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ON THE CUT POINT CONJECTURE

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Abstract. We sketch a proof of the fact that the Gromov boundary of a hyperbolic group does not have a global cut point if it is connected. This implies, by a theorem of Bestvina and Mess, that the boundary is locally connected if it is connected.

The object of this note is to sketch a proof of the following theorem:

Theorem 1. *If Γ is a one ended hyperbolic group, then the Gromov boundary $\partial\Gamma$ of Γ does not have a global cut point.*

It seems that several people independently thought of using treelike structures in this context, among them Bill Grosso in an unpublished manuscript [8]. The most significant advances in this direction were carried out by Brian Bowditch in a brilliant series of papers ([4]-[7]). We draw heavily from his work. He proved Theorem 1 in the case when Γ is one ended and does not split over a two ended group [6], and in the case when Γ is strongly accessible and one ended [5]. Our strategy is to relativize some of the arguments of [4], [5], use Levitt's construction [9] to obtain an R-tree on which a subgroup of Γ acts isometrically, and then use a relative version of the main theorem of [2] to arrive at a contradiction. One of the main results of [4] asserts:

Theorem 2 (Bowditch [4])**.** *If $\partial\Gamma$ has a global cut point, then $\partial\Gamma$ has a nontrivial, equivariant dendrite quotient $D(\partial\Gamma)$ on which Γ acts as a discrete convergence group.*

We refer to [4] for definitions; a dendrite is a compact separable R-tree. We assume that $\partial\Gamma$ has a global cut point, and we start with a graph of group decomposition of Γ over two ended subgroups in which none of the vertex groups splits over a finite or two ended subgroup relative to the edge groups in it. We may assume that the action of Γ on the associated tree Σ is minimal, reduced, without inversions, and Σ/Γ is compact. Since we are assuming that $\partial\Gamma$ has a global cut point, such splittings of Γ exist by [6] and [1].

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By [5], [7], there is a natural decomposition $\partial\Gamma = \partial_0\Gamma \cup \partial_\infty\Gamma$, where

$$\partial_0\Gamma = \bigcup_{v \in V(\Sigma)} \Lambda\Gamma(v)$$

and $\partial_\infty\Gamma$ is naturally identified with $\partial\Sigma$ (ΛG for a subgroup of Γ denotes the limit set of G in $\partial\Gamma$ and may be identified with ∂G if G is quasiconvex in Γ). Our starting point was the observation that the end points of an edge group cannot be separated by a global cut point, and it follows from the description in [4] of the quotient map $\partial\Gamma \to D(\partial\Gamma)$ that:

Proposition 3. *The fixed points of $\Gamma(e)$, for each edge e, are identified in the dendrite quotient $D(\partial\Gamma)$.*

This was first observed by the methods of [10] and the generalized accessibility of [1], but Bowditch pointed out that it follows immediately from Proposition 4 below. We now consider the images in $D(\partial\Gamma)$ of $\Lambda(\Gamma(v)$, for $v \in V(\Sigma)$. Bowditch considers for each directed edge $\vec{e}$ of Σ a subset $\Psi(\vec{e})$ of $\partial\Gamma$ defined as follows. Let $\Phi(\vec{e})$ denote the connected component of Σ minus the interior of e which contains the tail of $\vec{e}$. Then, $\Psi(\vec{e})$ is the part of the limit set of Γ to the left of $\vec{e}$:

$$\Psi(\vec{e}) = \partial\Phi(\vec{e}) \cup \bigcup_{v \in V(\Phi(\vec{e}))} \Lambda\Gamma(v).$$

Let $\vec{\Delta}(v)$ denote the set of directed edges of Σ with head at v.

Proposition 4 (Bowditch, Lemma 3.3 and Proposition 5.1 of [5]). *With the notation above, we have*

$$\Psi(\vec{e}) \cup \Psi(-\vec{e}) = \partial\Gamma = \Lambda\Gamma(v) \cup \bigcup_{\vec{e} \in \vec{\Delta}(v)} \Psi(\vec{e}),$$

$$\Psi(\vec{e}) \cap \Psi(-\vec{e}) = \partial\Gamma(e),$$

and $\Psi(\vec{e})$ is closed, $\Gamma(e)$ invariant, and connected.

The fact that $\Psi(\vec{e})$ is connected requires some argument. Since $\Psi(\vec{e})$ and $\partial\Gamma$ are connected, it follows that by identifying the two points in $\Lambda\Gamma(e)$ for $e \ni v$ we obtain a connected quotient of $\partial\Gamma(v)$. This together with Proposition 3 shows that the image of $\Lambda\Gamma(v)$ in $D(\partial\Gamma)$ is connected for each vertex v of Σ. Denote this by $D(M(v))$. At least one of these must be nontrivial (not a point) from the decomposition of $\partial\Gamma$ described before Proposition 3. Choose one such vertex v; we denote the quotient $D(M(v))$ by $D(M)$, $\Gamma(v)$ by G, and call $\Gamma(e)$ for $e \ni v$ the peripheral subgroups of G. Restricting the action of Γ to G we see that:

Proposition 5. *G acts as a discrete convergence group on $D(M)$, and the peripheral subgroups of G fix points of M. Moreover, $D(M)$ is a nontrivial dendrite.*

By removing the terminal points of $D(M)$ one obtains as in [4] an R-tree T on which the induced action of G is nonnesting (see Section 7 of [4]; nonnesting means that for any compact interval A, if $g(A) \subset A$, then $g(A) = A$). Since the action of G on $D(M)$ is a discrete convergence group action, the edge stabilizers for the action on T are finite. A peripheral subgroup H of G fixes a point of $D(M)$ and thus fixes either a point of T or exactly one end of T. Since a nonnesting action on a real tree does not have parabolic elements (in the usual sense; see [9]), it follows that the peripheral subgroups of G fix points of T. In summary, we have:

Proposition 6. *If $\partial\Gamma$ has a global cut point, then some vertex group $\Gamma(v) = G$ admits a nonnesting action on an R-tree T with finite edge stabilizers and such that each peripheral subgroup of G fixes some point of T.*

Under these conditions Levitt ([9], see also [6] — perhaps it is possible to use [6] too for the rest of the argument) constructs an action of G by isometries on an R-tree T_0 such that a subgroup of G fixing an arc of T_0 also fixes an arc of T. The construction is natural, and since we have only finitely many conjugacy classes of peripheral subgroups, we see that the peripheral subgroups of G still fix points of T_0 (this needs enlarging K in Levitt's construction; see Corollary 6 of [9]). Thus we have the analogue of Proposition 6 with the action of G by isometries rather than just homeomorphisms. The condition on edge stabilizers shows that the action is stable in the sense of Bestvina and Feighn [2]. Thus by the relative version of the main theorem of [2] (this is Theorem 9.6 of [2]), we conclude that there is a nontrivial decomposition of G along a virtually cyclic group such that each peripheral subgroup is conjugate to a subgroup of a vertex group. Since we started with a $G = \Gamma(v)$ which does not admit such a splitting, we have the desired contradiction to complete the proof of Theorem 1. By [3], it follows that $\partial\Gamma$ is locally connected and the theory of [7] goes through for all hyperbolic groups. This leads to a precise characterization of when $\mathrm{Out}(\Gamma)$ is infinite.

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