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THE COHOMOLOGICAL EQUATION FOR AREA-PRESERVING FLOWS ON COMPACT SURFACES

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ABSTRACT. We study the equation $Xu = f$ where X belongs to a class of area-preserving vector fields, having saddle-type singularities, on a compact orientable surface M of genus $g \geq 2$. For a "full measure" set of such vector fields we prove the existence, for any sufficiently smooth complex valued function f in a finite codimensional subspace, of a finitely differentiable solution u. The loss of derivatives is finite, but the codimension increases as the differentiability required for the solution increases, so that there are a countable number of necessary and sufficient conditions which must be imposed on f, in addition to infinite differentiability, to obtain infinitely differentiable solutions. This is related to the fact that the "Keane conjecture" (proved by several authors such as H.Masur, W.Veech, M.Rees, S.Kerckhoff, M.Boshernitzan), which implies for "almost all" X the unique ergodicity of the flow generated by X on the complement of its singularity set, does not extend to distributions. Indeed, our approach proves that, for "almost all" X, the vector space of invariant distributions not supported at the singularities has infinite (countable) dimension, while according to the Keane conjecture the cone of invariant measures is generated by the invariant area form ω.

§1. INTRODUCTION

In this announcement we describe results on the *cohomological* equation $Xu = f$, where X is a smooth area-preserving vector field on a compact orientable surface M of genus $g \geq 2$. Topological reasons force X to have singularities, which will be assumed to be of a canonical polynomial saddle type (not necessarily non-degenerate). The question we answer can be stated as follows: *given a smooth complex valued function f on M, is it possible to find a (smooth) solution u on M to the equation $Xu = f$?*

The study of cohomological equations is mainly motivated by the problem of describing time-changes for flows. In fact, the time-change induced in the flow generated by a vector field X by a positive function f is *trivial* if and only if the equation $Xu = f - 1$ has (smooth) solutions. In this case the flow produced by the time-change is (smoothly) conjugated to the original one [K-H, §2.2]. The first observation on cohomological equations is that each invariant measure μ for the flow ϕ_t^X generated by a vector field X on a compact manifold M gives a necessary

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condition on f for the existence of a continuous solution of the equation $Xu = f$. In fact, if μ is a probability measure invariant for ϕ_t^X,

$$\int_M u \circ \phi_t^X \, d\mu \equiv \int_M u \, d\mu \tag{1.1}$$

which implies, by taking the derivative with respect to t at $t = 0$, $\int Xu \, d\mu = 0$.

In the case of hyperbolic (Anosov) flows (or diffeomorphisms) the *Livshitz property* holds [L-M-M]. Suppose X is an Anosov flow on a compact manifold M. Then, for any $f \in C^\infty(M)$ satisfying the condition

$$\int_M f d\mu = 0 \quad \text{for all invariant measures } \mu \text{ supported on periodic orbits,} \tag{1.2}$$

the cohomological equation $Xu = f$ has a C^∞ solution which is unique up to additive constants. The Livshitz property is also true in the Hölder classes C^α ($\alpha \in \mathbb{R}^+$). In fact, if f is assumed to be only C^α, then the cohomological equation is still solvable and the solution $u \in C^{\alpha'}$, for any $\alpha' < \alpha$ [L-M-M, (2.3)]. These results were obtained for $0 < \alpha < 1$ in the pioneering work of Livshitz in the early seventies [Lv], while the C^∞ case is due to Guillemin-Kazhdan [G-K1-2] for the special case of geodesic flows on some negatively curved manifold and to de la Llave-Marko-Moriyon [L-M-M] for more general Anosov flows. In the Anosov case the range of the Lie derivative operator on functions is *closed* in C^α ($\alpha > 0$) and C^∞ by the Livshitz property. In fact, it is described by the conditions (1.2) which are all produced by *invariant measures*. Secondly, there is essentially no loss of derivatives, i.e. the solution u is almost as regular as the given f. However, the hyperbolic nature of the dynamical system forces the existence of a very large (countable) number of periodic orbits so that the codimension of the cohomological equation (i.e. the codimension of the Lie derivative operator on functions) is infinite in all the function spaces where it holds.

A different behaviour is expected for uniquely ergodic flows. A flow (or a homemorphism) on a compact space M is *uniquely ergodic* if it has a unique ergodic invariant probability measure. Indeed, for the simplest and best known examples of uniquely ergodic flows, the irrational translations on tori, the cohomological equation is *generically* solvable with codimension 1 in C^∞. Another important example of uniquely ergodic flow is given by the horocycle flow on a surface of negative curvature. In this case not very much is known on the properties of the cohomological equation, but in the constant curvature case it is not difficult to show by $SL(2,\mathbb{R})$ Fourier analysis that the codimension is infinite because of the existence of infinitely many linearly independent *invariant distributions*. It is worth examining a bit more closely the properties of the cohomological equation on the torus, since the results which we have obtained can be seen as a generalization to higher genus. Furthermore the torus case will serve as a model. Without loss of generality one can consider the two-dimensional torus. Let

$$X_\alpha := \alpha_1 \frac{\partial}{\partial x_1} + \alpha_2 \frac{\partial}{\partial x_2} \tag{1.3}$$

be the generator of a translation on the two-torus T^2. By the Weyl equidistribution theorem, the corresponding flow on T^2 is uniquely ergodic iff the ratio α_1/α_2 is irrational. However, in such generality no Livshitz type property holds: the range may not be closed even in C^∞ or in the analytic category. In fact, after the pioneering paper by C.L. Siegel [Sg] and the later landmark papers by A.N. Kolmogorov [Kl1-2] and J.Moser [Mo] which started the so-called KAM theory, it is well known that a number theoretic condition of *Diophantine* type must be imposed on the ratio α_1/α_2 to overcome the difficulties created by the appearance of *small divisors* in the formal series expansion of the solutions. The situation can be explained in this simple context. It goes as follows. Any $f \in C^s(T^2)$ can be expanded in Fourier series with respect to an orthonormal basis of eigenfunctions of the *flat* Laplacian $\Delta := -(\partial^2/\partial x_1^2 + \partial^2/\partial x_2^2)$. It is possible to choose a Fourier basis of eigenfunctions of Δ consisting at the same time of eigenfunctions of X_α. This is due to the fact that the vector field X_α *commutes* with Δ and the eigenfunctions of any *elliptic* partial differential operator such as the Laplacian Δ are smooth. Indeed,

$$X_\alpha e_k = 2\pi i (k,\alpha)\, e_k \ , \quad \text{if} \quad e_k = \exp 2\pi i (k_1 x_1 + k_2 x_2) \ , \tag{1.4}$$

where $(k,\alpha) := k_1\alpha_1 + k_2\alpha_2$, for $k = (k_1,k_2) \in \mathbb{Z}^2$. The cohomological equation $X_\alpha u = f$ can therefore be immediately solved, and the Fourier coefficients u_k of the solution u are:

$$u_k = \frac{f_k}{2\pi i (k,\alpha)} \ , \tag{1.5}$$

where f_k are the Fourier coefficients of the function f. It follows from the irrationality of the ratio α_1/α_2 that, if $f_0 = 0$, the equation is *formally* solvable. The condition $f_0 = 0$ is not surprising since $f_0 = \int_{T^2} f\omega$, where $\omega = dx_1 \wedge dx_2$ is the invariant area form which spans the (1-dimensional) cone of invariant measures for X_α. However, the convergence of the formal solution requires further conditions of Diophantine type on the "small divisors" (k,α), $k \in \mathbb{Z}^2 \setminus \{0\}$, for example the existence of constants $C > 0$, $\gamma > 1$ such that

$$|(k,\alpha)| \geq \frac{C}{|k|^\gamma} \ , \quad \text{for all} \quad k \in \mathbb{Z}^2 \setminus \{0\} \, . \tag{DC}$$

Assuming a Diophantine condition (DC) the following standard result holds. There exists a natural number $l > 1$ (which is related to the constant $\gamma > 1$ in (DC)) such that, if $s \geq l$, for any $f \in C^s(T^2)$ having zero average with respect to area-form ω, there exists a solution $u \in C^{s-l}$ (unique up to additive constants) of the equation $X_\alpha u = f$. The proof of this statement depends on the fact that the Fourier coefficients of a C^s function decay polynomially fast (as $|k|^{-s}$) as $|k| \to +\infty$ and are therefore able to compensate the effect of the small divisors (which is kept under control by (DC)) if s is sufficiently large.

Thus, irrational flows on the torus give a very important example of finite codimension cohomological equation, where the only condition for solvability is the obvious necessary condition produced by the invariant area. However, this "improvement", with respect to the case of hyperbolic flows, has the drawback that one loses a finite number of derivatives in the solution. In fact, the phenomenon of *finite loss of differentiability* is typical of "small divisors" problem. Solvability properties (with estimates) for the cohomological equation in the case of irrational translations are the cornerstone of the classical KAM theory, which leads to finite codimension smooth stability (with respect to smooth conjugacies) of such flows under "small" perturbations and to the famous results concerning the persistence of invariant tori in Hamiltonian systems (the reader can consult [Bs] for a review of the theory). Similarly the Livshitz property was the key step in establishing the canonical perturbation theory for Anosov flows, which describes a complete set of invariants for the smooth stability of such flows under "small" hamiltonian perturbations ([C-E-G] in the case of the geodesic flow on surfaces of constant negative curvature, [L-M-M] for more general Anosov flows).

A class of recurrent flows on compact orientable surfaces of genus $g \geq 2$, having non degenerate saddle singularities, was studied in a pioneering paper by A.Katok [Kt], who proved that they always have at most a *finite* number of non-trivial invariant probability measures. Later the notion of a *measured foliation* on a compact orientable surface was introduced by Thurston [Th] in his classification of diffeomorphisms of surfaces. Duality with respect to a fixed smooth area form gives a one-to-one correspondence between *orientable* measurable foliations and a class of area-preserving flows to which Katok's results apply. Since the space of measured foliations can be interpreted as a boundary of Teichmüller space (Thurston's boundary), Teichmüller theory began to play a role in understanding the ergodic properties of such flows and related dynamical systems (such as interval exchange transformations and rational polygonal billiards). The approach based on Thurston's theory (described in detail in [F-L-P]) and Teichmüller theory led in the eighties to a major breakthrough. The work of several authors such as H.Masur, W.Veech, S.Kerckhoff, M.Rees, J.Smillie, M.Boshernitzan settled the problem of the generic (in the sense of measure) *unique ergodicity* of such systems, known (for interval exchange transformations) as the *Keane conjecture* [Kn1-2]. Masur [Ms] and Veech [Vc] solved the Keane conjecture affirmatively, giving independent proofs of unique ergodicity for "almost all" measured foliations (or interval exchange transformations). Later, different proofs were given by Rees [Rs], Kerckhoff [Kr], Boshernitzan [Bn] and others. The methods of Teichmüller theory developed by Masur were applied to rational polygonal billiards by Kerckhoff-Masur-Smillie [K-M-S]. The meaning of these results for area-preserving flows on surfaces of higher genus (whose singularities are saddle-type) is the following. For a "full measure" set of such flows there are no non-trivial probability invariant measures besides the normalized invariant area. A natural question is then to determine the properties of the corresponding cohomological equation. The results we have obtained show how the presence of singularities generates a sequence of *invariant distributions* which represent additional obstructions to the solvability of the cohomological equation, besides the obvious one given by the invariant area as explained above. However, the finite codimension property still holds if we seek finitely differentiable solutions. We believe a natural and interesting direction of research is to pursue the study of solvability properties of the cohomological equation for uniquely ergodic (non-hyperbolic) flows preserving a smooth volume form and of the related KAM-type stability properties. Our approach seems to indicate that *the choice of a well adapted Fourier analysis is a key step in this kind of problem*, due to the subtle nature of the *cancellations* involved in the existence of solutions.

§2. The results

Let M be a compact orientable surface of genus $g \geq 2$. Let ω be a smooth area form on M. Let $\mathcal{E}_\omega(M,\Sigma)$ be the set of smooth vector fields X on M which preserve the area (i.e. the Lie derivative $\mathcal{L}_X\omega = 0$) and have $\ell \geq 1$ hyperbolic singularities (saddles) with (negative) indices $(i_1, ..., i_\ell)$ (satisfying $i_1 + ... + i_\ell = 2 - 2g$) at $\Sigma := \{p_1, ..., p_\ell\} \subset M$ (modeled on the singularities of the 1-forms $\mathrm{Im}(z^{-i_k}\, dz)$). The standard L^2 Sobolev spaces on the compact manifold M will be denoted by $H^s(M)$, $s \in \mathbb{Z}$. A natural measure class can be introduced in a standard way on $\mathcal{E}_\omega(M,\Sigma)$ by considering the Lebesgue measure class on the fundamental cohomology classes associated to the vector fields as follows. The *fundamental class* of a vector field $X \in \mathcal{E}_\omega(M,\Sigma)$ is the cohomology class of the closed 1-form $\eta_X := \imath_X\omega$ in $H^1(M,\Sigma;\mathbb{R})$, which is a finite dimensional vector space carrying the standard Lebesgue measure class. The following results, concerning the solvability of the cohomological equation for vector fields in $\mathcal{E}_\omega(M,\Sigma)$ are proved:

Theorem A. *There exists a "full measure" set $\mathcal{F}_\omega(M,\Sigma) \subset \mathcal{E}_\omega(M,\Sigma)$ such that, for every $X \in \mathcal{F}_\omega(M,\Sigma)$, the following holds. There exists a natural number $l > 0$ such that, if $f \in H^l(M)$ is supported in a compact set $K \subset M \setminus \Sigma$ and satisfies $\int_M f\,\omega = 0$, then the differential equation $Xu = f$ has a solution $u \in L^2_{loc}(M \setminus \Sigma)$.*

Theorem B. *If $X \in \mathcal{F}_\omega(M,\Sigma)$, then for any $s > l$ there exists a finite number $n_s > 0$ of distributions on $M\setminus\Sigma$, $\mathcal{D}^X_1, ..., \mathcal{D}^X_{n_s} \in H^{-s}_{loc}(M\setminus\Sigma)$, such that the following holds. If $f \in H^s(M)$ is supported in a compact set $K \subset M \setminus \Sigma$ and satisfies*

$$\int_M f\,\omega = 0\ ,\quad \mathcal{D}^X_\ell(f) = 0\ ,\ \ell = 1, ..., n_s\,,$$

then the differential equation $Xu = f$ has a solution $u \in H^{s-l}(M)$.

The above solvability results are obtained through "a priori" estimates in Sobolev spaces. Therefore the argument proving Theorem B also proves the following: there exists a constant $C^s_{X,K} > 0$ such that the solution u of the equation $Xu = f$, whose existence is the content of Theorem B, satisfies the Sobolev estimate:

$$(*)\qquad \|u - \mathrm{vol}\,(M)^{-1}\int_M u\,\omega\|_{s-l} \leq C^s_{X,K}\,\|f\|_s\,.$$

It should also be noticed that the estimated loss of derivatives $l > 0$ in Theorem B is uniform for all vector fields X in the "full measure" set $\mathcal{F}_\omega(M,\Sigma)$ (in fact the argument shows that $l = 7$ will do). Estimates such as $(*)$ are also a key ingredient of the Nash-Moser iteration scheme, which is the core of the KAM method in the smooth category. They are therefore of interest in the attempt of applying the KAM method to the smooth conjugacy problem for the orbit foliations associated to the class of area-preserving flows we are considering. This program will hopefully be completed in a forthcoming paper.

For area-preserving vector fields on M the *Keane conjecture*, which establishes the unique ergodicity of "almost all" measured foliations on compact higher genus surfaces, in the sense of Thurston [Th], (or, equivalently, of "almost all" interval exchange transformations), can be stated as follows:

Keane conjecture. *For "almost all" $X \in \mathcal{E}_\omega(M,\Sigma)$, the flow ϕ^X of X is uniquely ergodic on $M \setminus \Sigma$, i.e. the cone of invariant measures for ϕ^X on $M \setminus \Sigma$ is generated by the area induced by the invariant 2-form ω.*

Thus, for "almost all" $X \in \mathcal{E}_\omega(M,\Sigma)$, the flow ϕ_X has no other invariant measures besides the delta measures supported at its singularities and the area ω. In particular, the Keane conjecture implies that the dynamical system (ϕ_X, ω) is *ergodic* or, equivalently, that the Lie derivative $\mathcal{L}_X$ as a differential operator on $L^2(M,\omega)$ has trivial kernel (consisting only of constant functions), for "almost all" $X \in \mathcal{E}_\omega(M,\Sigma)$. This weaker form of the Keane conjecture, which is sufficient for our analysis, can be obtained as a direct consequence of Theorem A by standard ergodic theory. All the results that we prove in the paper are therefore *independent* of earlier proofs of the Keane conjecture, such as [Ms], [Vc], [Rs], [Bn], [Kr] and others. The approach developed in this paper also shows that the Keane conjecture (in the form stated above) does not extend to invariant distributions. In fact, the following result is proved:

Theorem C. *For "almost all" $X \in \mathcal{E}_\omega(M,\Sigma)$, the vector space of invariant distributions on $M \setminus \Sigma$ (i.e. those distributions $\mathcal{D}$ satisfying the equation $X\mathcal{D} = 0$ on $M \setminus \Sigma$) has infinite (countable) dimension.*

§3. A short description of the proofs

Measured foliations and quadratic differentials. As mentioned, the orbit foliation of a vector field $X \in \mathcal{E}_\omega(M,\Sigma)$ is a *measured foliation* in the sense of Thurston [Th], i.e. it has a transverse measure given by $|\eta_X|$ where $\eta_X := \imath_X\omega$ is the closed dual 1-form. The closedness of η_X is equivalent to the area preserving property by the identity:

$$d\imath_X\omega + \imath_X d\omega = \mathcal{L}_X\omega = 0 \ , \tag{3.1}$$

where $d\omega = 0$, since ω is a 2-form. A pair of transverse measured foliations having the same singularities induces a *complex structure* on the complement of the singularity set, which can be uniquely extended to the whole surface M, and defines a *holomorphic quadratic differential* on M having the two given foliations as *horizontal* and *vertical* foliations in the following sense. A (holomorphic) quadratic differential q is a holomorphic quadratic form on M. With respect to a local holomorphic coordinate $z = x_1 + ix_2$, q can be written as $q = \phi(z)dz^2$, where ϕ is a locally defined holomorphic function. Then, the horizontal foliation $\mathcal{F}_q$ and the vertical foliation $\mathcal{F}_{-q}$ associated to q are defined as follows: $\mathcal{F}_q$ is given by $\operatorname{Im} q^{1/2} = 0$ with transverse measure $|\operatorname{Im} q^{1/2}|$; similarly, $\mathcal{F}_{-q}$ is given by $\operatorname{Re} q^{1/2} = 0$ with transverse measure $|\operatorname{Re} q^{1/2}|$. A measured foliation $\mathcal{F}$ is said to be *realizable* if $\mathcal{F} = \mathcal{F}_q$ for some holomorphic quadratic differential q. A measured foliation is realizable iff it is possible to find another measured foliation transverse to it. By results contained in [H-M], "almost all" measured foliations are realizable. In fact, a measured foliation is realizable if it has a 1-dimensional dense leaf (hence by [Kt] and references therein all 1-dimensional leaves are dense).

A holomorphic quadratic differential q vanishing at a finite set of points Σ induces a *flat structure* with cone type singularities at Σ. There is in fact a flat metric R_q canonically associated to q, defined as $R_q := |q|^{1/2}$ and in coordinates $R_q =$

$|\phi(z)|^{1/2}|dz|$. This metric has *cone-type singularities* in the sense that, at each $p \in \Sigma$, there exists a neighborhood U_p of p in M such that the metric can be written on $U_p \setminus \{p\}$, with respect to coordinates (ρ, θ), as

$$R_q = \left(d\rho^2 + (c\rho\, d\theta)^2\right)^{1/2} , \tag{3.2}$$

where c is a positive real number ($2\pi c$ is called the *cone angle* at p). In our case, the cone angle is always $> 2\pi$ and depends on the order of vanishing of the quadratic differential q at p. The reader can consult [St] for properties of the geometry given by the metric R_q. In the particular case of the two-torus there exists a non-vanishing holomorphic quadratic differential defined by $q = dz^2$ with respect to a global holomorphic coordinate for any complex structure. The horizontal (resp. vertical) foliations are spanned by the vector fields $S = \partial/\partial x_1$ (resp. $T = \partial/\partial x_2$) and the metric R_q is the standard flat Riemannian metric. In analogy with the torus case we proceed to establish the basic properties of Fourier analysis for any holomorphic quadratic differential on a compact Riemann surface M.

Fourier analysis for quadratic differentials. Since the metric R_q is flat, there exists on $M \setminus \Sigma$ an orthonormal frame $\{S, T\}$ of the tangent bundle TM . The vector fields S, T are not defined at the singularity set Σ of the metric R_q (the set Σ corresponds to the set of zeros of the quadratic differential q and to the singularity set of its horizontal and vertical foliations). In fact, if written in coordinates at a singular point $p \in \Sigma$, their coefficients are divergent at p. The *Laplace-Beltrami operator* associated to the metric R_q can be written as $\Delta_q := -(S^2 + T^2)$. It is a well defined second order *elliptic* differential operator on $M \setminus \Sigma$, but (as for S, T) its expression in coordinates diverges at the singularities. It is natural to introduce adapted Sobolev spaces $H^s_q(M)$ as follows. The space $H^0_q(M)$, denoted by $L^2_q(M)$, is simply the L^2 space defined with respect to the area-element ω_q of the metric R_q. The space $H^s_q(M)$ (for $s \in \mathbb{N}$) is defined as completion of $C^\infty(M)$ with respect to the norm

$$|u|_s := \Big(\sum_{i+j\le s} |S^i T^j u|_0^2\Big)^{1/2} . \tag{3.3}$$

The Laplace-Beltrami operator Δ_q has a L^2_q orthonormal basis of *weak* eigenfunctions. In fact, one can consider its *Dirichlet form*

$$\mathcal{Q}(u, v) := (Su, Sv)_q + (Tu, Tv)_q , \tag{3.4}$$

(where $(\cdot,\cdot)_q$ denotes the L^2_q inner product) and prove the following:

Theorem 3.1 (Spectral Theorem). *The hermitian form $\mathcal{Q}$ on $L^2_q(M)$ has the following properties:*

1. *$\mathcal{Q}$ is positive semi-definite and the set $EV(\mathcal{Q})$ of its eigenvalues is a discrete subset of $[0, +\infty)$;*
2. *Each eigenvalue has finite multiplicity, in particular the eigenvalue 0 has multiplicity 1 and the kernel of $\mathcal{Q}$ consists only of constant functions;*
3. *The space $L^2_q(M)$ decomposes as an orthogonal sum of eigenspaces. Furthermore, the eigenfunctions $\{e_k\}_{k\in\mathbb{N}}$ are C^∞ (real analytic) on M.*

It is also important to establish the *Weyl asymptotics* for the eigenvalues of the hermitian form $\mathcal{Q}$. This formula gives the information which we need on the rate of growth of the eigenvalues. For any $\Lambda > 0$, let $N_q(\Lambda) := \text{card}\,\{\lambda \in \text{EV}(\mathcal{Q}) \,|\, \lambda \le \Lambda\}$, where each eigenvalue λ is counted according to its multiplicity.

Theorem 3.2 (Weyl asymptotics). *There exists a constant $C > 0$ such that*

$$\lim N_q(\Lambda)/\Lambda \;=\; C\,\mathrm{vol}_q(M)\ , \quad \textit{as}\ \ \Lambda \to +\infty\ .$$

Theorem 3.1 and 3.2 can be proved by standard methods (the reader can consult [Ch]). Theorem 3.1 was proved by J.Cheeger in [Cg] for any compact Riemann manifold with cone-type singularities. Theorem 3.2 can also be deduced in the general case by Cheeger methods. Theorem 3.2 says that the growth of the eigenvalues is linear in $k \in \mathbb{N}$. As a simple consequence we obtain that the Fourier coefficients $f_k := (f, e_k)_q$ of a function $f \in H^s_q(M)$ decay polynomially (as k^{-s}) as in the torus case. However, there is an obstruction (and indeed it must be so) to carrying over the argument sketched above for the case of irrational translations on the torus: although the vector fields S and T commute (hence they commute with the Laplacian Δ_q), the eigenfunctions $e_k \in H^1_q(M)$ but in general they *do not belong* to $H^2_q(M)$. Therefore, they *are not* eigenfunctions of the vector fields S, T or their linear combinations. Thus, a different strategy must be found.

A unitary operator. The circle group acts on the space of (holomorphic) quadratic differentials as follows: $(\theta, q) \to q_\theta := (e^{-i\theta})^2 q$, for all $\theta \in S^1$. The horizontal foliation $\mathcal{F}_\theta$ of the quadratic differential q_θ (defined by the closed 1-form $\mathrm{Im}(q_\theta)^{1/2}$) is spanned by a vector field S_θ, unitary with respect to the metric R_q, which can be obtained through a rotation by the angle θ from the vector field S spanning the horizontal foliation of q. It is essentially an application of the Fubini theorem to show that all statements concerning a "full measure" class of area-preserving vector fields can be reduced to the corresponding statements about the family S_θ for "almost all" $\theta \in S^1$ (with respect to the 1-dimensional Lebesgue measure). In terms of the orthonormal frame $\{S, T\}$ introduced earlier:

$$S_\theta := \{e^{-i\theta}(S + iT) + e^{i\theta}(S - iT)\}/2\ . \tag{3.5}$$

One is led therefore to the study of the simpler *Cauchy-Riemann* operators $S \pm iT$ (these are *elliptic* first order differential operators), for which the following properties can be proved:

Proposition 3.3. *The Cauchy-Riemann operators $S \pm iT$ are closable operators on $L^2_q(M)$, whose closures (denoted by the same symbols) have the following properties. Let $\mathcal{M}_\Sigma$ (resp. $\overline{\mathcal{M}}_\Sigma$) be the (finite dimensional) vector spaces of meromorphic (resp. anti-meromorphic) L^2_q functions.*

1. *$D(S \pm iT) = H^1_q(M)$ and $N(S \pm iT) = \mathbb{C}$.*
2. *$R^+ := Ran\,(S + iT) = \overline{\mathcal{M}}_\Sigma^\perp$ and $R^- := Ran\,(S - iT) = \mathcal{M}_\Sigma^\perp$.*
3. *$S \pm iT : (H, \mathcal{Q}) \to (R^\pm, (\,\cdot\,,\,\cdot\,)_q)$ are unitary operators.*

Consequently the operator $U_q : R^- \to R^+$, defined as $U_q := (S + iT)(S - iT)^{-1}$, is a partial isometry (with respect to the L^2_q scalar product). Therefore it can be extended by any isometry $J : \mathcal{M}_\Sigma \to \overline{\mathcal{M}}_\Sigma$ to a unitary operator U_J on $L^2_q(M)$ for which the following key identities hold:

$$S_\theta = e^{-i\theta}\Big(U_J + e^{2i\theta}\Big)(S - iT) = e^{i\theta}\Big(U_J^{-1} + e^{-2i\theta}\Big)(S + iT)\ . \tag{3.6}$$

Estimates for the solution of the equation $S_\theta u = f$ in the Sobolev spaces $H^s_q(M)$ will therefore be related to estimates for the resolvents $\mathcal{R}^\pm_J(z)$ of the unitary operators

U_J (resp. U_J^{-1}) at points on the circle S^1 (which contains the spectrum). It turns out that such estimates can be obtained in the *weak sense* for *any* unitary operator, by applying *Fatou's theory* on boundary values of holomorphic functions. In fact, for any $u,\, v \in L^2_q(M)$, the function $z \to (\mathcal{R}_J(z)u, v)_q$ is holomorphic on the unit disk and, by the spectral theorem for unitary operators, it can be represented as a Cauchy integral over the spectral measure $d(F(t)u, v)_q$:

$$(\mathcal{R}_J(z)u, v)_q = \frac{1}{2\pi}\int_0^{2\pi} (z - e^{it})^{-1}\, d(F(t)u, v)_q \ , \quad |z| < 1 \ , \tag{3.7}$$

It is a classical theorem [Zy, VII.9] that, for any Borel complex measure, the corresponding Cauchy integral $I_\mu(z)$ (defined as in (3.7) by replacing the spectral measure by the measure μ) has the following property:

Lemma 3.4. *The non-tangential limit $I_\mu(z) \to I^*_\mu(\theta)$ as $z \to e^{i\theta}$ exists almost everywhere with respect to the Lebesgue 1-dimensional measure $\mathcal{L}$ on S^1. Furthermore, there exists a constant $C > 0$ such that the following estimates hold:*

$$\mathcal{L}\{\theta \in S^1 \mid |I^*_\mu(\theta)| > \lambda\} \leq \frac{C}{\lambda}\, \|\mu\| \ ,$$

where $\|\mu\|$ denotes the total mass of the measure μ.

(In fact, we need a refined version of this, whose proof can be obtained by applying results from [Rd], [Sn] and [S-W]).

Sobolev estimates and existence of invariant distributions. The refined version of Lemma 3.4 can be applied to find a *distributional* solution of the equation $S_\theta u = f$ for any given $f \in L^2_q(M)$ having zero average and for a full measure set of $\theta \in S^1$ (which will depend on the function f considered). This is obtained essentially by proving certain estimates in the Sobolev spaces $H^s_q(M)$. The dependence on f can be eliminated by Fourier series decomposition. Here the information on the decay rate of the Fourier coefficients of H^s_q functions plays a crucial role (thus motivating the Fourier analysis of Theorems 3.1 and 3.2). The solution is then regularized through a procedure based on properties of Cauchy-Riemann operators. Finally, the existence of non-trivial distributions is proved by the following idea. By the Riemann-Roch theorem, the dimension of the space $\mathcal{M}_\Sigma$ of L^2_q meromorphic function has dimension equal to the genus $g \geq 2$. Hence, there always exists a meromorphic function Φ having zero average. By the previous construction the equation $S_\theta U = \Phi$ has a distributional solution for almost all $\theta \in S^1$. By the commutation property $\mathcal{D} := (S + iT)U$ is an invariant distribution, i.e. $S_\theta \mathcal{D} = 0$ in distributional sense.

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