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UNITONS AND THEIR MODULI

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Abstract. We sketch the proof that unitons (harmonic spheres in $\mathrm{U}(N)$) correspond to holomorphic 'uniton bundles', and that these admit monad representations analogous to Donaldson's representation of instanton bundles. We also give a closed-form expression for the unitons involving only matrix operations, a finite-gap result (two-unitons have energy ≥ 4), computations of fundamental groups of energy ≤ 4 components, new methods of proving discreteness of the energy spectrum and of Wood's Rationality Conjecture, a discussion of the maps into complex Grassmannians and some open problems.

1. Harmonic maps

Harmonic maps between Riemannian manifolds M and N are critical values of an energy functional

$$\operatorname{energy}(S : M \to N) = \frac{1}{2}\int_M |dS|^2.$$

They generalise geodesics, which are the local minima of the length functional. (Length is energy with respect to the induced metric.) Harmonic maps of surfaces are likewise branched minimal immersions (locally least-area surfaces with $z \mapsto z^k$ type singularities).

In the case of surfaces in a matrix group, with the standard (bi-invariant) metric, the energy takes the form

$$\operatorname{energy}(S) = \frac{1}{2}\int_{\mathbb{R}^2}\left(|S^{-1}\frac{\partial}{\partial x}S|^2 + |S^{-1}\frac{\partial}{\partial y}S|^2\right) dx \wedge dy. \tag{1.1}$$

Unitons are harmonic maps $S : \mathbb{S}^2 \to \mathrm{U}(N)$. Some authors call them multi-unitons. Since the energy is conformally invariant in the case of surfaces, it is natural to stick to coordinates x and y on $\mathbb{R}^2$ and derive the Euler-Lagrange equations,

$$\frac{\partial}{\partial x}(S^{-1}\frac{\partial}{\partial x}S) + \frac{\partial}{\partial y}(S^{-1}\frac{\partial}{\partial y}S) = 0, \tag{1.2}$$

which we will refer to as the uniton equations. From [SaUhl, Theorem 3.6], we know that harmonic maps from $\mathbb{R}^2 \to \mathrm{U}(N)$ extend to $\mathbb{S}^2$ iff they have finite energy, and

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that such maps are always smooth. So working in terms of coordinates x and y on $\mathbb{R}^2$ or $z \in \mathbb{C}$ poses no real limitation.

Remark on metrics. Since harmonic maps $\mathbb{S}^2 \to \mathrm{U}(N)$ have constant determinant, and $\mathrm{SU}(N)$ has an essentially bi-invariant metric, nothing is gained by considering other bi-invariant metrics on $\mathrm{U}(N)$. On the other hand, bi-invariance of the metric is essential in what follows.

1.3 Based unitons. Unitons are determined by the pullback of the Maurer-Cartan form on $\mathrm{U}(N)$,

$$A \overset{\text{def}}{=} \frac{1}{2} S^{-1} dS = A_z\, dz + A_{\bar{z}} d\bar{z} \tag{1.4}$$

and a choice of a basepoint, $S(\infty) \in \mathrm{U}(N)$, as we can see by thinking of $d + 2A$ as a flat connection and S as a gauge transformation. We thus have a decomposition of the $\mathrm{U}(N)$-unitons,

$$\mathrm{Harm}(\mathbb{S}^2, \mathrm{U}(N)) = \mathrm{U}(N) \times \mathrm{Harm}^*(\mathbb{S}^2, \mathrm{U}(N)), \tag{1.5}$$

into initial values $S_0 \in \mathrm{U}(N)$ and *based* unitons (which we take to have $S(\infty) = \mathbb{I}$).

Since $\mathrm{Harm}^*(\mathbb{S}^2, \mathrm{U}(N)) = \left\{A = dS : S \in \mathrm{Harm}(\mathbb{S}^2, \mathrm{U}(N))\right\}$, it is useful to have equations for A as well. Two $\mathrm{U}(N)$-valued maps A_x, A_y come from a map $S : \mathbb{R}^2 \to \mathrm{U}(N)$ in this way iff $d + 2A$ has zero curvature. They come from a *harmonic* map if, in addition, $d^* A = 0$. This has a zero-curvature formulation:

Theorem 1.6 [Po, Uhl]. *Let $\Omega \subset \mathbb{S}^2$ be a simply connected neighbourhood and $A : \Omega \to T^*(\Omega) \otimes \mathrm{U}(N)$. Then $2A = S^{-1} dS$, with S harmonic iff the curvature of the connection*

$$\mathcal{D}_\lambda = \left(\frac{\partial}{\partial \bar{z}} + (1 + \lambda) A_{\bar{z}}, \frac{\partial}{\partial z} + (1 + \lambda^{-1}) A_z\right) \tag{1.7}$$

vanishes for all $\lambda \in \mathbb{C}^*$.

After Uhlenbeck, we will write $E_\lambda : \Omega \times \mathbb{C}^* \to \mathrm{Gl}(N)$ for flat sections of $\mathcal{D}_\lambda$ and call them extended solutions. In the setting of [BuRa], they are the twistor lifts associated to the harmonic maps.

2. Holomorphic methods

In investigating unitons and their moduli, we will be using an idea that goes back to Weierstraß' description of minimal surfaces in $\mathbb{R}^3$ via a pair of holomorphic functions and their derivatives. His starting point was Enneper's closed form expression for these minimal surfaces in terms of analytic functions and quadratures.

Uhlenbeck proved [Uhl] that all unitons can be constructed via quadratures (i.e. Bäcklund transformations, written explicitly as quadratures by Wood [Wo]), and our results giving an equivalence between unitons and holomorphic bundles are analogous to Weierstraß' description, but the uniton story is more complicated than the history of minimal surfaces in $\mathbb{R}^3$.

The first[1] modern use of complex methods, and the starting point for twistor methods in harmonic map theory, is Calabi's association to minimal immersions $\mathbb{S}^2 \to \mathbb{S}^{2n} = \mathrm{SO}(2n+1)/\mathrm{SO}(2n)$ of a holomorphic curve in $\mathrm{SO}(2n+1)/\mathrm{U}(n)$ (a Kähler manifold). Analysis of the holomorphic curves led him to a classification

[1] The referee has pointed out that Bochner did work in this directon; see Trans. Amer. Math. Soc. **47** (1940), 146–164.

of the minimal spheres and the discovery that the area spectrum was discrete. Calabi's work was followed by work of physicists and mathematicians on maps into complex projective spaces and Grassmannians, and then Uhlenbeck's work on maps into $\mathrm{U}(N)$. Since Grassmannians are symmetric spaces in $\mathrm{U}(N)$ (they are the components of $\{S \in \mathrm{U}(N) : S^2 = \mathbb{I}\}$) and are totally geodesically embedded via the Cartan embedding ($V \subset \mathbb{C}^N \mapsto \pi_V - \pi_V^\perp; \pi_V =$ projection onto V), Uhlenbeck's results subsume the previous work. Finally, symmetric spaces contain $\mathrm{U}(N)$ as a symmetric space of $\mathrm{U}(N) \times \mathrm{U}(N)$ and unitons can be treated in this most general context (see [BuRa]), although we will not do so now.

3. Uniton bundles

The present work rests on the observation of Ward and Uhlenbeck that the harmonic map equations for maps $\mathbb{R}^2 \to \mathrm{U}(N)$ are dimensionally reduced Bogomolny equations on $\mathbb{R}^{2+1}$. Ward ([Wa]) additionally observed that Hitchin's ([Hi]) equivalence of Bogomolny solutions on $\mathbb{R}^3$ and holomorphic bundles on $T\mathbb{P}^1 = T\mathbb{S}^2 = \{\text{oriented lines in } \mathbb{R}^3\}$ applied to maps $\mathbb{R}^2 \to \mathrm{SU}(N)$, and showed how this construction naturally 'compactified' for maps $\mathbb{S} \to \mathrm{SU}(2)$. In [An2] we show that this equivalence can be 'compactified', identifying finite-energy harmonic maps $\mathbb{S}^2 \to \mathrm{U}(N)$ with bundles which are the restriction of bundles on the fibrewise compactification of $T\mathbb{P}^1$, a Hirzebruch surface.

The uniton bundles are bundles on $\widetilde{T\mathbb{P}}^1 \stackrel{\text{def}}{=} \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(2))$, the fibrewise compactification of the tangent bundle $T\mathbb{P}^1$ of the *complex* projective line.

Let (λ, η) and $(\hat{\lambda} = 1/\lambda, \hat{\eta} = \eta/\lambda^2)$ be coordinates on $T\mathbb{P}^1 \cong \mathcal{O}_{\mathbb{P}^1}(2)$, where λ is the usual coordinate on $\mathbb{P}^1$ and η is the coordinate associated to $d/d\lambda$. Meromorphic sections (s) of $T\mathbb{P}^1$ give all the holomorphic sections of $\widetilde{T\mathbb{P}}^1$ ($[s, 1]$ in projective coordinates on $\widetilde{T\mathbb{P}}^1$), save one. We fix notation for the lines on $\widetilde{T\mathbb{P}}^1$:

$$
\begin{aligned}
P_\lambda &= \pi^{-1}(\lambda) = \text{a pfibre (silent p)}, \\
G_{a,b,c} &= \left\{(\lambda, [a - 2b\lambda - c\lambda^2, 2]\right\}, \text{ for } (a, b, c) \in \mathbb{C}^3 \cong H^0(\mathbb{P}^1, T\mathbb{P}^1), \\
G_\infty &= \{(\lambda, [1, 0])\} = \text{infinity section of } \widetilde{T\mathbb{P}}^1.
\end{aligned} \tag{3.1}
$$

To encode unitarity, we need the real structure

$$
\sigma^*(\lambda, \eta) = (1/\bar{\lambda}, -\bar{\lambda}^{-2}\bar{\eta}) \tag{3.2}
$$

acting on $\widetilde{T\mathbb{P}}^1$ and compatibly on $\mathbb{C}^3 \cong H^0(\mathbb{P}^1, \mathcal{O}(2))$, the space of finite sections:

$$
\sigma^*(a, b, c) = (\bar{c}, -\bar{b}, \bar{a}). \tag{3.3}
$$

We similarly define time translation

$$
\begin{aligned}
\delta_t : (\lambda, \eta) &\mapsto (\lambda, \eta - 2t\lambda), \\
(a, b, c) &\mapsto (a, b + t, c).
\end{aligned} \tag{3.4}
$$

Definition (Uniton Bundles). A rank N, or $\mathrm{U}(N)$, *uniton bundle*, $\mathcal{V}$, is a holomorphic rank N bundle on $\widetilde{T\mathbb{P}}^1$ which is a) trivial when restricted to the following curves in $\widetilde{T\mathbb{P}}^1$:

1. the section at infinity (G_∞),
2. nonpolar fibres (i.e. P_λ for $\lambda \in \mathbb{C}^*$),
3. real sections of $T\mathbb{P}^1$ (sections invariant under σ, $G_{(z,it,\bar{z})}$);

b) is equipped with bundle lifts

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{\tilde{\delta}_t} & \mathcal{V} \\ \downarrow & & \downarrow \\ \widetilde{T\mathbb{P}}^1 & \xrightarrow{\delta_t} & \widetilde{T\mathbb{P}}^1 \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{V} & \xrightarrow{\tilde{\sigma}} & \mathcal{V}^* \\ \downarrow & & \downarrow \\ \widetilde{T\mathbb{P}}^1 & \xrightarrow{\sigma} & \widetilde{T\mathbb{P}}^1 \end{array}$$

1. $\tilde{\delta}_t$ a one-parameter family of holomorphic transformations fixing $\mathcal{V}$ above the section at infinity and lifting δ_t, and
2. $\tilde{\sigma}$ a norm-preserving, antiholomorphic lift of σ such that the induced hermitian metric on $\mathcal{V}$ restricted to a fixed point of σ is positive definite. Equivalently, the induced lift to the principal bundle of frames acts on fibres of fixed points of σ by $X \mapsto X^{*-1}$; and

c) has a framing, $\phi \in H^0(P_{-1}, Fr(\mathcal{V}))$, of the bundle $\mathcal{V}$ restricted to the fibre $P_{-1} = \{\lambda = -1\} \subset \widetilde{T\mathbb{P}}^1$ such that $\tilde{\sigma}(\phi) = \phi$.

Remark. An argument similar to the one in [ADHM] shows that real triviality is implied by the other bundle properties. The proof depends essentially on the monad construction in §5.

Theorem (The Equivalence). *The space of based unitons,* $\mathrm{Harm}^*(\mathbb{S}^2, \mathrm{U}(N))$, *is isomorphic to the space of rank N uniton bundles, with energy corresponding to the second Chern class.*

Sketch of Proof. The elements of Hitchin's bundles are solutions of a linear differential operator on the corresponding line of $\mathbb{R}^3$. A $\bar{\partial}$-operator defines a Koszul-Malgrange holomorphic structure on the bundle. In adapting the construction one must show that a singularity in this $\bar{\partial}$-operator along G_∞ can be removed by a continuous gauge change. Away from $\lambda = 0, \infty$, any extended solution lifts to give a trivialisation of $\mathcal{V}|_{\widetilde{T\mathbb{C}}^*}$, so the analytic difficulties are only at two points.

Going the other way, Hitchin reconstitutes the Bogomolny system via 'null' connections on lines in $\mathbb{R}^3$ corresponding to the set of sections of $T\mathbb{P}^1$ mutually tangent at a fixed point. (Evaluation at the point gives a flat trivialisation along the line.) The extension involves

1. pushing the holomorphic bundle down to the one-point compactification of $T\mathbb{P}^1$,
2. realising this as a quadric in $\mathbb{P}^3$,
3. showing that pulling and pushing (taking a direct image of) the bundle gives a bundle on $\mathbb{R}P^2 \times \mathbb{R} \subset \mathbb{P}^{3\mathrm{dual}}$, and
4. showing that the reconstituted solution on $\mathbb{R}P^2 \times \mathbb{R}$ gives a harmonic map from $\mathbb{S}^2$ (this involves calculating the energy of the map on the big open cell $\mathbb{R}^2 \hookrightarrow \mathbb{R}P^2$ in well-chosen affine coordinates on $\mathbb{R}P^2$, and using Sacks' and Uhlenbeck's result on extendibility of finite energy maps on $\mathbb{R}^2$).

Finally, one must check that the extra conditions on the bundle are necessary and sufficient. We explain the energy calculation below.

4. Monodromy construction

The above constructions depend heavily on the twistor correspondence

$$\mathbb{R}^3 \leftarrow \mathbb{S}^2 \times \mathbb{R}^3 \cong \mathbb{R} \oplus T\mathbb{S}^2 \rightarrow T\mathbb{P}^1 \tag{4.1}$$

which relates points on $\mathbb{R}^3$ to sections of $T\mathbb{P}^1$ and points of $T\mathbb{P}^1$ to null lines of $\mathbb{C}^3 = \mathbb{R}^3_{\mathbb{C}}$. We use it to construct flat sections or trivialisations of the bundles on $\mathbb{R}^3$ and $T\mathbb{P}^1$, and their compact counterparts.

The most important example of this is the G_∞-trivialisation of $\mathcal{V}|_{\widetilde{T\mathbb{C}}^*}$, which can be intrinsically defined using the trivialisation of $\mathcal{V}|_{G_\infty}$. (Since $\mathcal{V}|_{G_\infty}$ is holomorphically trivial, $\mathcal{V}|_{pt} \cong H^0(G_\infty, \mathcal{V})$, giving a notion of parallel translation along holomorphically trivial curves in $\widetilde{T\mathbb{P}}^1$, G_∞, G_{real}, $P_\lambda : \lambda \in \mathbb{C}^*$ for example.)

The uniton bundle $\mathcal{V} \to \widetilde{T\mathbb{P}}^1$ is constructed generically as the kernel of a differential operator on $\mathbb{R}^3 \times \mathbb{S}^2$; $\mathcal{D}_\lambda$ is the projection of that operator from $\mathbb{R}^2 \times \mathbb{C}^*$. The extended solution spans the kernel of $\mathcal{D}_\lambda$. Pulling E_λ back to $\mathbb{R}^3 \times \mathbb{C}^*$ gives a trivialisation of $\mathcal{V}$ restricted to the open set $\{\lambda \in \mathbb{C}^*\}$. Intrinsically, the G_∞-infinity trivialisation is the trivialisation on each fibre P_λ, $\lambda \in \mathbb{C}^*$, which agrees with a trivialisation of $\mathcal{V}|_{G_\infty}$. We 'evaluate' this trivialisation and get the extended solution in terms of the constant trivialisation of $\mathbb{C}^N \times \mathbb{S}^2$, by comparing it to the trivialisation of $\mathcal{V}$ on the real sections. We obtain

$$E_\lambda(z, \bar{z}) \in \mathrm{Hom}(\mathbb{C}^N) = \mathrm{Hom}(H^0(P_{-1}, \mathcal{V})) \tag{4.2}$$

by composing the cycle of maps

$$\begin{array}{ccccc} \mathcal{V}_{\lambda,\infty} & \xleftarrow{\text{restr}} & H^0(G_\infty, \mathcal{V}) & \xrightarrow{\text{restr}} & \mathcal{V}_{-1,\infty} \\ \text{restr}\big\uparrow & & & & \big\uparrow\text{restr} \\ H^0(P_\lambda, \mathcal{V}) & & & & H^0(P_{-1}, \mathcal{V}) \\ \text{restr}\big\downarrow & & & & \big\downarrow\text{restr} \\ \mathcal{V}_{(\lambda, z/2 - t\lambda - \lambda^2\bar{z}/2)} & \xleftarrow{\text{restr}} & H^0(G_{(z,\bar{z},t)}, \mathcal{V}) & \xrightarrow{\text{restr}} & \mathcal{V}_{(-1, z/2 - t\lambda - \bar{z}/2)} \end{array} \tag{4.3}$$

counterclockwise. The existence of the bundle isomorphism $\tilde{\delta}_t$ (time translation) ensures that the result does not depend on t. Finiteness, i.e. extension to $\mathbb{S}^2$, follows from the compactness of $\widetilde{T\mathbb{P}}^1$.

Since any extended solution encodes the uniton as $S = E_{-1}^{-1}E_1$, this gives a construction of the uniton which does not involve quadratures. If we define the bundle as a collection of transition functions, it reduces to solving the Riemann-Hilbert problem. This is Ward's construction. (We illustrate this for a two-uniton in [An2].) While solving the Riemann-Hilbert problem is not as routine as taking derivatives, the monodromy construction is quite useful as a theoretical tool. We can use it to affirm Wood's conjecture:

Corollary (Wood's Conjecture). *If $S : \mathbb{S}^2 \to \mathrm{U}(N)$ is a uniton, then the composition with $\mathrm{U}(N) \hookrightarrow \mathrm{Gl}(N)$ is rational, i.e. the functions in x and y which make up the matrix $S \in \mathrm{U}(N)$ are rational.*

We use it to show energy$(S) = c_2(\mathcal{V})$ (and hence that the energy spectrum is discrete) by computing the Chern-Weil integral for c_2 for a constructed connection. The connection is pieced together with a partition of unity from connections defined in terms of flat frames: take the G_∞-trivialisation/extended solution to be flat away from $\lambda = 0, \infty$, and the standard (nonholomorphic!) frame flat near $\lambda = 0$ or ∞. The integral is then reduced to the energy integral by using the zero-curvature equations for E_λ and integration by parts.

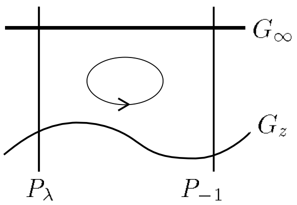
Previous proofs of energy discreteness and Wood's conjecture [Va1], [Va2] are technically quite different and do not have interpretations in terms of the uniton bundles.

4.4 Grassmannian solutions. The union of Grassmannians in $\mathbb{C}^N$ is realised as $\{S \in \mathrm{U}(N) : S^2 = \mathbb{I}\}$. We will call $S \in \mathrm{Harm}(\mathbb{S}^2, \mathrm{U}(N))$ a *Grassmannian solution* if there exists a $Q \in \mathrm{U}(N)$ such that $(QS)^2 = \mathbb{I}$. Let E_λ now be an extended solution for S and assume $E_\lambda(\infty)$ is conjugate to $\begin{pmatrix} \mathbb{I} & \\ & -\lambda\mathbb{I} \end{pmatrix}$. Uhlenbeck shows $\widetilde{E}_\lambda = E_{-\lambda}E_1^{-1}$ is an extended solution for S^{-1}.

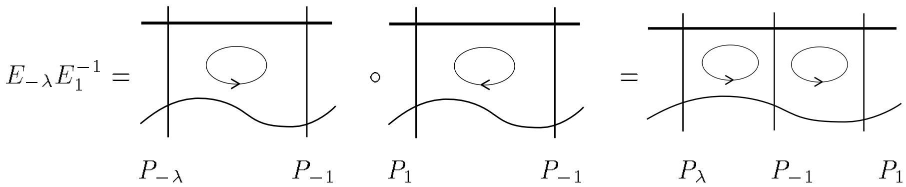
Now define an involution by

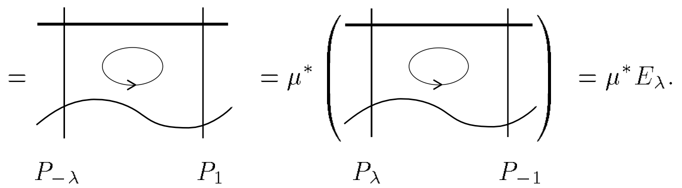
$$\mu^*\lambda = -\lambda, \quad \mu^*\eta = \eta \quad (\text{equivalently } \mu^*z = z),$$

and fix generators $\pi_1(G_\infty \cup P_{-1} \cup G_z \cup P_\lambda)$ above which we calculate the monodromies which determine E_λ:

$E_\lambda =$ 

Then the formula for $\widetilde{E}_\lambda$ has the interpretation





Since $\widetilde{E}_\lambda$ determines the uniton bundle and vice versa, the uniton bundle for S^{-1} is the μ-pullback of the bundle for S, up to the choice of framing. To see the effect on the framing, note that when $E_\lambda(\infty) = \begin{pmatrix} \mathbb{I} & \\ & -\lambda\mathbb{I} \end{pmatrix}$ (the difference between the chosen framing of $\mathcal{V}|_{G_\infty}$ and the canonical one) μ carries a frame at $P_{-1} \cap G_\infty$ to $\begin{pmatrix} \mathbb{I} & \\ & -\mathbb{I} \end{pmatrix}$ times itself (after 'transporting' it back using evaluation of the canonical frame).

So uniton bundles $(\mathcal{V}, \Phi)$ correspond to Grassmannian solutions iff μ lifts to $\tilde{\mu} : \mathcal{V} \to \mathcal{V}$ and the signature of $\mu^* \phi \phi^{-1} : \mathbb{C}^N \to \mathbb{C}^N$ determines the component (rank of the image Grassmannian).

5. Horrocks' monad construction

Atiyah et al ([ADHM]) used Horrocks' monad representation for holomorphic bundles on projective spaces to represent Yang-Mills' instantons. Donaldson [Do] then used the monads themselves to equate the real instanton bundles on $\mathbb{P}^3$ with bundles on $\mathbb{P}^2$. See [OSS] for a general account of such representations. For our purposes, it is enough to know that holomorphic bundles on $\mathbb{P}^2$ which are trivial on generic lines can be represented as

$$\underline{\ker K}/\underline{\operatorname{im} J},$$

where

$$\mathcal{O}(-1)^k \xrightarrow{J} \mathcal{O}^{2k+N} \xrightarrow{K} \mathcal{O}(1)^k,$$

i.e. by two $k \times (2k+N)$ matrices of degree 1 homogenous polynomials in three variables. By blowing up a point on $\widetilde{T\mathbb{P}^1}$ and blowing down two exceptional divisors, one arrives at a birational equivalence of $\widetilde{T\mathbb{P}^1}$ and $\mathbb{P}^2$ along which one can push and pull uniton bundles. A few technical difficulties aside, we obtain

Theorem (Monad representation). *The space of based unitons* $\operatorname{Harm}^*(\mathbb{S}^2, \mathrm{U}(N))$ *is isomorphic to the set of monad data*

$$\gamma, \alpha_1', \delta \in \mathrm{gl}(k/2), \quad \gamma \textit{ nilpotent},$$
$$a' \in \mathrm{M}_{k/2+N, k/2}, \quad b' \in \mathrm{M}_{k/2, k/2+N}$$

satisfying

(nondegeneracy) $$\operatorname{rank}\begin{pmatrix} \gamma \\ \alpha_1' + z \\ a' \end{pmatrix} = \operatorname{rank}\begin{pmatrix} \gamma & \alpha_1' + z & b' \end{pmatrix} = k/2 \qquad \forall z \in \mathbb{C},$$

(monad equation) $$[\gamma, \alpha_1'] + b'a' = 0,$$

(time invariance) $$\begin{aligned} [\delta, \gamma] &= 0, \\ [\delta, \alpha_1'] &= \gamma, \\ a'\delta &= 0, \\ \delta b' &= 0, \end{aligned}$$

quotiented by the action of $g \in \mathrm{Gl}(k/2)$*:*

$$\gamma \mapsto g\gamma g^{-1}, \quad \alpha_1' \mapsto g\alpha_1' g^{-1}, \quad \delta \mapsto g\delta g^{-1},$$
$$a' \mapsto a'g^{-1}, \quad b' \mapsto gb'.$$

Reinterpreting the monodromy construction of E_λ, we obtain

Theorem (Closed Form). *Based rank N unitons of energy $8\pi k$ are all of the form*

$$S = \mathbb{I} + a\alpha_2^{-1}(\alpha_1 - 2(x + iy\alpha_2))^{-1} b.$$

(Multiplication is matrix multiplication.)

The primed monad data determine the uniton bundle over a hemisphere. Reality determines it over the other hemisphere. The unprimed data describing the whole bundle and appearing in the closed form are

$$\begin{aligned} \alpha_1 &= 2\begin{pmatrix} \alpha_1'^* & \phi_1 \\ \phi_2 & \alpha_1' \end{pmatrix}, & \alpha_2 &= \begin{pmatrix} -\mathbb{I} - 2\gamma^* & \\ & \mathbb{I} + 2\gamma \end{pmatrix}, \\ a &= 2\begin{pmatrix} b'^* & a' \end{pmatrix}, & b &= 2\begin{pmatrix} a'^* \\ b' \end{pmatrix}, \end{aligned} \tag{reality}$$

where ϕ_1 and ϕ_2 are functions of γ, a' and b' determined by the big monad equation $[\alpha_1, \alpha_2] + ba = 0$.

Remark. Because homogenous monads give simple bases of sections above lines in $\mathbb{P}^2$, computing the monodromy expression for E_λ is much easier on $\mathbb{P}^2$. Such bases are very difficult to compute for $\widetilde{T\mathbb{P}^1}$ monads (see [An1]).

The monads representing uniton bundles are stratified by the order of the nilpotent matrix γ. When it is zero, the resulting solutions factor through a holomorphic map, via a translation of the Cartan embedding of a Grassmannian into $\mathrm{U}(N)$. (Putting the U(2) monads in a normal form recovers the poles and principal parts description of rational maps.) Examination of the corresponding monad data reveals that the $\gamma = 0$ maps have $\mathbb{N}$ components given by energy, so the map

$$\mathrm{Harm}(\mathbb{S}^2, \mathrm{Gr}(\mathbb{C}^N)) \hookrightarrow \mathrm{Harm}(\mathbb{S}^2, \mathrm{U}(N))$$

is not injective, as it maps degree components onto each other. When γ is not zero, on the other hand, the existence of time translation is obstructed and generic γ's do not occur.

That $\gamma = 0$ maps are 1-unitons (i.e. the extended solutions can be written as degree 1 polynomials, [Uhl]) and that the extended solution produced from the monad data never has terms of degree (as a Laurent polynomial) greater than the degree of γ, lead us to

Conjecture (Uniton Number). *The uniton number is the smallest $n \in \mathbb{Z}$ such that $\gamma^n = 0$.*

In particular, this is the case when the uniton bundle has a section in a neighbourhood of $\lambda = 0$ which has a zero on the nth formal neighbourhood (i.e. as a truncated nth order polynomial).

6. Applications

Corollary (Finite Gap). *Maps with energy ≤ 3 are 1-unitons; therefore, $\mathrm{Harm}_k^*(\mathbb{S}^2,\mathrm{U}(N))$, $k \leq 3$ (the first three nontrivial energy components) are connected.*

This bound on the uniton number is interesting in that it marks a distinction between unitons and instantons. To the author's knowledge, it was not predicted. The bound is sharp as the U(3) uniton from [An2] has energy 4 and is a 2-uniton.

Finally, to demonstrate the potential for naive approaches to moduli topology we calculate

Proposition (Moduli Topology).

$$\pi_0\left(\mathrm{Harm}_4^*(\mathbb{S}^2, \mathrm{U}(N))\right) = 0, \quad \pi_1\left(\mathrm{Harm}_4^*(\mathbb{S}^2, \mathrm{U}(N))\right) = 0,$$

for $N \geq 3$.

Martin Guest is also able to make these computations by extending the ideas of [FGKO].

7. Prospects

One should be able to calculate the homology of low energy components using the L-stratification spectral sequence [BHMM] or GIT techniques. One could also try to compute the relative invariants of $\mathrm{Harm}(\mathbb{S}^2, \mathrm{Gr}_k(\mathbb{C}^N)) \hookrightarrow \mathrm{Harm}(\mathbb{S}^2, \mathrm{U}(N))$ (based and unbased) to understand the relationship between these spaces, and get information about the Grassmannian maps. These questions are the subject of joint enquiry with A. Crawford.

Related to the conjecture on the uniton number are the problems of trying to represent the bundle transformations which correspond to

1. composing the harmonic map with $z \mapsto 1/z$ (since translations act nicely on the monad, this would give us an action of $\mathrm{Conf}(\mathbb{S}^2, \mathbb{S}^2)$ on the bundles) and
2. adding a uniton/performing a flag transformation.

These operations should be interesting in their own right and understanding the second is a first step toward possible generalisations to maps into the symmetric spaces as in [BuRa].

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