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GEOMETRIC TOPOLOGY OF STRATIFIED SPACES

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ABSTRACT. A theory of tubular neighborhoods for strata in manifold stratified spaces is developed. In these topologically stratified spaces, manifold stratified approximate fibrations and teardrops play the role that fibre bundles and mapping cylinders play in smoothly stratified spaces. Applications include a multiparameter isotopy extension theorem, neighborhood germ classification and a topological version of Thom's First Isotopy Theorem.

1. INTRODUCTION

Often spaces are studied which are not manifolds, but which are composed of manifold pieces, those pieces being called the *strata* of the space. Examples include polyhedra, algebraic varieties, orbit spaces of many group actions on manifolds, and mapping cylinders of maps between manifolds.

The purpose of this note is to announce recent progress in understanding the structure of neighborhoods of strata in certain spaces, namely, the stratified spaces proposed by Frank Quinn [18] and called by him 'manifold homotopically stratified sets'. Quinn's objective was 'to give a setting for the study of purely topological stratified phenomena' as opposed to the smooth and piecewise linear phenomena previously studied.

Roughly, the stratified spaces of Quinn are spaces X together with a finite filtration by closed subsets

$$X = X^m \supseteq X^{m-1} \supseteq \cdots \supseteq X^0 \supseteq X^{-1} = \emptyset,$$

such that the strata $X_i = X^i \setminus X^{i-1}$ are manifolds with neighborhoods in $X_i \cup X_k$ (for $k > i$) which have the local homotopy properties of mapping cylinders of fibrations. These spaces include the smoothly stratified spaces of Whitney [28], Thom [24] and Mather [16] (see e.g. [9]) as well as the locally conelike stratified spaces of Siebenmann [21] and, hence, orbit spaces of finite groups acting locally linearly on manifolds.

Smoothly stratified spaces have the property that strata have neighborhoods which are mapping cylinders of fibre bundles, a fact which is often used in arguments involving induction on the number of strata. Such neighborhoods fail to exist in general for Siebenmann's locally conelike stratified spaces. For example, it

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is known that a (topologically) locally flat submanifold of a topological manifold (which is an example of a locally conelike stratified space with two strata) may fail to have a tubular neighborhood [20]. However, Edwards [7] proved that such submanifolds do have neighborhoods which are mapping cylinders of manifold approximate fibrations (see also [14]). On the other hand, examples of Quinn [17] and Steinberger-West [23] show that strata in orbit spaces of finite groups acting locally linearly on manifolds may fail to have mapping cylinder neighborhoods. In Quinn's general setting, mapping cylinder neighborhoods may fail to exist even locally.

Our main result (Theorem 3.2) gives an effective substitute for neighborhoods which are mapping cylinders of bundles. Instead of fibre bundles, we use 'manifold stratified approximate fibrations,' and instead of mapping cylinders, we use 'teardrops'. This result should be thought of as a tubular neighborhood theorem for strata in manifold stratified spaces.

Applications are discussed in Section 5. They include a classification of neighborhood germs, a multiparameter isotopy extension theorem, the local contractibility of the homeomorphism group of a compact stratified space, a topological version of Thom's First Isotopy Theorem, and a generalization of Anderson-Hsiang pseudoisotopy theory.

In related recent work, Weinberger [27] has developed a surgery theoretic classification of manifold stratified spaces. In fact, one of the proofs envisioned by Weinberger for his theory relies on our main result (Theorem 3.2). The main result was first discovered in the case of two strata in the course of joint work with Taylor, Weinberger, and Williams [15] where the application to Weinberger's surgery theory is mentioned. The book by Hughes and Ranicki [12] contains a proof of the main result in the special case of two strata with the lower stratum consisting of a single point, and should be consulted for further background, examples, historical remarks and applications. The work of Steinberger and West [22], [23] has been very influential on the stratified point of view. Other important recent work includes the thesis of Yan [29], the speculations of Quinn [19] and the paper by Connolly and Vajiac [5].

Complete proofs will appear elsewhere, most notably in [11].

2. Stratified spaces and stratified approximate fibrations

We begin by recalling some definitions from Quinn [18] (see also [12], [15]). A subset $Y \subset X$ is *forward tame* in X if there exist a neighborhood U of Y in X and a homotopy $h : U \times I \to X$ such that $h_0 = \text{inclusion} : U \to X$, $h_t|Y = \text{inclusion} : Y \to X$ for each $t \in I, h_1(U) = Y$, and $h((U \setminus Y) \times [0,1)) \subseteq X \setminus Y$.

Define the *homotopy link* of Y in X by

$$\text{holink}(X,Y) = \{\omega \in X^I \mid \omega(t) \in Y \text{ iff } t = 0\}.$$

Evaluation at 0 defines a map $q : \text{holink}(X,Y) \to Y$ called *holink evaluation*.

Let $X = X^m \supseteq X^{m-1} \supseteq \cdots \supseteq X^0 \supseteq X^{-1} = \emptyset$ be a space with a finite filtration by closed subsets. Then X^i is the *i-skeleton*, the difference $X_i = X^i \setminus X^{i-1}$ is called the *i-stratum*, and X is said to be a *space with a stratification*. A subset A of a space X with a stratification is called a *pure* subset if A is closed and a union of components of strata of X. For example, the skeleta are pure subsets.

The *stratified homotopy link* of Y in X, denoted by $\text{holink}_s(X,Y)$, consists of all ω in $\text{holink}(X,Y)$ such that $\omega((0,1])$ lies in a single stratum of X. The stratified homotopy link has a natural filtration with i–skeleton $\mathrm{holink_s}(X,Y)^i = \{\omega | \omega(1) \in X^i\}$. The holink evaluation (at 0) restricts to a map $q : \mathrm{holink_s}(X,Y) \to Y$.

If X is a filtered space, then a map $f : Z \times A \to X$ is *stratum preserving along* A if for each $z \in Z$, $f(\{z\} \times A)$ lies in a single stratum of X. In particular, a map $f : Z \times I \to X$ is a *stratum preserving homotopy* if f is stratum preserving along I.

A filtered space X is a *manifold stratified space* if the following four conditions are satisfied:

(i) **Manifold strata.** X is a locally compact, separable metric space and each stratum X_i is a topological manifold (without boundary).

(ii) **Forward tameness.** For each $k > i$, the stratum X_i is forward tame in $X_i \cup X_k$.

(iii) **Normal fibrations.** For each $k > i$, the holink evaluation $q : \mathrm{holink}(X_i \cup X_k, X_i) \to X_i$ is a fibration.

(iv) **Finite domination.** For each i there exists a closed subset K of the stratified homotopy link $\mathrm{holink_s}(X, X^i)$ such that the holink evaluation map $K \to X^i$ is proper, together with a stratum preserving homotopy

$$h : \mathrm{holink_s}(X, X^i) \times I \to \mathrm{holink_s}(X, X^i),$$

which is also fibre preserving over X^i (i.e., $qh_t = q$ for each $t \in I$), such that $h_0 = \mathrm{id}$ and $h_1(\mathrm{holink_s}(X, X^i)) \subseteq K$.

For $x \in X^i$, the subset $q^{-1}(x) \subseteq \mathrm{holink_s}(X, X^i)$ is called the *stratified local holink at* x. Note that condition (iv) implies that the stratified local holinks are finitely dominated.

Quinn [18] calls a filtered space satisfying (ii) and (iii) a 'homotopically stratified set', and he calls such a space a 'manifold homotopically stratified set' if it additionally satisfies (i) (he also allows the manifold strata to have boundaries). These four conditions are not independent; in fact, (iv) follows from the other conditions assuming some fundamental group properties. Quinn implicitly assumes these properties so that our manifold stratified spaces are essentially the same as Quinn's manifold homotopically stratified sets (cf. [12, 9.15–18, 10.13–14]).

Next we generalize the definition of an approximate fibration (as given in [13]) to the stratified setting. Let $X = X^m \supseteq \cdots \supseteq X^0$ and $Y = Y^n \supseteq \cdots \supseteq Y^0$ be filtered spaces and let $p : X \to Y$ be a map (p is not assumed to be stratum preserving). Then p is said to be a *stratified approximate fibration* provided given any space Z and any commuting diagram

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "\times 0"'] & X \arrow[d, "p"] \\ Z \times I \arrow[r, "F"] & Y \end{tikzcd}$$

where F is a stratum preserving homotopy, there exists a *stratified controlled solution*, i.e., a map $\tilde{F} : Z \times I \times [0,1) \to X$ which is stratum preserving along $I \times [0,1)$ such that $\tilde{F}(z,0,t) = f(z)$ for each $(z,t) \in Z \times [0,1)$ and the function $\bar{F} : Z \times I \times [0,1] \to Y$ defined by $\bar{F}|Z \times I \times [0,1) = p\tilde{F}$ and $\bar{F}|Z \times I \times \{1\} = F \times \mathrm{id}_{\{1\}}$ is continuous.

A stratified approximate fibration between manifold stratified spaces is a *manifold stratified approximate fibration* if, in addition, it is a proper map (i.e., inverse images of compact sets are compact). The following result suggests that there is a relationship between manifold stratified spaces and manifold stratified approximate fibrations.

Proposition 2.1. *Let $p : X \to Y$ be a map between manifold stratified spaces. Then the open mapping cylinder $\overset{\circ}{\mathrm{cyl}}(p)$ is a manifold stratified space if and only if p is a manifold stratified approximate fibration.*

If $Y = Y^n \supseteq \cdots \supseteq Y^0$ and $X = X^m \supseteq \cdots \supseteq X^0$, then in the proposition above $\overset{\circ}{\mathrm{cyl}}(p)$ is filtered so that the strata are given by

$$(\overset{\circ}{\mathrm{cyl}}(p))_i = \begin{cases} Y_i & \text{if } 0 \le i \le n, \\ X_{i-n-1} \times (0,1) & \text{if } n+1 \le i \le m+n+1. \end{cases}$$

Note that Y is a pure subset of the open mapping cylinder $\overset{\circ}{\mathrm{cyl}}(p)$.

3. Teardrop structure on neighborhoods

Given spaces X, Y and a map $p : X \to Y \times \mathbb{R}$, the *teardrop* of p (see [15]) is the space denoted by $X \cup_p Y$ whose underlying set is the disjoint union $X \amalg Y$ with the minimal topology such that

(i) $X \subseteq X \cup_p Y$ is an open embedding, and
(ii) the function $c : X \cup_p Y \to Y \times (-\infty, +\infty]$ defined by

$$c(x) = \begin{cases} p(x) & \text{if } x \in X, \\ (x, +\infty) & \text{if } x \in Y, \end{cases}$$

is continuous.

The map c is called *the tubular map of the teardrop* or *the teardrop collapse*. The tubular map terminology comes from the smoothly stratified case (see [16], [25], [6]). This is a generalization of the construction of the open mapping cylinder of a map $g : X \to Y$. Namely, $\overset{\circ}{\mathrm{cyl}}(g)$ is the teardrop $(X \times \mathbb{R}) \cup_{g \times \mathrm{id}} Y$. The following result is an analogue of Proposition 2.1 for teardrops.

Theorem 3.1. *If X and Y are manifold stratified spaces and $p : X \to Y \times \mathbb{R}$ is a manifold stratified approximate fibration, then $X \cup_p Y$ is a manifold stratified space with Y a pure subset.*

In this statement, $Y \times \mathbb{R}$ and $X \cup_p Y$ are given the natural stratifications.

The main result is a kind of converse to this proposition. First, some more definitions. A subset Y of a space X has a *teardrop neighborhood* if there exist a neighborhood U of Y in X and a map $p : U \setminus Y \to Y \times \mathbb{R}$ such that the natural function $(U \setminus Y) \cup_p Y \to U$ is a homeomorphism. In this case, U is the *teardrop neighborhood* and p is the restriction of the tubular map.

Theorem 3.2 (Teardrop neighborhood existence)**.** *Let X be a manifold stratified space such that all components of strata have dimension greater than 4, and let Y be a pure subset. Then Y has a teardrop neighborhood whose tubular map $c : U \to Y \times (-\infty, +\infty]$ is a manifold stratified approximate fibration.*

Note that it follows that the restriction $c| : U \setminus Y \to Y \times \mathbb{R}$ is also a manifold stratified approximate fibration. This is what was established for the two strata case in [15], so that 3.2 is a slight improvement of [15] even for two strata. Recall from the introduction that Y need not have a mapping cylinder neighborhood in X.

4. The main tools

There are two tools which are important in the proof of the Teardrop Neighborhood Existence Theorem, 'stratified sucking' and 'stratified straightening.' These generalize unstratified results of Chapman [4], Hughes [10], and Hughes-Taylor-Williams [13]. Stratified sucking gives a condition for a map to be close to a manifold stratified approximate fibration, whereas stratified straightening can be thought of as a uniqueness result which gives a condition for two manifold stratified approximate fibrations to be controlled homeomorphic.

Let $X = X^m \supseteq \cdots \supseteq X^0$ and $Y = Y^n \supseteq \cdots \supseteq Y^0$ be filtered spaces. Let β be an open cover of Y. Then a map $p : X \to Y$ is a *stratified β–fibration* provided given any space Z and any commuting diagram

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ {\scriptstyle \times 0}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ Z \times I & \xrightarrow{F} & Y \end{array}$$

where F is a stratum preserving homotopy, there exists a stratum preserving homotopy $\tilde{F} : Z \times I \to X$ such that $\tilde{F}(z, 0) = f(z)$ for each $z \in Z$ and $p\tilde{F}$ is β -close to F.

For the remainder of the section, suppose X and Y are manifold stratified spaces such that all components of strata have dimension greater than 4. For undefined terms related to the *controlled category* see [13].

Theorem 4.1 (Stratified sucking). *For every open cover α of Y there exists an open cover β of Y such that if $p : X \to Y$ is a proper stratified β-fibration, then p is properly α-homotopic to a manifold stratified approximate fibration.*

We remark that there are also relative and Δ^k-parameter versions of stratified sucking.

Theorem 4.2 (Stratified straightening). *Suppose $p : X \times \Delta^k \to Y \times \Delta^k$ is a map which is fibre preserving over Δ^k and stratum preserving along Δ^k. Suppose further that for each $t \in \Delta^k$, $p_t : X \times \{t\} \to Y \times \{t\}$ is a manifold stratified approximate fibration. Then there exists a homeomorphism $h : X \times \Delta^k \times [0, 1) \to X \times \Delta^k \times [0, 1)$ such that*

(i) *h is fibre preserving over $\Delta^k \times [0, 1)$,*
(ii) *h is stratum preserving along $\Delta^k \times [0, 1)$,*
(iii) *$h(x, 0, s) = (x, 0, s)$ for each $(x, s) \in X \times [0, 1)$,*
(iv) *h is a controlled map from $p_0 \times \Delta^k$ to p where $0 \in \Delta^k$ is a vertex; i.e., the function $\bar{h} : X \times \Delta^k \times [0, 1] \to Y \times \Delta^k$ defined by $\bar{h}|X \times \Delta^k \times [0, 1) = (p_0 \times \mathrm{id}_{\Delta^k}) \circ \mathrm{proj} \circ h$ and $\bar{h}|X \times \Delta^k \times \{1\} = p$ is continuous.*

A useful consequence of the straightening principle is the fact that manifold stratified approximate fibrations have an isotopy covering property which we now state.

Corollary 4.3 (Controlled stratified isotopy covering). *Let $p : X \to Y$ be a manifold stratified approximate fibration, and let $H : Y \times \Delta^k \to Y \times \Delta^k$ be a stratum* *preserving isotopy; i.e., H is a homeomorphism such that H is fibre preserving over Δ^k, stratum preserving along Δ^k, and $H_0 = \mathrm{id} : Y \times \{0\} \to Y \times \{0\}$. Then there exists a homeomorphism*

$$\tilde{H} : X \times \Delta^k \times [0,1) \to X \times \Delta^k \times [0,1)$$

such that

(i) *$\tilde{H}$ is fibre preserving over $\Delta^k \times [0,1)$,*
(ii) *$\tilde{H}$ is stratum preserving along $\Delta^k \times [0,1)$,*
(iii) *$\tilde{H}(x,0,s) = (x,0,s)$ for all $(x,s) \in X \times [0,1)$,*
(iv) *$\tilde{H}$ is a controlled map from $p \times \mathrm{id}_{\Delta^k}$ to $H \circ (p \times \mathrm{id}_{\Delta^k})$.*

Proof. Apply Theorem 4.2 to the map $H \circ (p \times \mathrm{id}_{\Delta^k})$. □

In the two stratum case the proof in [15] of the Teardrop Neighborhood Existence Theorem relied on the sucking principle whereas the corresponding uniqueness result relied on the straightening principle, both principles in the manifold (unstratified) case. The proof of 3.2 in the multiply stratified case involves a complicated induction on the number of strata in the pure subset $Y \subseteq X$ and the number of strata in the complement $X \setminus Y$, and the stratified straightening principle must be proved as part of the induction.

5. Applications

One of Quinn's main results in [18] is an isotopy extension theorem for manifold stratified spaces, a result which is quite useful for the theory of group actions (see [26] and [2]). Quinn's methods only apply to a single isotopy at a time. On the other hand, Siebenmann [21] had earlier established a *multiparameter* isotopy extension theorem for locally conelike stratified spaces. Our first application is a generalization to manifold stratified spaces.

Theorem 5.1 (Multiparameter isotopy extension). *Let X be a manifold stratified space such that all components of strata have dimension greater than 4, let Y be a pure subset of X, let U be a neighborhood of Y in X, and let $h : Y \times \Delta^k \to Y \times \Delta^k$ be a k-parameter stratum preserving isotopy. Then there exists a k-parameter stratum preserving isotopy $\tilde{h} : X \times \Delta^k \to X \times \Delta^k$ extending h and supported on $U \times \Delta^k$.*

Siebenmann's main goal in studying locally conelike stratified spaces was to provide a setting for generalizing the Edwards-Kirby [8] and Cernavskii [3] result on the local contractibility of the homeomorphism group of a compact manifold. Siebenmann's proof is adequate for manifold stratified spaces in general, so we have the following result.

Theorem 5.2 (Local contractibility). *Let X be a compact manifold stratified space such that all components of strata have dimension greater than 4. Then the group of all stratum preserving self-homeomorphisms of X is locally contractible in the compact-open topology.*

The next result is a topological analogue of Thom's First Isotopy Theorem [24]. This can be viewed as a first step towards a topological theory of topological stability.

Theorem 5.3 (First topological isotopy). *Let X be a manifold stratified space and let $p : X \to \mathbb{R}^n$ be a map such that*

(i) *p is proper,*
(ii) *for each stratum X_i of X, $p| : X_i \to \mathbb{R}^n$ is a topological submersion,*
(iii) *for each $t \in \mathbb{R}^n$, the filtration of X restricts to a filtration of $p^{-1}(t)$ giving $p^{-1}(t)$ the structure of a manifold stratified space such that all components of strata have dimension greater than 4.*

Then p is a bundle and can be trivialized by a stratum preserving homeomorphism; that is, there exists a stratum preserving homeomorphism $h : p^{-1}(0) \times \mathbb{R}^n \to X$ such that ph is projection.

The next application concerns the classification of neighborhoods of pure subsets of a manifold stratified space. Given a manifold stratified space Y, a *stratified neighborhood* of Y consists of a manifold stratified space containing Y as a pure subset. Two stratified neighborhoods X, X' of Y are *equivalent* if there exist neighborhoods U, U' of Y in X, X', respectively, and a stratum preserving homeomorphism $h : U \to U'$ such that $h|Y = \mathrm{id}$. A *neighborhood germ* of Y is an equivalence class of stratified neighborhoods of Y.

Theorem 5.4 (Neighborhood germ classification)**.** *Let Y be a manifold stratified space such that all components of strata have dimension greater than 4. Then the teardrop construction induces a one-to-one correspondence from controlled, stratum preserving homeomorphism classes of manifold stratified approximate fibrations over $Y \times \mathbb{R}$ to neighborhood germs of Y.*

When Y has just one stratum (i.e., Y is a manifold), one can use the following result (which generalizes [13], [14]) to give a classifying space description of the manifold stratified approximate fibrations which occur in the theorem above. For notation, let B be a connected i-manifold and let $q : V \to \mathbb{R}^i$ be a manifold stratified approximate fibration where all components of strata of V have dimension greater than 4. A manifold stratified approximate fibration $p : X \to B$ has *fibre germ* q if there exists an embedding $\mathbb{R}^i \subseteq B$ such that $p| : p^{-1}(\mathbb{R}^i) \to \mathbb{R}^i$ is controlled, stratum preserving homeomorphic to q. Fibre germs are unique up to controlled homeomorphism if the embedding $\mathbb{R}^i \subseteq B$ is orientation preserving in the case B is orientable. Let $\mathrm{TOP}^{\mathrm{level}}(q)$ denote the simplicial group of self-homeomorphisms of the mapping cylinder $\mathrm{cyl}(p)$ which preserve the mapping cylinder levels and are stratum preserving with respect to the induced stratification of $\mathrm{cyl}(q)$. Note that there is a restriction homomorphism $\mathrm{TOP}^{\mathrm{level}}(q) \to \mathrm{TOP}_i$.

Theorem 5.5 (MSAF classification)**.** *Controlled, stratum preserving homeomorphism classes of manifold stratified approximate fibrations over B with fibre germ q are in one-to-one correspondence with homotopy classes of lifts of the map $\tau : B \to \mathrm{BTOP}_i$ which classifies the tangent bundle of B, to $\mathrm{BTOP}^{\mathrm{level}}(q)$.*

Actually, Theorems 5.4 and 5.5 are just corollaries of deeper theorems which give homotopy equivalences between simplicial sets. Then 5.4 and 5.5 are just the statements of the results on the π_0 level.

Finally, we mention that the teardrop technology announced in this paper allows the stratified pseudoisotopy theory of Anderson-Hsiang [1] (which is valid in the locally conelike case) to be generalized to manifold stratified spaces. Like those of Anderson-Hsiang, our results are valid for the full space of pseudoisotopies, whereas Quinn's work [18] only gives information about π_0 of that space.

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