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FLOWS ON S^3 SUPPORTING ALL LINKS AS ORBITS

ROBERT W. GHRIST

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ABSTRACT. We construct counterexamples to some conjectures of J. Birman and R. F. Williams concerning the knotting and linking of closed orbits of flows on 3-manifolds. By establishing the existence of "universal templates," we produce examples of flows on S^3 containing closed orbits of all knot and link types simultaneously. In particular, the set of closed orbits of any flow transverse to a fibration of the complement of the figure-eight knot in S^3 over S^1 contains representatives of every (tame) knot and link isotopy class. Our methods involve semiflows on branched 2-manifolds, or *templates*.

In this announcement, we answer some questions raised by Birman and Williams in their original examination of the link of closed orbits in the flow on S^3 induced by the fibration of the complement of a fibred knot or link [4]. In their work, they proposed the following conjecture:

Conjecture 0.1 (Birman and Williams, 1983). *The figure-eight knot (K_8) does not appear as a closed orbit of the flow induced by the fibration of the complement of the figure-eight knot.*

By "the" flow is meant the flow obtained by integrating the gradient of p, where $p : S^3 \setminus K_8 \to S^1$ is the unique fibration over S^1 whose monodromy is the pseudo-Anosov representative of its isotopy class, with respect to the Nielsen-Thurston classification [16, 9].

We have resolved this conjecture in the negative and, in so doing, have discovered interesting examples of flows on 3-manifolds and semiflows on branched 2-manifolds. More specifically, in Theorem 3.1, we show that any fibration of the complement of the figure-eight knot in S^3 over S^1 induces a flow on S^3 containing every tame knot and link as closed orbits. We include only those definitions and results which are relevant for this announcement, leaving details to a separate work [11].

1. TEMPLATE THEORY

Periodic orbits of a flow are embedded circles. When the flow is three-dimensional, periodic orbits are knots and the collection of periodic orbits forms a link which often is nontrivial. A valuable tool for examining knotted periodic orbits in three dimensional flows is the template construction of Birman and Williams [3, 4].

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Definition 1.1. A *template* (also known as a *knotholder*) is a compact branched two-manifold with boundary and smooth expanding semiflow built from a finite number of *branch line charts*, as given in Figure 1(a).

For a more detailed definition, along with examples, see [3, 4]. Each *branch line* of a template appears (locally) as in Figure 1(a): there will be a certain number (≥ 2) of incoming strips which completely cover the branch line (expanding the incoming semiflow), and a certain number (≥ 2) of outgoing strips. The incoming and outgoing strips of all the branch lines are then connected bijectively. In Figure 1(b), we display the simplest example of a template: the Lorenz template [3], consisting of one branch line and two strips.

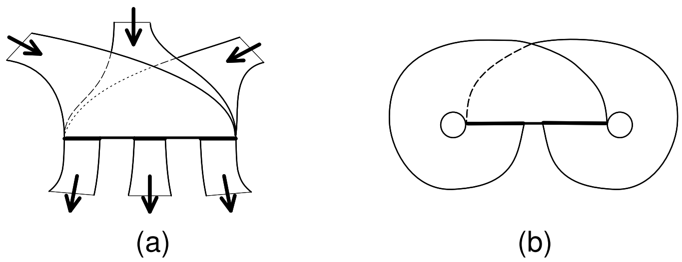

FIGURE 1. (a) Strips meet at a branch line; (b) the (embedded) Lorenz template.

Upon embedding a template in S^3, the periodic orbits of the semiflow form links. The relationship between embedded templates and links of periodic orbits in three dimensional flows is expressed in the Template Theorem of Birman and Williams [4, 3]:

Theorem 1.2 (Birman and Williams, 1983). *Given a flow on a 3-manifold M having a hyperbolic chain-recurrent set (i.e., Axiom A plus no-cycle), the link of periodic orbits is in bijective correspondence with the link of periodic orbits on a particular embedded template $\mathcal{T} \subset M$ (with at most two exceptions). On any finite sublink, this correspondence is via ambient isotopy.*

Aside from their relevance to the dynamics of flows, templates are in their own right a fascinating class of objects. Birman and Williams [3, 4], Holmes and Williams [13], M. Sullivan [14], and others have discovered a number of remarkable properties of templates and template knots. One outstanding conjecture about templates was posed in [4] and revisited in [14]:

Conjecture 1.3 (Birman and Williams, 1983). *There does not exist an embedded template which supports all (tame) knots as periodic orbits of the semiflow: i.e., a* universal template.

We will discuss the resolution of Conjecture 1.3 in §2 and its relation to Conjecture 0.1 in §3.

Our primary focus in the investigation of templates is on *subtemplates*:

Definition 1.4. A *subtemplate* $\mathcal{S}$ of a template $\mathcal{T}$ is a subset of $\mathcal{T}$ which, under the induced semiflow, is itself a template: we write $\mathcal{S} \subset \mathcal{T}$.

Of particular interest to us are subtemplates which are diffeomorphic to their "parents," for they induce a map from the template to the subtemplate:

Definition 1.5. A *template inflation* of a template $\mathcal{T}$ is a map $\mathbf{R} : \mathcal{T} \hookrightarrow \mathcal{T}$ taking orbits to orbits which is a diffeomorphism onto its image.

Remark 1.6. The image of a template inflation $\mathbf{R} : \mathcal{T} \hookrightarrow \mathcal{T}$ is a subtemplate.

For $\mathcal{T}$ an embedded template, a template inflation $\mathbf{R} : \mathcal{T} \hookrightarrow \mathcal{T}$ induces a topological action on periodic orbits of $\mathcal{T}$. Inflations which preserve the isotopy class of the link of periodic orbits are of particular importance.

Definition 1.7. Let $\mathbf{R} : \mathcal{T} \hookrightarrow \mathcal{T}$ be an inflation of a template $\mathcal{T} \subset S^3$, and let $i_{\mathcal{T}}$ denote inclusion of $\mathcal{T}$ into S^3. If $i_{\mathcal{T}}$ and $i_{\mathcal{T}} \circ \mathbf{R}$ are isotopic embeddings of $\mathcal{T}$ in S^3, then $\mathbf{R}$ is an *isotopic inflation*.

Remark 1.8. The image of an isotopic inflation $\mathbf{R}$ is a subtemplate $\mathbf{R}(\mathcal{T})$ isotopic in S^3 to $\mathcal{T}$.

Example 1.9. The first example of template containing an isotopic subtemplate was given by M. Sullivan [14], for a template $\mathcal{V}$, pictured in Figure 2(a). In Figures 2(b) and 2(c) we show a subtemplate within $\mathcal{V}$ and removed from $\mathcal{V}$ respectively. The reader may verify that the template of Figure 2(c) is isotopic to $\mathcal{V}$.

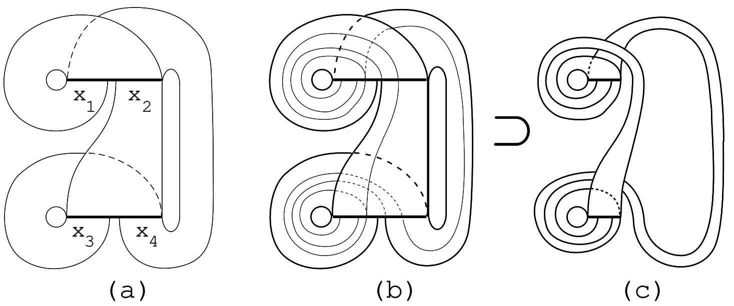

FIGURE 2. (a) The template $\mathcal{V}$; (b) a subtemplate of $\mathcal{V}$; (c) removed from $\mathcal{V}$.

Our primary tool in the analysis of templates is the application of symbolic dynamics [4, 13]. By crushing out the transverse direction of the semiflow, a template $\mathcal{T}$ becomes a directed graph; hence, orbits on the template $\mathcal{T}$ may be placed in a bijective correspondence with the space of symbolic itineraries $\Sigma_{\mathcal{T}}$ in a subshift of finite type [7].

With this symbolic (Markov) structure, template inflations can be represented as a symbolic action on the associated Markov partition: each symbol in the sequence is inflated to a particular word. This allows for the efficient computation of very "deep" subtemplates through composing the inflations and their associated symbolic actions.

Example 1.10. In Example 1.9, the subtemplate of $\mathcal{V}$ induces an isotopic template inflation $\mathbf{D} : \mathcal{V} \hookrightarrow \mathcal{V}$ which in turn induces an action on $\Sigma_{\mathcal{V}}$ given by,

$$\mathbf{D} : \Sigma_{\mathcal{V}} \hookrightarrow \Sigma_{\mathcal{V}} \qquad \begin{cases} x_1 \mapsto x_1 \\ x_2 \mapsto x_1 x_2 \\ x_3 \mapsto x_3 \\ x_4 \mapsto x_3 x_4 \end{cases}, \tag{1.1}$$

where $\{x_i : i = 1 \ldots 4\}$ is the Markov partition for $\mathcal{V}$ represented in Figure 2(a).

Our strategy involves using combinations of isotopic inflations along with the induced actions on symbol sequences. This allows us to keep track of the topology of orbits on complicated subtemplates, while also permitting symbolic "coordinates" for tracking deeply embedded orbits.

2. Universal Templates

The following result resolves Conjecture 1.3 in the negative:

Theorem 2.1. *The template $\mathcal{V}$ of Example 1.9 contains an isotopic copy of every (tamely embedded) knot and link as periodic orbits of the semiflow.*

Idea of proof: Consider the templates $\mathcal{W}_q$, $q > 0$, pictured in Figure 3. These templates are embedded q-fold covers of $\mathcal{V}$ which have an alternating sequence of $2q$ "ears".

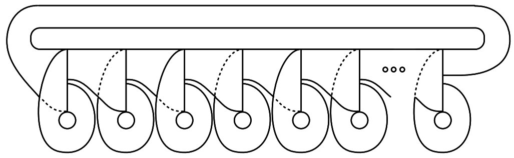

Figure 3. The template $\mathcal{W}_q$ is a q-fold cover of $\mathcal{V}$.

Lemma 2.2. *Let $b \in B_n$ be a braid on n-strands. Then $\bar{b}$, the closure of b, appears as a (set of) periodic orbit(s) on $\mathcal{W}_q$ for sufficiently large $q < \infty$.*

Proof. The concatenation of alternating positive and negative ears on $\mathcal{W}_q$ mimics the group operation of B_n. It is simple to find a generating set for B_n, each element of which "fits" on a finite concatenation of alternating ears, as occurs in $\mathcal{W}_q$. □

Given $\mathcal{T}$ a subtemplate of $\mathcal{V}$, we can consider the space of orbit symbol sequences $\Sigma_{\mathcal{T}} \subset \Sigma_{\mathcal{V}}$. By specifying a form of "symbolic surgery" on $\Sigma_{\mathcal{T}}$, we modify $\mathcal{T}$ to create a new subtemplate $\mathcal{T}^+$ which has an additional "ear": the topological action of this surgery is depicted in Figure 4.

Proposition 2.3. *The template $\mathcal{W}_q$ appears as a subtemplate of $\mathcal{V}$ for all $q > 0$.*

Proof. The technical part of the proof is to show that we can append ears to a copy of $\mathcal{V}$ in alternating fashion. To do so, we first must map $\mathcal{V}$ into itself in such a way as to avoid certain edges of $\mathcal{V}$, as suggested in Figure 4. Then, after appending the appropriate ears, we send this latter $\mathcal{V}$ back into itself in such a way that we may iterate the procedure. The procedure is diagrammatically represented in

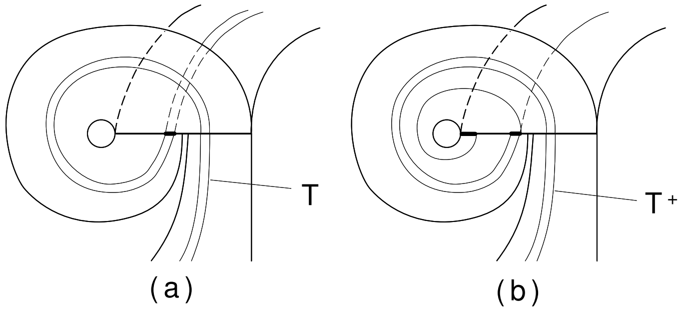

FIGURE 4. (a) A closeup of the top ear of $\mathcal{V}$ and a subtemplate $\mathcal{T}$; (b) appending an ear to $\mathcal{T}$ yields $\mathcal{T}^+$.

Equation (2.1). This step depends greatly on our ability to work with subtemplates symbolically.

$$\begin{array}{ccccccccc} \mathcal{W}_1 & \xrightarrow{\text{append}} & \mathcal{W}_2 & \xrightarrow{\text{append}} & \mathcal{W}_3 & \xrightarrow{\text{append}} & \mathcal{W}_4 & \xrightarrow{\text{append}} & \\ \downarrow = & & \downarrow \subset & & \downarrow \subset & & \downarrow \subset & & \cdots \\ \mathcal{V} & \hookrightarrow & \mathcal{V} & \hookrightarrow & \mathcal{V} & \hookrightarrow & \mathcal{V} & \hookrightarrow & \end{array} \tag{2.1}$$

$\square$

Proof of Theorem 2.1. By Lemma 2.2, we can find any closed braid as a periodic orbit set on some $\mathcal{W}_q$. By Proposition 2.3, every $\mathcal{W}_q \subset \mathcal{V}$; hence, $\mathcal{V}$ contains all closed braids. By a theorem of Alexander (see [1, 5]), $\mathcal{V}$ contains all knots and links. $\square$

Remark 2.4. The proof is constructive, but does not necessarily yield the "simplest" version of a closed braid in $\mathcal{V}$. Consider Conjecture 0.1 concerning the existence of a figure-eight knot (K_8) in a particular flow. A careful attempt to draw K_8 on $\mathcal{V}$ will frustrate the reader. By computing symbol sequences, we calculate a representation of the figure-eight knot in $\mathcal{V}$ (the simplest known example) which crosses the branch lines $11,358,338$ times (i.e., this is the minimal period of the itinerary). The symbolic methods used in these proofs extract very deep information.

By employing a version of Alexander's Theorem due to Franks and Williams [10] along with our techniques, we can extend Theorem 2.1:

Theorem 2.5. *The template $\mathcal{V}$ contains all orientable templates as subtemplates. These may be chosen to be disjoint and completely unlinked.*

This hints at a classification of (orientable) templates: a template $\mathcal{T}$ is *universal* $\Leftrightarrow$ $\mathcal{T}$ contains $\mathcal{V}$ $\Leftrightarrow$ $\mathcal{T}$ contains all templates.

3. Fibrations of Knot Complements

Besides applications to template theory, Theorem 2.1 has implications for flows induced by fibrations of knot and link complements whose monodromies are of pseudo-Anosov type (in the Thurston classification [16, 9]). The complement of the figure-eight knot in S^3 is fibred with fibre a punctured torus and monodromy isotopic to the pseudo-Anosov map [15]

$$\Phi = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} : \mathbb{R}^2 \setminus \mathbb{Z}^2 \to \mathbb{R}^2 \setminus \mathbb{Z}^2.$$

The closed orbits of the induced flow (the embedded suspension of Φ) are the subject of [4] and of Conjecture 0.1, which we now resolve.

Theorem 3.1. *Any transverse flow to a fibration of the complement of the figure-eight knot in S^3 over S^1 contains every tame knot and link as closed orbits.*

Proof. In [4], Birman and Williams derived a template for the fibration-induced flow on the complement of the figure-eight knot. In [14], it was shown that this template contains $\mathcal{V}$ as a subtemplate. By Theorem 2.1, the figure-eight template is universal. This corresponds to the flow induced by the gradient of the particular fibration which has pseudo-Anosov monodromy; however, pseudo-Anosov maps minimize dynamics, so any other flow transverse to a fibration in the isotopy class has at least the periodic orbit link that the pseudo-Anosov case has [2, 16]. □

Remark 3.2. As in Remark 2.4, the simplest-known copy of the figure-eight knot which appears as a closed orbit of the flow induced by the monodromy has period $11,358,338$ in the monodromy.

Knowing which fibred knots (links) do *not* support every link as closed orbits of the fibration-induced flow would be useful, since, for fibred knots (links), the [minimal] link of closed orbits of the fibration-induced flow forms an invariant for the knot (link)[4]. We have found, using branched covering techniques of Birman [6], an infinite family of fibred links which support a universal template in the induced flow on the complement:

Proposition 3.3. *The closure of any braid of the form $\left(\sigma_1\sigma_2^{-1}\right)^k$ for $|k| > 1$ is a fibred link with fibration whose induced flow on the complement supports all knots and links as closed orbits. In particular, the Borromean rings ($k = 3$) shares this property.*

Remark 3.4. Since pseudo-Anosov maps have dense periodic point sets, the flows induced on the complements of these closed braids can be chosen so that the closed orbits fill up S^3 densely.

4. Third-Order ODEs

There are other implications which we do not describe in this announcement, mostly in connection with third-order ODEs and global bifurcations of periodic orbits in parametrized flows. Notable among them is the following, explored in detail in [12]:

Theorem 4.1. *There exists an open set of parameters $\beta \in [6.5, 10.5]$ for which periodic solutions to the differential equation*

$$\begin{aligned} \dot{x} &= 7[y - \phi(x)], \\ \dot{y} &= x - y + z, \\ \dot{z} &= -\beta y, \\ \phi(x) &= \frac{2}{7}x - \frac{3}{14}\left[|x+1| - |x-1|\right], \end{aligned} \tag{4.1}$$

contain representatives from every knot and link equivalence class.

Equation (4.1) is a PL-vector field modeling an electric circuit [8]. This is but one example of a class of ODEs supporting a particular type of homoclinic connection which induces the existence of a universal template within the flow.

References

1. J. W. Alexander, *A lemma on systems of knotted curves*, Proc. Nat. Acad. Sci. USA **9** (1923), 93–95.
2. D. Asimov and J. Franks, *Unremovable closed orbits*, Geometric Dynamics, Lecture Notes in Mathematics 1007 (J. Palis, ed.), Springer-Verlag, 1983.
3. J. Birman and R. F. Williams, *Knotted periodic orbits in dynamical systems–I : Lorenz's equations*, Topology **22(1)** (1983), 47–82.
4. ———, *Knotted periodic orbits in dynamical systems–II : knot holders for fibered knots*, Cont. Math. **20** (1983), 1–60.
5. J. S. Birman, *Braids, links, and mapping class groups*, Princeton University Press, Princeton, N.J., 1974.
6. ———, *On the construction of fibred knots and their monodromy maps*, Topology of Low-Dimensional Manifolds, Lecture Notes in Mathematics 722 (R. Fenn, ed.), Springer-Verlag, 1979.
7. R. Bowen, *On Axiom A diffeomorphisms*, Regional Conference Series in Mathematics 35.
8. L. Chua, M. Komuro, and T. Matsumoto, *The double scroll family*, IEEE Trans. on Circuits and Systems **33** (1986), 1073–1118.
9. A. Fathi, F. Laudenbach, and V. Poenaru et al., *Travaux de Thurston sur les surfaces*, Astérique **66-67** (1979), 1–284.
10. J. Franks and R. F. Williams, *Entropy and knots*, Trans. Am. Math. Soc. **291(1)** (1985), 241–253.
11. R. Ghrist, *Branched two-manifolds supporting all links*, Submitted for publication, December 1994.
12. R. Ghrist and P. Holmes, *An ODE whose solution contains all knots and links*, To appear in *Intl. J. Bifurcation and Chaos*, 1995.
13. P. J. Holmes and R. F. Williams, *Knotted periodic orbits in suspensions of Smale's horseshoe: torus knots and bifurcation sequences*, Archive for Rational Mech. and Anal. **90(2)** (1985), 115 –193.
14. M. C. Sullivan, *The prime decomposition of knotted periodic orbits in dynamical systems*, J. Knot Thy. and Ram. **3(1)** (1994), 83–120.
15. W. P. Thurston, *Three dimensional manifolds, Kleinian groups, and hyperbolic geometry*, Bull. Am. Math. Soc. **6(3)** (1982), 357–381.
16. ———, *On the geometry and dynamics of diffeomorphisms of surfaces*, Bull. Am. Math. Soc. **19(2)** (1988), 417–431.

Center for Applied Mathematics, Cornell University, Ithaca NY, 14853

Current address: Program in Applied and Computational Mathematics, Princeton University, Princeton NJ, 08544-1000; Institute for Advanced Study, Princeton NJ, 08540

E-mail address: rwghrist@math.princeton.edu; robg@math.ias.edu