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MÖBIUS TRANSFORMATIONS AND MONOGENIC FUNCTIONAL CALCULUS

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ABSTRACT. A new way of doing functional calculi is presented. A functional calculus $\Phi : f(x) \to f(T)$ is not an algebra homomorphism of a functional algebra into an operator algebra, but an intertwining operator between two representations of a group acting on the two algebras (as linear spaces).

This scheme is shown on the newly developed monogenic functional calculus for an arbitrary set of non-commuting self-adjoint operators. The corresponding spectrum and spectral mapping theorem are included.

1. INTRODUCTION

Functional calculus and the corresponding spectral theory belong to the heart of functional analysis [8, 16, 25]. J. L. TAYLOR [22] formulated a problem of functional calculus in the following terms:

For a given element a of a Banach algebra $\mathfrak{A}$ the correspondence $x \mapsto a$ gives rise to a representation of the algebra of polynomials in the variable x in $\mathfrak{A}$. It is necessary to extend the representation to a larger functional algebra.

The scheme works perfectly for several commuting variables x_i and commuting elements $a_i \in \mathfrak{A}$, and gives rise to the analytic Taylor calculus [21, 22]. But for the non-commuting case the situation is not so simple: searching for an appropriate substitution for a functional algebra in commuting variables was done in classes of Lie algebras [22], freely generated algebras [23], cumbersome $\times$-algebras [14], etc. For the Weyl calculus [2] no algebra homomorphism is known.

The main point of our approach (firstly presented in the paper) is the following reformulation of the problem, where the shift from an algebra *homomorphism* to group *representations* is made:

Definition 1.1. Let a group algebra $L_1(G)$ have linear representations $\pi_F(g)$ in a space of functions F and $\pi_B(g)$ in a Banach algebra B (depending on a set of elements $T \subset B$). One says that a linear mapping $\Phi : F \to B$ is a *functional*

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calculus if Φ is an intertwining operator between π_F and π_B, namely

$$\Phi[\pi_F(g)f] = \pi_B(g)\Phi[f],$$

for all $g \in G$ and $f \in F$. Additional conditions will be stated later.

One may ask whether there is a connection between these two formulations at all. It is possible to check that the classic Dunford-Riesz calculus (see [17, Chap. IX] and [8, Chap. 2]) is generated by the group of fractional-linear transformations of the complex line; the Weyl [2] and continuous functional calculi [16, Chap. VII] are generated by the affine group of the real line; the analytic functional calculus [21, 25] is based on biholomorphic automorphisms of a domain $U \subset \mathbb{C}^n$; and the monogenic (Clifford-Riesz) calculus [14] is generated by the group of Möbius transformations in $\mathbb{R}^n$. Moreover, both the Weyl and the analytic calculi can be obtained from the monogenic one by restrictions of the symmetry group (see Theorem 3.8 and Corollary 3.17). Such connections are based on the role of group representations in function theories (particularly on links with integral representations [10, 12]).

We will postpone the general results [13] about functional calculi. The subject of this paper is the construction of the monogenic calculus from the Möbius transformations (for other calculi see Remark 4.3). The monogenic and the Riesz-Clifford [14] calculi are cousins but not twins: besides having conceptually different starting points, they are technically different too. For example, for a function $\mathbb{R}^n \to \mathbf{Cl}(n)$ we construct a calculus for n-tuple of operators, not for $(n-1)$-tuples as it was done in [14].

Due to the brief nature of present paper, "proofs" mean "hints" or "sketches of proof".

2. Clifford analysis and the conformal group

2.1. **The Clifford algebra and the Möbius group.** Let $\mathbb{R}^n$ be a Euclidean n-dimensional vector space with a fixed frame $a_1, a_2, \ldots, a_n$ and let $\mathbf{Cl}(n)$ be the *real Clifford algebra* [24, § 12.1] generated by 1, e_j, $1 \leq j \leq n$ and the relations

$$e_i e_j + e_j e_i = -2\delta_{ij}, \qquad 1e_i = e_i 1 = e_i.$$

We put $e_0 = 1$. The embedding $\mathfrak{i} : \mathbb{R}^n \to \mathbf{Cl}(n)$ is defined by the formula:

$$\mathfrak{i} : x = \sum_{j=1}^{n} x_j a_j \mapsto \mathbf{x} = \sum_{j=1}^{n} x_j e_j. \tag{1}$$

We identify $\mathbb{R}^n$ with its image under $\mathfrak{i}$ and call its elements *vectors*. There are two linear anti-automorphisms $*$ and $-$ of $\mathbf{Cl}(n)$ defined on its basis $A_\nu = e_{j_1} e_{j_2} \cdots e_{j_r}$, $1 \leq j_1 < \cdots < j_r \leq n$ by the rule:

$$(A_\nu)^* = (-1)^{\frac{r(r-1)}{2}} A_\nu, \qquad \bar{A}_\nu = (-1)^{\frac{r(r+1)}{2}} A_\nu.$$

In particular, for vectors, $\bar{\mathbf{x}} = -\mathbf{x}$ and $\mathbf{x}^* = \mathbf{x}$. It is easy to see that $\mathbf{xy} = \mathbf{yx} = 1$ for any $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} = \bar{\mathbf{x}}\,\|\mathbf{x}\|^{-2}$, which is the *Kelvin inverse* of $\mathbf{x}$. Finite products of vectors are invertible in $\mathbf{Cl}(n)$ and form the *Clifford group* Γ_n. Elements $a \in \Gamma_n$ such that $a\bar{a} = \pm 1$ form the $\mathrm{Pin}(n)$ group—the double cover of the group of orthogonal rotations $\mathrm{O}(n)$.

Let (a, b, c, d) be a quadruple from $\Gamma_n \cup \{0\}$ with the properties:

1. $(ad^* - bc^*) \in \mathbb{R} \setminus 0$;
2. ab^*, cd^*, c^*a, d^*b are vectors.

Then the *Vahlen* [1, 4, 19] 2×2-matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ form a semisimple [9, § 6.2] group $V(n)$ under the usual matrix multiplication. It has a representation $\pi_{\mathbb{R}^n}$ by transformations of $\mathbb{R}^n \cup \{\infty\}$ given by:

$$\pi_{\mathbb{R}^n}\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \mathbf{x} \mapsto (a\mathbf{x}+b)(c\mathbf{x}+d)^{-1}, \tag{2}$$

which form the *Möbius* (or the *conformal* [24, Chap. 10]) group of $\mathbb{R}^n$. The analogy with fractional-linear transformations of the complex line $\mathbb{C}$ is useful, as well as representations of shifts $\mathbf{x} \mapsto \mathbf{x}+y$, orthogonal rotations $\mathbf{x} \mapsto k(a)\mathbf{x}$, dilatations $\mathbf{x} \mapsto \lambda\mathbf{x}$, and the Kelvin inverse $\mathbf{x} \mapsto \mathbf{x}^{-1}$ by the matrices $\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} a & 0 \\ 0 & a^{*-1} \end{pmatrix}$, $\begin{pmatrix} \lambda^{1/2} & 0 \\ 0 & \lambda^{-1/2} \end{pmatrix}$, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, respectively.

2.2. **Möbius transformations of unit balls and Clifford function theory.** One usually says that the conformal group in $\mathbb{R}^n$, $n > 2$ is not so rich as the conformal group in $\mathbb{R}^2$. Nevertheless, the conformal covariance has many applications in Clifford analysis [19]. Notably, groups of conformal mappings of open unit balls $\mathbb{B}^n \subset \mathbb{R}^n$ *on*to itself are similar for all n and as sets can be parametrized by the product of $\mathbb{B}^n$ itself and the group of isometries of its boundary $\mathbb{S}^{n-1}$.

Lemma 2.1. *Let $a \in \mathbb{B}^n$, $b \in \Gamma_n$; then the Möbius transformations of the form*

$$\phi_{(a,b)} = \begin{pmatrix} b & 0 \\ 0 & b^{*-1} \end{pmatrix}\begin{pmatrix} 1 & -a \\ a^* & -1 \end{pmatrix} = \begin{pmatrix} b & -ba \\ b^{*-1}a^* & -b^{*-1} \end{pmatrix}$$

constitute the group B_n of conformal mappings of the open unit ball $\mathbb{B}^n$ onto itself. B_n acts on $\mathbb{B}^n$ transitively. Transformations of the form $\phi_{(0,b)}$ constitute a subgroup isomorphic to $\mathrm{O}(n)$. The homogeneous space $B_n/\mathrm{O}(n)$ is isomorphic as a set to $\mathbb{B}^n$. Moreover:

1. *$\phi^2_{(a,1)} = 1$ identically on $\mathbb{B}^n$ ($\phi^{-1}_{(a,1)} = \phi_{(a,1)}$).*
2. *$\phi_{(a,1)}(0) = a$, $\phi_{(a,1)}(a) = 0$.*

Obviously, conformal mappings preserve the space of harmonic functions, i.e., null solutions to the *Laplace* operator $\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$. They also preserve the space of *monogenic* functions, i.e., null solutions $f : \mathbb{R}^n \to \mathbf{Cl}(n)$ of the *Dirac* [3, 7] operator $D = \sum_{j=1}^n e_i \frac{\partial}{\partial x_j}$ (note that $\Delta = D\bar{D} = -D^2$). The group B_n is sufficient for construction of the Poisson integral representation of harmonic functions and the Cauchy and Bergman formulas in Clifford analysis by the formula [10]

$$K(x,y) = c\int_G [\pi_g f](x)\overline{[\pi_g f](y)}\, dg, \tag{3}$$

where π_g is an irreducible unitary square integrable representation [9, § 9.3] of a group G, $f(x)$ is an arbitrary non-zero function, and c is a constant.

3. Monogenic calculus

3.1. **Representations of the Möbius group in C^*-algebras.** Fix an n-tuple of *bounded self-adjoint* elements $T = (T_1, \ldots, T_n)$ of a C^*-algebra $\mathfrak{A}$. Let $\boldsymbol{\mathfrak{A}} =$

$\mathfrak{A} \otimes \mathbf{Cl}(n)$. We can associate with T an element $\mathbf{T} \in \mathfrak{A}$ by the formula [15]

$$\mathbf{T} = \sum_{j=1}^{n} T_j \otimes e_j$$

in analogy with the embedding i of (1). One notes the naturality of the following

Definition 3.1. The *Clifford resolvent set* $R(\mathbf{T})$ of an n-tuple T is the maximal open subset of $\mathbb{R}^n$ such that for $\lambda \in R(\mathbf{T})$ the element $\mathbf{T} - \lambda I$ is invertible in $\mathfrak{A}$. The *Clifford spectrum* $\sigma(\mathbf{T})$ is the completion of the Clifford resolvent set $\mathbb{R}^n \setminus R(\mathbf{T})$.

For example, the joint spectrums of one $\{j_1\}$, two $\{j_1, j_2\}$, three $\{j_1, j_2, j_3\}$ Pauli matrices [24, § 12.4] are the unit sphere $\mathbb{S}^0$ $(= \{-1, 1\})$, the origin $(0,0)$ (= a sphere of zero radius), the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^1$, $\mathbb{R}^2$, and $\mathbb{R}^3$, respectively. The canonical connection between the Grassmann and Clifford algebras gives (see [25, § III.6])

Lemma 3.2. *Let an n-tuple T consist of mutually commuting operators in $\mathfrak{A}$. Then the Taylor joint spectrum*[1] [21] *and the Clifford spectrum coincide.*

Let V be a subgroup of the Vahlen group $V(n)$ such that for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in V$ we have $c^{-1}d \in R(\mathbf{T})$ (i.e., $c\mathbf{T} + d$ is invertible in $\mathfrak{A}$). Then there is a (non-linear) representation π_T of V:

$$\pi_T \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \mathbf{T} \mapsto (a\mathbf{T} + b)(c\mathbf{T} + d)^{-1}. \tag{4}$$

The following easy lemma has a useful corollary.

Lemma 3.3. *A vector λ belongs to the resolvent set $R(\mathbf{T})$ if and only if $\phi_{(\lambda,1)}(\mathbf{T})$ is well defined.*

Corollary 3.4. *A vector λ belongs to the spectrum $\sigma(\mathbf{T})$ if and only if the vector $\phi_{(\lambda,1)}(\beta)$ belongs to the spectrum $\sigma(\phi_{(\beta,1)}(\mathbf{T}))$.*

Having a representation π of the group G in a *linear* space X one can define the *linear* representation π_L of the convolution algebra $L_1(G)$ in the space of mappings $X \to X$ by the rule

$$\pi_L(f) = \int_G f(g)\pi(g)\, dg, \tag{5}$$

where dg is Haar measure on G.

We denote such an extension of $\pi_{\mathbb{R}^n}$ of (2) by π_F and the extension of π_T of (4) by $\pi_{\mathfrak{A}}$. Let $F(\mathbb{R}^n) = \pi_F(L_1(G))$, $\widehat{f}(\mathbf{x}) = \pi_F(f)$, $\widehat{f}(\mathbf{T}) = \pi_{\mathfrak{A}}(f)$. Following Definition 1.1 we give

Definition 3.5. A *monogenic functional calculus* $\Phi_V : F(\mathbb{R}^n) \to \mathfrak{A}$ for a subgroup V and n-tuple T is a linear operator with the following properties:

1. It is an intertwining operator between the two representations π_F and $\pi_{\mathfrak{A}}$:
$$\Phi_V \pi_F = \pi_{\mathfrak{A}} \Phi_V.$$
2. $\Phi\widehat{1} = I$, i.e., the image of the function identically equal to 1 is the unit operator.
3. $\Phi\widehat{\delta_e} = \mathbf{T}$, where δ_e is the delta function centered at the group unit.

[1] There are less used non-commutative versions of the Taylor joint spectrum [22].

The following theorem is group-independent:

Theorem 3.6. *Let $k(g)$ and $l(g)$ be functions on V such that both $\widehat{l}(\mathbf{T})$ and $\widehat{k}(\widehat{l}(\mathbf{T}))$ are well defined. Then $\widehat{k}(\widehat{l}(\mathbf{T})) = [\widehat{k * l}](\mathbf{T})$ ($*$ is convolution on the group). In particular, $\pi_{\mathfrak{A}}(g)[\widehat{l}(\mathbf{T})] = [\widehat{\pi_l(g) l}](\mathbf{T})$.*

Theorem 3.7 (Uniqueness)**.** *If the representation π_F is irreducible, then the functional calculus (if any) is unique. Particularly, for a given simply connected domain Ω and an n-tuple of operators T, there exists no more than one monogenic calculus.*

Theorem 3.8. *For any n-tuple T of bounded self-adjoint operators there exists a monogenic calculus on $\mathbb{R}^n$. This calculus coincides on monogenic functions with the Weyl functional calculus from* [2]. *Particularly, the polynomial functions of operators are the symmetric (Weyl) polynomials.*

Proof. Möbius transformations of $\mathbb{R}^n$ (without the point ∞) are the affine transformations of $\mathbb{R}^n$, which are represented via upper-triangular Vahlen matrices. But the Weyl calculus is defined uniquely by its affine covariance [2, Theorem 2.4(a)] according to the following lemma, which is given without proof. □

Lemma 3.9. *The one-dimensional Weyl calculus is uniquely defined by its affine covariance.*

The following result is a direct counterpart of the classic one.

Theorem 3.10. *For the existence of the monogenic functional calculus in the unit ball $\mathbb{B}^n$ it is necessary that $\sigma(\mathbf{T}) \subset \mathbb{B}^n$.*

In the next subsection we obtain via an integral representation that this condition is also sufficient.

3.2. **Integral representation.** The best way to obtain an integral formula for the monogenic calculus is to use (3). To save space we will now use another straightforward approach, which however could be obtained from (3) by a suitable representation of $\mathbb{Z}$. We already know from Theorem 3.8 how to construct monogenic functions for polynomials. Thus, by the linearity (via decomposition [7, Chap. II, (1.16)]) there is only one way to define the Cauchy kernel of the operators T_j.

Definition 3.11. Let us define (we use notation from [7])

$$E(y,T) = \sum_{j=0}^{\infty} \left(\sum_{|\alpha|=j} V_\alpha(T) W_\alpha(y) \right), \tag{6}$$

where

$$W_\alpha(x) = (-1)^{|\alpha|} \partial^\alpha E(x) = \frac{(-1)^{|\alpha|}\Gamma(\frac{n+1}{2})}{2\pi^{(n+1)/2}} \partial^\alpha \frac{\overline{x}}{|x|^{n+1}},$$

$$V_\alpha(T) = \frac{1}{\alpha!} \sum_\sigma (T_1 e_{\sigma(1)} - T_{\sigma(1)})(T_1 e_{\sigma(2)} - T_{\sigma(2)}) \cdots (T_1 e_{\sigma(n)} - T_{\sigma(n)}), \tag{7}$$

where the summation is taken over all possible permutations.

Lemma 3.12. *Let $|T| = \lim_{j\to\infty} \sup_\sigma \left\| T_{\sigma(1)} \cdots T_{\sigma(j)} \right\|^{1/j}$, $1 \le \sigma(i) \le n$ be the Rota-Strang joint spectral radius* [18]. *Then for a fixed $|y| > |T|$, equation* (6) *defines a bounded operator in $\mathfrak{A}$.*

As usual, by Liouville's theorem [15, Theorem 5.5] for the function $E(y,T)$ the Clifford spectrum is not empty; by definition it is closed and by the previous lemma it is bounded. Thus

Lemma 3.13. *The Clifford spectrum $\sigma(\mathbf{T})$ is compact.*

Lemma 3.14. *Let $r > |T|$ and let Ω be the ball $\mathbb{B}(0,r) \in \mathbb{R}^n$. Then we have*

$$P(T) = \int_{\partial \mathbb{B}} E(y,T)\, dn_y\, P(y), \tag{8}$$

where $P(T)$ is the symmetric polynomial (7) of the n-tuple T.

Lemma 3.15. *For any domain Ω that does not contain $\sigma(\mathbf{T})$ and any $f \in H(\Omega)$, we have*

$$\int_{\partial \Omega} E(y,T)\, dn_y\, f(y) = 0. \tag{9}$$

Due to this lemma we can replace the domain $\mathbb{B}(0,r)$ in Lemma 3.14 with an arbitrary domain Ω containing the spectrum $\sigma(\mathbf{T})$. An application of Lemma 3.14 gives the main

Theorem 3.16. *Let $T = (T_1, \ldots, T_n)$ be an n-tuple of bounded self-adjoint operators. Let the domain Ω with piecewise smooth boundary have a connected complement and suppose the spectrum $\sigma(\mathbf{T})$ lies inside the domain Ω. Then the mapping*

$$\Phi : f(x) \mapsto f(T) = \int_{\partial \Omega} E(y,T)\, dn_y\, f(y) \tag{10}$$

defines the monogenic calculus for T.

Because holomorphic functions are also monogenic, we can apply our construction to them. For a commuting n-tuple T the calculus constructed above will coincide with the analytic one [25] on the polynomials. This gives

Corollary 3.17. *Restriction of the monogenic calculus for a commuting n-tuple T to holomorphic functions produces the analytic functional calculus.*

3.3. **Spectral mapping theorem.** A good notion of functional calculus should be connected with a good notion of spectrum via the spectral mapping theorem. In the analytic calculus [8, II.2.23] and the functional calculus of self-adjoint operators [16, Theorem VII.1(e)] the homomorphism property plays a crucial role in proofs of spectral mapping theorems. Nevertheless, in our versions these results are also preserved.

It is easy to check that formula [1, (3.5)] can be adapted in the following way:

Lemma 3.18. *Let $(-c^{-1}d) \notin \sigma(\mathbf{T})$ and $\mathbf{x} \neq -c^{-1}d$. Let g be the action of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathfrak{A}$. Then*

$$g(\mathbf{T}) - g(\mathbf{x}) = (c\mathbf{x}+d)^{*-1}(\mathbf{T}-\mathbf{x})(c\mathbf{T}+d)^{-1}.$$

Theorem 3.19 (Spectral mapping theorem)**.**

$$\sigma(\widehat{f}(\mathbf{T})) = \{\widehat{f}(\mathbf{x}) \mid \mathbf{x} \in \sigma(\mathbf{T})\}.$$

Proof. Let $x \in \sigma(\mathbf{T})$ and $\mathbf{y} = f(\mathbf{x})$ then

$$\begin{aligned}\widehat{f}(\mathbf{T}) - \mathbf{y} &= \widehat{f}(\mathbf{T}) - \widehat{f}(\mathbf{x})\\ &= \int_V f(g)g(\mathbf{T})\,dg - \int_V f(g)g(\mathbf{x})\,dg\\ &= \int_V f(g)\left[g(\mathbf{T}) - g(\mathbf{x})\right]\,dg\\ &= \int_V f(g)(c\mathbf{x}+d)^{*-1}(\mathbf{T}-\mathbf{x})(c\mathbf{T}+d)^{-1}\,dg.\end{aligned}$$

Thus $f(\mathbf{T}) - \mathbf{y}$ belongs to the closed proper left ideal generated by $\mathbf{T} - \mathbf{x}$.

Otherwise let $\widehat{f}(\mathbf{x}) \neq \mathbf{y}$ for all $\mathbf{x} \in \Omega \supset \sigma(\mathbf{T})$. Analogously to Lemma 3.3 with the help of Theorem 3.6 we conclude that the function

$$[\widehat{\phi_{(y,1)}f}](\mathbf{x}) = \phi_{(y,1)} \int_V f(g)g(\mathbf{x})\,dg = \int_V f(\phi_{(y,1)}g)g(\mathbf{x})\,dg$$

is well defined on Ω. Thus we can define an operator

$$[\widehat{\phi_{(y,1)}f}](\mathbf{T}) = \int_V f(\phi_{(y,1)}g)g(\mathbf{T})\,dg.$$

Due to Lemma 3.3 the existence of such an operator provides us with the invertibility of the operator $\widehat{f}(\mathbf{T}) - \mathbf{y}$. □

4. Concluding remarks

Remark 4.1. It seems worthwhile (if not very natural) to connect the two main motives of functional analysis: functional calculus and group representations. Such a connection causes changes in applications based on the functional calculi (quantum mechanics [11]): to quantize one could use not algebra homomorphisms (observables are usually believed to form an algebra) but representations of groups (symmetry of the system under consideration).

Remark 4.2. There is the Cayley transformation [1] of the unit ball to the "upper half-plane" $x_n \geq 0$. This allows one to make a straightforward modification of the monogenic calculus for an n-tuple of operators where only T_n is semibounded (Hamiltonian!) and the other operators are just unbounded.

Remark 4.3. The classic Dunford-Riesz [17, Chap. IX] calculus can be obtained similarly using the representation of $\mathrm{SL}(2,\mathbb{R})$ by fractional-linear mappings [24] of the complex line $\mathbb{C}$. The functional calculus of a self-adjoint operator [16, Chap. VII] can be obtained from affine transformations of the real line. But a non-formal integral formula is possible only through an integral representation of the δ-function: in this way we have arrived, for example, at the Weyl functional calculus [2]. Biholomorphic automorphisms of unit ball in $\mathbb{C}^n$ also form a sufficiently large group (subgroup of the Möbius group) to construct a function theory [12]. Thus, we can develop a functional calculus based on the theory of several complex variables. Due to its biholomorphic covariance [5, 6] it must coincide with the well-known analytic calculus [25].

Remark 4.4. Our Definition 1.1 is connected with *Arveson-Connes spectral theory* [20]. The main differences are: (i) most of our groups are non-commutative; (ii) we use a space of analytic functions (not functions on the group directly) as a model of functional calculus and definition of a spectrum. It seems that the first feature is mainly due to the second one.

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