}{X^{2g-2-2j}\sin|\sigma|X/2}\Big),$$

*where the normalization is such that the class* $a_2 = \Phi(X^2)$.

*Proof:* This is true since

$$\int_{M(2,1)} a_2^j e^{if_2} = \int_{M(2,1)} \Phi\Big(X^{2j} e^{i\bar\omega}\Big)$$

(by the remarks in the paragraph above (29), and noting that $\tau_2(X) = X^2$)

$$= \int_N s_\xi^* \Phi(X^{2j+1} e^{i\bar\omega}) \text{ (by Lemma 5.3).}$$

(Notice that $\mathcal{D}(X) = \gamma(X) = 2X$ in our chosen normalization.) We then combine this fact with (33). □

A short calculation using the results of [16] (particularly the description of the symplectic form using the formulas (7.11)-(7.12) for the fundamental class) shows that

$$\int_{T^2} \omega = -2.$$

This gives $vol_\omega(T^{2g}) = (-1)^g 2^g$. So, recalling from Proposition 4.1(c) that $\sigma = -2$, we find finally

$$\int_{M(2,1)} a_2^j e^{if_2} = \frac{i^{g-1}}{2^{g-1}} \mathrm{Res}_{X=0}\Big(\frac{1}{X^{2g-2-2j}\sin X}\Big). \tag{34}$$

Thus we have recovered Witten's formula (see (37) below): this result was first obtained by Donaldson [7] and Thaddeus [24].

**Theorem 5.11.** *The intersection numbers of the classes $a_2$ and $f_2$ on $M(2,1)$ are given by*

$$\int_{M(2,1)} a_2^j e^{f_2} = \frac{(-1)^{g-1-j}}{2^{g-1}} \mathrm{Res}_{X=0}\Big(\frac{1}{X^{2g-2-2j}\sin X}\Big). \tag{35}$$

*Remark:* Notice that formally we can express the result of Theorem 5.10 as

$$\int_{\mathcal{M}(2,1)} a_2^j e^{if_2} = (-1)^g \mathrm{Res}_{X=0}\Big(\sum_{k\geq 0} \frac{(2X)^2 X^{2j} e^{(2k+1)i\sigma X/2} \int_{T^{2g}} e^{i\omega}}{(2X)^{2g}}\Big),$$

which is what we would expect from applying the residue formula of [17] formally to $M(c)$. However this sum does not converge in a neighbourhood of 0 and the sum of the residues at 0 does not converge.

Witten's result (using (4.29) and (4.44) of [26]) may be expressed in the following form for $j \leq g-2$:

$$\int_{M(2,1)} \Theta^j e^{\omega} = \frac{(-1)^{g-1-j}}{2^{g-2}\pi^{2(g-1-j)}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{2(g-1-j)}} \tag{36}$$

(taking into account orientations and the fact that $M(2,1)$ is a $2^{2g}$-fold cover of the moduli space Witten studies). To show this agrees with our result (35), we use the following fact which can be established by an elementary contour integral argument:

**Lemma 5.12.** *For any positive integer $m$, we have*

$$\sum_{n>0} \frac{(-1)^{n+1}}{n^{2m}} = \pi i \mathrm{Res}_{X=0} \frac{e^{i\pi X}}{X^{2m}(e^{2i\pi X}-1)} = \frac{\pi^{2m}}{2} \mathrm{Res}_{Y=0} \frac{1}{Y^{2m}\sin Y}.$$

Using Lemma 5.12, (36) becomes

$$\int_{M(2,1)} \Theta^j e^{\omega} = \frac{(-1)^{g-1-j}}{2^{g-1}} \mathrm{Res}_{Y=0} \frac{1}{Y^{2(g-1-j)}(\sin Y)}, \tag{37}$$

in agreement with (35).

The sum in (36) may be expressed in terms of zeta functions through the following fact which follows from an easy computation:

**Lemma 5.13.** *For any positive integer $m$, we have*

$$\sum_{n>0} \frac{(-1)^{n+1}}{n^{2m}} = (1 - \frac{1}{2^{2m-1}})\zeta(2m).$$

*Remark:* The proof of Theorem 5.10 given above can be generalized to give formulas for intersection pairings involving the $b_2^j$ as well as $a_2$ and $f_2$. These were already treated in Thaddeus' work [24].

*Remark:* Lemma 5.12 establishes directly that for rank 2 and degree 1 the formula (13) obtained by Witten (expressing intersection numbers in terms of a sum over irreducible representations of $SU(2)$) may be rewritten as the residue that appears in our proof. We can generalize this argument to cover $n$ higher than 2, getting formulas involving iterated 1-variable residues similar to the residue in Theorem 5.10. Details will appear in a later paper [18]. For general coprime $n$ and $d$ the work of Szenes [23] establishes the equivalence of the sum (9) obtained by Witten and the residue (28) which appears in our proof. Szenes' proof is a multidimensional generalization of the contour integral argument that establishes Lemma 5.12.

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