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ON THE EXISTENCE OF POSITIVE SOLUTIONS OF YAMABE-TYPE EQUATIONS ON THE HEISENBERG GROUP

L. BRANDOLINI, M. RIGOLI, AND A. G. SETTI

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Abstract. We study nonexistence, existence and uniqueness of positive solutions of the equation $\Delta_{H^n} u + a(x)u - b(x)u^\sigma = 0$ with $\sigma > 1$ on the Heisenberg group H^n. Our results hold, with essentially no changes, also for the Euclidean version of the above equation. Even in this case they appear to be new.

Introduction

Let H^n be the Heisenberg group of real dimension $2n+1$, i.e. the nilpotent Lie group which as a manifold is the product

$$H^n = \mathbb{C}^n \times \mathbb{R}$$

and whose group structure is given by

$$(z,t)\circ(z',t') = (z+z', t+t'+2\,\mathrm{Im}(z,z')),$$
$$(z,t),\ (z',t') \in H^n,$$

where $(\ ,\ )$ denotes the usual Hermitian product on $\mathbb{C}^n$.

A (real) basis for the Lie algebra of left-invariant vector fields on H^n is given by

$$X_j = 2\,\mathrm{Re}\,\frac{\partial}{\partial z_j} + 2\,\mathrm{Im}\, z_j\frac{\partial}{\partial t}, \quad Y_j = 2\,\mathrm{Im}\,\frac{\partial}{\partial z_j} - 2\,\mathrm{Re}\, z_j\frac{\partial}{\partial t}, \quad \frac{\partial}{\partial t},$$

for $j = 1, 2, \ldots, n$. The above basis satisfies Heisenberg's canonical commutation relations for position and momentum

$$[X_j, Y_k] = -4\delta_{jk}\frac{\partial}{\partial t},$$

all other commutators being 0. It follows that the vector fields X_j, Y_k satisfy Hörmander's condition, and the real part of the Kohn-Spencer Laplacian, defined by

$$\Delta_{H^n} = \sum_{j=1}^{n}\left(X_j^2 + Y_j^2\right), \tag{1}$$

is hypoelliptic by Hörmander's theorem ([5]).

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In H^n one has a natural origin $0 = (0, 0)$ and a distinguished distance function from 0 defined by

$$\rho(x) = \rho(z,t) = \left(|z|^4 + t^2\right)^{1/4},$$

which is homogeneous of degree one with respect to the Heisenberg dilations $(z,t) \to (\delta z, \delta^2 t)$. The distance between two points x, $x' \in H^n$ is then given by $d(x,x') = \rho(x^{-1}x')$.

We also define the density function with respect to 0 by

$$\psi(x) = \psi(z,t) = \frac{|z|^2}{\rho(z,t)^2}, \quad \text{for } x \neq 0,$$

and note that $0 \leq \psi(x) \leq 1$. If u is a "radial function", that is, $u(z,t) = f\left(\rho(z,t)\right)$ for $f : [0, +\infty) \to \mathbb{R}$ of class C^2, then

$$\Delta_{H^n} u = \psi \left\{ f''(\rho) + \frac{2n+1}{\rho} f'(\rho) \right\}.$$

In this paper we consider the equation

$$\Delta_{H^n} u + a(x)u - b(x)|u|^{\sigma-1}u = 0, \tag{2}$$

with $\sigma > 1$ constant, and determine conditions on the coefficients $a(x)$, $b(x)$ in order to guarantee the existence (resp., nonexistence) of positive solutions on H^n.

Our problem is motivated by the following geometric fact. The vector fields $Z_j = X_j + iY_j$ span a subbundle $T_{1,0}$ of the complexified tangent bundle of H^n, and give rise to its canonical CR structure with contact form θ, which is determined modulo the transformation

$$\tilde{\theta} = u^{2/n}\theta \tag{3}$$

for $0 < u \in C^\infty(H^n)$. The choice of θ specifies a pseudo-Hermitian structure on H^n, and

$$\theta_o = dt + \sum_{j=1}^{n} \left(iz^j d\bar{z}^j - i\bar{z}^j dz^j \right)$$

defines the canonical one.

A contact form on a CR manifold M induces a scalar curvature (the Webster scalar curvature) R_θ, which under the change (3) of contact forms transforms according to the equation

$$\frac{2n+2}{n}\Delta_\theta u + R_\theta u = R_{\tilde{\theta}} u^{(n+2)/n},$$

where Δ_θ is the hypoelliptic Laplacian of the pseudo-Hermitian manifold (M, θ).

The Webster scalar curvature of the canonical pseudo-Hermitian structure θ_o on H^n is identically zero, and Δ_{θ_o} is the operator defined in (1). Therefore, when $a(x) \equiv 0$, equation (2) arises as the transformation law for the Webster scalar curvature of (H^n, θ_o) under the conformal change $\tilde{\theta} = u^{2/n}\theta_o$ of contact forms. In this sense, (2) is the CR analogue of the Yamabe equation on Euclidean space that has been extensively investigated over the past fifteen years (see, for instance, [3], [4], [9]).

The CR-Yamabe equation has been studied in a series of papers by D. Jerison and J. Lee ([6], [7] and [8]). In particular, they consider the problem of conformally changing the contact form to one having constant Webster curvature in the compact setting. As in the Riemannian case, their analysis is based on variational techniques, which, however, are not straightforwardly extendable to the noncompact case.

We stress that the presence of the linear term in (2) introduces additional difficulties and gives rise to new interesting phenomena. Most notably, our results suggest the existence of a critical behaviour for the coefficient $a(x)$. On the one hand, the linear term essentially does not affect existence and nonexistence of positive solutions if $a(x)$ decays faster than quadratically in the distance (see Theorem 5 and Theorem 2, respectively). On the other hand, if $a(x)$ decays more slowly than quadratically, Theorem 3 guarantees the existence of positive solutions under minimal conditions on $b(x)$.

The presence of the linear term is also reflected in the uniqueness result contained in Theorem 7. Without it the uniqueness would follow by a direct application of the maximum principle.

In our analysis of equation (2) we use a variety of different techniques. The nonexistence results are for the most part based on applications of maximum principle techniques and geometric ideas to ordinary differential equations, which yield results that might be of independent interest.

The key ingredient in the proof of our existence results is the method of super- and subsolutions, which is the standard approach to existence problems when lack of compactness of the ambient space makes a variational treatment unnatural. Again, geometric ideas play a crucial role in the construction of super- and subsolutions.

Our results are based on the assumption that the coefficients $a(x)$ and $b(x)$ in (2) satisfy bounds of the form

$$\begin{aligned} &\psi(x)\, a_1\left(\rho(x)\right) \leq a(x), \quad \text{respectively } a(x) \leq \psi(x)\, a_2\left(\rho(x)\right), \\ &\psi(x)\, b_1\left(\rho(x)\right) \leq b(x), \quad \text{respectively } b(x) \leq \psi(x)\, b_2\left(\rho(x)\right), \end{aligned} \tag{4}$$

for suitable, nonnegative, continuous functions a_i and b_i on $[0, +\infty)$. In some sense, the presence of the factor $\psi(x)$ in (4) reflects the anisotropic nature of the Heisenberg group. It allows us to "radialise" the problem and apply ordinary differential equation methods.

Since $0 \leq \psi(x) \leq 1$, its occurrence as a factor in the lower bounds in (4) is not restrictive. On the other hand, the presence of ψ in the upper bounds is a genuine restriction, and since $\psi(x)$ vanishes for $x = (z, 0)$, it forces the corresponding coefficient to vanish along the t-axis. However, an analysis of our results shows that the restriction so introduced is necessary in our nonexistence and uniqueness theorems (Theorems 1, 2, and 7, respectively) at least if one is willing to assume that the bounds on the coefficients are expressed in terms of the distance function.

Moreover, the presence of the factor ψ in the existence theorems allows us to obtain solutions whose behaviour at infinity can be controlled by radial functions (see Theorems 3 and 5).

We also remark that all our results hold, with essentially no changes, for the Euclidean version of equation (2). It suffices to set $\psi \equiv 1$ throughout, and replace Δ_{H^n} with the standard Laplace operator of $\mathbb{R}^m$. Even in this case our results appear to be new. (For related results in the setting of complete noncompact Riemannian manifolds, however, see [10] and [1].)

Next sections contain the statements of our main results. All the proofs will be included in the forthcoming paper [2].

1. Nonexistence results

Theorem 1. *Let $\sigma > 1$, and let $a, b \in C^0(H^n)$ satisfy*

$$\left\{ \begin{array}{l} a(x) \le \psi(x) a_2(\rho(x)), \\ \\ b(x) \ge \psi(x) b_1(\rho(x)) \end{array} \right. \quad on \ H^n, \tag{5}$$

with a_2, $b_1 \in C^0([0, +\infty))$. Assume that for some constant $A \le n$,

$$a_2(t) \le \frac{A^2}{t^2},$$

that $b_1(t) \ge 0$ on $[0, +\infty)$, and that for some integer k,

$$\left\{ \begin{array}{ll} \liminf_{t\to+\infty} b_1(t) \dfrac{(\log t)^{\sigma+1} \log(\log t) \cdots \log^{(k)}(t)}{t^{n(\sigma-1)-2}} > 0 & if\ A = n, \\ \\ \liminf_{t\to+\infty} b_1(t) \dfrac{(\log t) \log(\log t) \cdots \log^{(k)}(t)}{t^{(n-\sqrt{n^2-A^2})(\sigma-1)-2}} > 0 & if\ A < n, \end{array} \right.$$

where $\log^{(k)}$ denotes the kth composition power of $\log$. Then the equation

$$\Delta_{H^n} u + a(x) u - b(x) u^\sigma = 0 \tag{6}$$

has no positive solution on H^n.

Theorem 2. *Assume that $a, b \in C^0(H^n)$ satisfy (5) in the statement of Theorem 1. Suppose that a_2 satisfies*

$$a_2(t) \le \min\left\{ \frac{n^2}{t^2}, \frac{A^2}{t^{2+\epsilon}} \right\},$$

for some constants A and $\epsilon > 0$, and that there exist an integer k and a constant $C > 0$ such that

$$\left\{ \begin{array}{l} b_1(t) \ge 0 \quad on\ [0, +\infty), \\ \\ \liminf_{t\to+\infty} t^2 \log t \log(\log t) \cdots \log^{(k)}(t)\, b_1(t) \ge C > 0. \end{array} \right.$$

Then (6) has no positive solution on H^n.

2. Existence results

Theorem 3. *Let a, $b \in C^\infty(H^n)$, $\mu \le 2$, $1 < \sigma \le \frac{n+2}{n}$,*

$$A_\mu > \left\{ \begin{array}{ll} 0 & if\ \ \mu < 2, \\ \dfrac{2n}{\sigma - 1} & if\ \ \mu = 2, \end{array} \right. ,$$

and let $\gamma \in \mathbb{R}$. Assume that a and b satisfy

$$\psi(x) a_1(\rho(x)) \le a(x) \le \psi(x) a_2(\rho(x)) \quad on\ H^n$$

and

$$\psi(x) b_1(\rho(x)) \le b(x) \le \psi(x) b_2(\rho(x)) \quad on\ H^n,$$

respectively, for suitable $a_1, a_2, b_1, b_2 \in C^0([0, +\infty))$ with

$$a_1(t) = A_\mu t^{-\mu} \qquad for\ t \gg 1,$$

i) $b_1(t) \geq 0$ *on* $[0, +\infty)$ *and* $b_1(t) > 0$ *in* $[t_0, +\infty)$,

ii) $b_2(t) \leq c_1 t^{-\frac{2}{n}\gamma - \mu}$ *for* $t \gg 1$,

with $c_1 > 0$ *and* t_0 *such that* $a_2(t) \leq \frac{n^2}{t^2}$ *on* $(0, t_0]$. *Then there exists a positive solution* u *of* (6) *on* H^n *satisfying the further requirement*

$$u(x) \geq c_2 \rho(x)^{\frac{2\gamma}{n(\sigma-1)}}$$

for some constant $c_2 > 0$ *and* $\rho(x)$ *sufficiently large.*

Definition 4. We say that a solution U of equation (6) on H^n is *maximal* if for any other solution u on H^n we have

$$u(x) \leq U(x) \quad \text{for } x \in H^n.$$

Theorem 5. *Let* $a, b \in C^0(H^n)$. *Suppose there exist* $a_2, b_1, b_2 \in C^0([0, +\infty))$ *satisfying*

$$a_2(t) \leq \frac{n^2}{t^2}$$

and

i) $b_1(t) \geq 0$ *on* $[0, +\infty)$ *and* $b_1(t) > 0$ *for* $t \gg 1$,

ii) $t b_2(t) \in L^1(+\infty)$,

such that

$$0 \leq a(x) \leq \psi(x) a_2(\rho(x)) \quad \text{on } H^n$$

and

$$\psi(x) b_1(\rho(x)) \leq b(x) \leq \psi(x) b_2(\rho(x)) \quad \text{on } H^n.$$

Then, there exists a unique, positive, maximal solution U *of* (6) *on* H^n *satisfying*

$$\lim_{\rho(x) \to +\infty} U(x) = +\infty.$$

Furthermore, if for some constant $c > 0$

$$b_1(t) \geq c t^{-k}, \text{ for } k > 2 \text{ and } t \gg 1,$$

then

$$U(x) \leq C \rho(x)^{\frac{k-2}{\sigma-1}}$$

for some constant $C > 0$ *and* $\rho(x) \gg 1$.

Similarly, if for some constant $c > 0$

$$b_2(t) \leq c t^{-h}, \text{ for } h > 2 \text{ and } t \gg 1,$$

then

$$U(x) \geq C \rho(x)^{\frac{h-2}{\sigma-1}}$$

for some constant $C > 0$ *and* $\rho(x) \gg 1$.

In particular, in the case where

$$C_1 \psi(x) \left[1 + \rho(x)\right]^{-k} \leq b(x) \leq C_2 \psi(x) \left[1 + \rho(x)\right]^{-k}$$

for some constants $C_1, C_2 > 0$, *we find that*

$$U(x) \asymp \rho(x)^{\frac{k-2}{\sigma-1}} \text{ as } \rho(x) \to +\infty.$$

Theorem 6. *Let $a, b \in C^0(H^n)$. Suppose there exist $a_2, b_1 \in C^0([0,+\infty))$ satisfying*

$$a_2(t) \le \frac{n^2}{t^2},$$

$$b_1(t) \ge 0 \text{ on } [0,+\infty) \text{ and } b_1(t) > 0 \text{ for } t \gg 1,$$

such that

$$0 \le a(x) \le \psi(x) a_2(\rho(x)) \quad \text{on } H^n$$

and

$$\psi(x) b_1(\rho(x)) \le b(x) \quad \text{on } H^n.$$

If there exists a positive subsolution u of (6), then there exists a unique, positive, maximal solution U of (6) on H^n satisfying

$$\lim_{\rho(x)\to+\infty} U(x) = +\infty.$$

3. A uniqueness result

Theorem 7. *Let $a(x)$, $b(x) \in C^0(H^n)$ satisfy $b(x) \ge 0$ on H^n, and*

$$a(x) \le \psi(x) a_2(\rho(x)) \quad \text{on } H^n,$$

where $a_2 \in C^0([0,+\infty)$ satisfies

$$a_2(t) \le \frac{A^2}{t^2} \qquad \text{on } (0,+\infty), \tag{7}$$

with $A \le n$. Let $u, v \in C^2(H^n)$ be positive solutions of equation (6). If

$$(u-v)(x) = \begin{cases} o\left(\rho(x)^{-n}\log(\rho(x))\right) & \text{for } A = n, \\ o\left(\rho(x)^{-n+\sqrt{n^2-A^2}}\right) & \text{for } A < n, \end{cases} \qquad \text{as } \rho(x) \to +\infty, \tag{8}$$

then $u \equiv v$ on H^n.

Moreover, if instead of (7) we assume that

$$a_2(t) \le \min\left\{\frac{n^2}{t^2}, \frac{A^2}{t^{2+\epsilon}}\right\}$$

for some positive constants A and ϵ, then the same conclusion holds with (8) replaced by

$$(u-v)(x) = o(1) \quad \text{as } \rho(x) \to +\infty.$$

References

1. B. Bianchini and M. Rigoli, *Nonexistence and uniqueness of positive solutions of Yamabe type equations on nonpositively curved manifold*, preprint.
2. L. Brandolini, M. Rigoli, and A. G. Setti, *Positive solutions of Yamabe-type equations on the Heisenberg group*, preprint.
3. K. S. Cheng and J. T. Lin, *On the elliptic equations $\Delta u = K(x)u^\sigma$ and $\Delta u = K(x)e^{2u}$*, Trans. Amer. Math. Soc. **304** (1987), 639–668. MR **88j**:35054
4. K. S. Cheng and W. M. Ni, *On the structure of the conformal scalar curvature equation on $\mathbb{R}^n$*, Indiana Univ. Math. J. **41** (1992), 261–278. MR **93g**:35040
5. L. Hörmander, *Hypoelliptic second order differential equations*, Acta Math. **119** (1968), 147–171. MR **36**:5526
6. D. Jerison and J. M. Lee, *A subelliptic, nonlinear eigenvalue problem and scalar curvature on CR manifolds*, Cont. Math. **27** (1984), 57–63. MR **85i**:58122
7. ———, *The Yamabe problem on CR manifolds*, J. Diff. Geom. **25** (1987), 167–197. MR **88i**:58162

8. ______, *Extremals for the Sobolev inequality on the Heisenberg group and the CR Yamabe problem*, J. Amer. Math. Soc. **1** (1988), 1–12. MR **89b:**53063
9. W. M. Ni, *On the elliptic equation $\Delta u + K(x)u^{(n+2)/(n-2)} = 0$, its generalizations, and applications in geometry*, Indiana Univ. Math. J. **31** (1982), 493–529. MR **84e:**35049
10. A. Ratto, M. Rigoli, and L. Véron, *Scalar curvature and conformal deformation of noncompact Riemannian manifolds*, Math. Z. (to appear).

Dipartimento di Matematica, Via Saldini 50, 20133 Milano, Italy
E-mail address: `brandolini@vmimat.mat.unimi.it`

Dipartimento di Matematica, Via Saldini 50, 20133 Milano, Italy
E-mail address: `rigoli@vmimat.mat.unimi.it`

Dipartimento di Matematica, Via Saldini 50, 20133 Milano, Italy
E-mail address: `setti@vmimat.mat.unimi.it`