ELECTRONIC RESEARCH ANNOUNCEMENTS
OF THE AMERICAN MATHEMATICAL SOCIETY
Volume 2, Number 3, December 1996
S 1079-6762(96)00015-7

THE EQUICHORDAL POINT PROBLEM

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(Communicated by Krystyna Kuperberg)

ABSTRACT. If C is a Jordan curve on the plane and $P, Q \in C$, then the segment $\overline{PQ}$ is called a *chord* of the curve C. A point inside the curve is called *equichordal* if every two chords through this point have the same length. Fujiwara in 1916 and independently Blaschke, Rothe and Weitzenböck in 1917 asked whether there exists a curve with two distinct equichordal points O_1 and O_2. This problem has been fully solved in the negative by the author of this announcement just recently. The proof (published elsewhere) reduces the question to that of existence of heteroclinic connections for multi-valued, algebraic mappings. In the current paper we outline the methods used in the course of the proof, discuss their further applications and formulate new problems.

1. INTRODUCTION

1.1. **The origins of the problem.** The Equichordal Point Problem ($\mathcal{EPP}$) was originally posed by Fujiwara in 1916 [3] and probably independently by Blaschke, Rothe and Weitzenböck in 1917 [1].

Problem 1.1. Let C be a Jordan curve on the plane and let O be a point inside the curve. We will call O an *equichordal point* if every chord of the curve C passing through O has the same length (cf. Figure 1). The $\mathcal{EPP}$ asks whether there is a curve with two distinct equichordal points.

Any Jordan curve C with two equichordal points O_1 and O_2 will be called an *equichordal curve.* By scaling, we may assume that all chords of C passing through O_1 have length 1. Since there is a chord passing through both O_1 and O_2, all chords passing through O_2 will also have length 1.

Once the above normalization has been made, the quantity $a = \|O_1 - O_2\|$ becomes a parameter of the $\mathcal{EPP}$ known as the *eccentricity* of C. Thus, the $\mathcal{EPP}$ can be considered for any particular value $a \in (0, 1)$.

Due in part to its elementary formulation the $\mathcal{EPP}$ has been studied in a number of papers, either using elementary methods or advanced analytical tools. Fujiwara [3] proved by elementary methods that there cannot be three equichordal points. Clearly, there are many curves with one equichordal point. Roughly speaking, they can be obtained by choosing a "half" of the curve almost arbitrarily and then "reflecting" in the point O. The reader will easily fill in the detailed conditions

Received by the editors September 15, 1996.
1991 *Mathematics Subject Classification.* Primary 52A10, 39A; Secondary 39B, 58F23, 30D05.
Key words and phrases. Equichordal, heteroclinic, convex, multi-valued.
This research has been supported in part by the National Science Foundation under grant no. DMS 9404419.

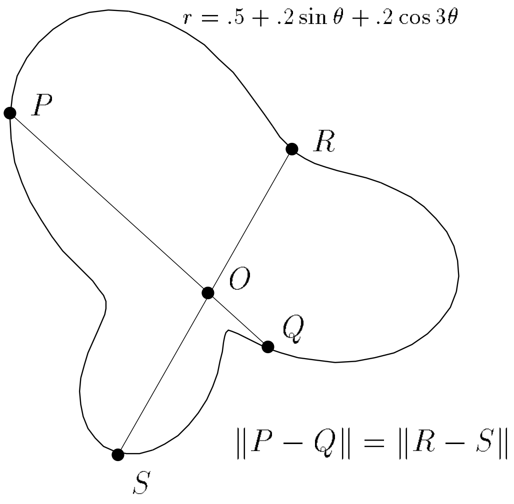

FIGURE 1. An equichordal point O of a curve given in polar coordinates.

needed in this construction. For instance, the origin is an equichordal point of a curve given in polar coordinates by the equation $r = f(\theta)$, where $f : \mathbb{R} \to \mathbb{R}$ is continuous, periodic with period 2π, $f(\theta+\pi) + f(\theta) = 1$ and $0 < f(\theta) < 1$ for all θ. These conditions imply that the Fourier series of f has only odd terms and a constant term, i.e.

$$f(\theta) = \frac{1}{2} + \sum_{n=0}^{\infty} a_{2n+1}\cos(2n+1)\theta + \sum_{n=0}^{\infty} b_{2n+1}\cos(2n+1)\theta. \tag{1.1}$$

A sufficient condition for a_{2n+1} and b_{2n+1} is that

$$\sum_{n=0}^{\infty} |a_{2n+1}| + \sum_{n=0}^{\infty} |b_{2n+1}| < \frac{1}{2} \tag{1.2}$$

(cf. Figure 1). Thus the only case left to consider is that of two equichordal points.

One may also ask about equichordal bodies in $\mathbb{R}^n$ for $n \geq 3$. However, two-dimensional sections of such bodies form equichordal figures in $\mathbb{R}^2$. Thus, a negative answer to the $\mathcal{EPP}$ in two dimensions implies a negative answer for all $n \geq 2$.

1.2. **Preliminary remarks on the solution of $\mathcal{EPP}$.** Most of the work on this problem has been aimed at finding a negative solution to $\mathcal{EPP}$. In [6] we finally found a complete negative solution. In the current research announcement we will discuss the techniques and problems related to our solution of the $\mathcal{EPP}$.

The most recent contribution to the solution of the $\mathcal{EPP}$ prior to our paper was made in [7] in 1992, based upon a subtle asymptotic analysis of the problem as $a \to 0$. With this approach one has been able to prove the absence of equichordal curves for small values of a and even that there exist only a finite number of values of a for which there exists an equichordal curve of eccentricity a. In principle, with the aid of a computer one could eliminate this finite number of values.

Our approach does not use asymptotic analysis but global analysis based on a complexification of the problem. This approach is conceptually simpler and it produces a complete solution without the assistance of a computer. Our main result about equichordal curves is contained in the following theorem:

Theorem 1.2. *For all eccentricities $a \in (0,1)$ there is no equichordal curve.*

The solution of the Equichordal Point Problem fits well into the context of contemporary mathematics. Our original motivation stems from the observation that the problem can be reformulated as a question about the existence of a heteroclinic connection in a certain discrete algebraic dynamical system. This question has been studied recently from the point of view of Dynamical Systems theory [4, 9].

1.3. **A precise formulation of the main result.** It will be convenient to assume that in cartesian coordinates $O_1 = (-b, 0)$ and $O_2 = (b, 0)$, where $b = a/2$. Let $B(O_i, 1)$, $i = 1, 2$, be the open unit disks about O_i. Let us consider the two transformations $T_i : B(O_i,1)\backslash\{O_i\} \to B(O_i,1)\backslash\{O_i\}$ defined by the property that if $P \in B(O_i, 1)$, then $O_i \in [P, T_i(P)]$ and the distance from P to $T_i(P)$ is exactly 1. The maps T_i are defined via the following formula:

$$T_i(P) = P - \frac{P - O_i}{||P - O_i||}, \tag{1.3}$$

or more explicitly, using cartesian coordinates:

$$T_i(x,y) = \left(x - \frac{x \pm b}{\sqrt{(x \pm b)^2 + y^2}}, y - \frac{y}{\sqrt{(x \pm b)^2 + y^2}}\right). \tag{1.4}$$

A straightforward generalization of the $\mathcal{EPP}$ to Jordan curves requires that any straight line passing through O_1 or O_2 should intersect C at exactly two points. A curve with this last property is called *strongly star-shaped* with respect to O_1 and O_2. The following lemma (proved in [6]) contains a useful characterization of equichordal curves.

Lemma 1.3. *Let C be a Jordan curve and O_1 and O_2 be two points with the following properties:*

1. *$C \subseteq B(O_1, 1) \cap B(O_2, 1)$.*
2. *For $i = 1, 2$ we have $T_i(C) \subseteq C$.*

Any Jordan curve C satisfying the above conditions is strongly star-shaped with respect to O_1 and O_2 and is an equichordal curve.

It is clear that every convex equichordal curve satisfies the hypothesis of this lemma. Convexity was an original assumption of Fujiwara.

2. The elementary analysis of the problem

We consider the $\mathcal{EPP}$ from the point of view of Dynamical Systems theory. We define the map $U = G \circ T_2$, where G is the reflection in the point O which is the center of the segment $[O_1, O_2]$. It is easy to verify that $U^2 = T$, where $T = T_1 \circ T_2$. The equichordal curve, if it exists, is invariant under U. If we use coordinates (x, y) such that $O = (0, 0)$, $O_1 = (-1/2, 0)$ and $O_2 = (1/2, 0)$, then the map U can be expressed in the form

$$U(x,y) = \left(-x + \frac{x - b}{\sqrt{(x - b)^2 + y^2}}, -y + \frac{y}{\sqrt{(x - b)^2 + y^2}}\right). \tag{2.1}$$

This formula simply reflects the fact that in the cartesian coordinates

$$U(x,y) = -T_1(x,y).$$

The main feature of this formula is that it is

1. *algebraic*, i.e. the coordinates of U are given as solutions of polynomial equations,
2. when extended to the complex domain, it is *multi-valued*, more precisely, 2-valued, provided that the branches of $\sqrt{\cdot}$ are chosen consistently in both coordinates.

Thus, we have arrived at the problem of studying invariant curves under *multi-valued complex algebraic mappings*. Such mappings have appeared in literature and have also been known as *correspondences*. We also use the term *nonsingular algebraic relations*.

3. The Dynamical Systems point of view

An essential part of our solution of the $\mathcal{EPP}$ is the development of the theory of algebraic relations treated as dynamical systems. In the end, the solution of the $\mathcal{EPP}$ is achieved by studying global obstructions to the existence of heteroclinic connections.

Our solution of the $\mathcal{EPP}$ starts with the observation that an equichordal curve C, if it existed, would be an *invariant curve*, i.e. $U(C) = C$. Let us discuss this observation in more detail. In the course of this discussion further properties of the hypothetical equichordal curve C will be revealed.

3.1. **The basic stability analysis.** In coordinates, it is easy to show that $A_1 = (-1/2, 0)$ and $A_2 = (1/2, 0)$ are the only fixed points of the map U. The standard stability analysis shows that the horizontal and vertical directions are eigendirections for $DU(A_i)$, $i = 1, 2$. The eigenvalues corresponding to the horizontal eigendirection are in both cases -1. The vertical direction has eigenvalue $\lambda_i = (1 \mp a)/(1 \pm a)$, $i = 1, 2$ and thus it is hyperbolic for $a \in (0, 1)$, which is the parameter range of interest.

Every point A of the straight line O_1O_2, with the exception of O_1 and O_2, is a periodic point of U of period 2. A stability analysis shows that the vertical direction is hyperbolic.

3.2. **The invariant manifolds.** The invariant manifold theory provides us with two *real-analytic* local invariant manifolds $W^s_{loc}(A_1)$ and $W^u_{loc}(A_2)$ defined as

$$\begin{aligned} W^s_{loc}(A_1) &= \left\{ P \in B(A_1, \epsilon) \mid \lim_{n\to\infty} U^n(P) = A_1 \right\}, \\ W^u_{loc}(A_2) &= \left\{ P \in B(A_2, \epsilon) \mid \lim_{n\to-\infty} U^n(P) = A_2 \right\}. \end{aligned}$$

Strictly speaking, this definition depends on the choice of a sufficiently small constant $\epsilon > 0$, but in the Dynamical Systems tradition, we will ignore this dependence in our notation.

Every point A of the line O_1O_2 with the exception of O_1 and O_2 also possesses a local invariant manifold $W^{s,u}(A)$. The character of stability depends on where the point is located. The invariant manifolds computed for A near A_1 and A_2 will foliate the neighborhoods of these points.

From the existence of a foliation it follows that if C is an equichordal curve, then $C \supset W^s_{loc}(A_1) \cup W^u_{loc}(A_2)$.

The *global* invariant manifolds of A_1 and A_2 are defined via the following formulas:

$$\begin{aligned} W^s(A_1) &= \left\{ P \in \mathbb{R}^2 \mid \lim_{n\to\infty} U^n(P) = A_1 \right\}, \\ W^u(A_2) &= \left\{ P \in \mathbb{R}^2 \mid \lim_{n\to-\infty} U^n(P) = A_2 \right\}. \end{aligned}$$

It is clear that

$$\begin{aligned} W^s(A_1) &= \bigcup_{n=0}^{\infty} U^{-n}\left(W^s_{loc}(A_1)\right), \\ W^u(A_2) &= \bigcup_{n=0}^{\infty} U^{n}\left(W^u_{loc}(A_2)\right). \end{aligned} \tag{3.1}$$

It is easy to see that

$$\begin{aligned} W^s(A_1) &= C\backslash\{A_2\}, \\ W^u(A_2) &= C\backslash\{A_1\}. \end{aligned}$$

For instance, $W^s(A_1)$ is an open arc of a Jordan curve, invariant under an orientation-preserving homeomorphism $U|C$. Thus, the endpoints of this arc are fixed points of $U|C$. But there is only one fixed point of U different from A_1, namely A_2. Hence, the complement of $W^s(A_1)$ in C is $\{A_2\}$.

In conclusion, an equichordal curve admits the formula

$$C = W^s(A_1) \cup W^u(A_2). \tag{3.2}$$

This proves that for any value of the eccentricity a there can be at most one equichordal curve. This formula and the symmetries of U imply that C is symmetric with respect to reflections in both axes.

More importantly, if C existed, then $W^s(A_1)$ and $W^u(A_2)$ would form a *heteroclinic connection*, i.e. there would be an arc in $W^s(A_1) \cup W^u(A_2)$ connecting A_1 to A_2.

We note that if C does not exist, the sets $W^s(A_1)$ and $W^u(A_2)$ may not even be well defined, as iterations of some points will leave the domain of U.

3.3. **A summary of the elementary results.** These results which can be obtained from the rather elementary Dynamical Systems considerations can be summarized in the following:

Theorem 3.1. *For any given value of the parameter a there exists at most one equichordal curve, up to rotations and dilations. This curve is a union of the invariant curves of the equichordal map T:*

$$C = W^s(A_1) \cup W^u(A_2),$$

where $A_1 = (-1/2, 0)$ and $A_2 = (1/2, 0)$. If C exists, then it is real-analytic and symmetric with respect to reflections about both axes.

The necessary and sufficient condition of the existence of an equichordal curve for a fixed a is that the sets (each consisting of two open arcs) $W^s(A_1)\backslash\{A_1\}$ and $W^u(A_2)\backslash\{A_2\}$, coincide, i.e. that there is a heteroclinic connection between A_1 and A_2.

3.4. **Oscillatory behavior and numerics.** It is known that if the set $W^s(A_1) \cap W^u(A_2)$ is discrete, then $W^s(A_1) \cup W^u(A_2)$ forms a topologically complex structure, schematically shown in Figure 2. This structure indicates that the invariant manifolds $\Gamma(A_1) = W^s(A_1)$ and $\Gamma(A_2) = W^u(A_2)$ oscillate near A_2 and A_1, respectively. The invariant curves in the real domain can be easily approximated

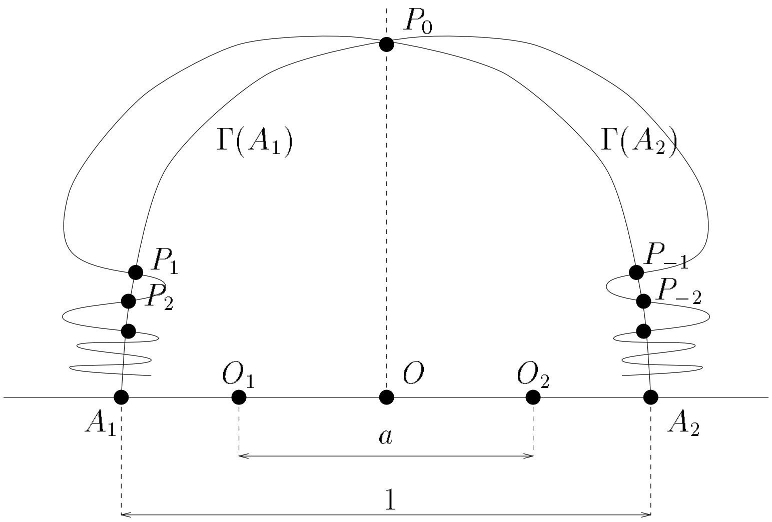

FIGURE 2. Intersecting invariant curves.

by numerical methods for values of a not close to 0. For instance, Figure 3 and Figure 4 created with $a = 0.6$ illustrate the fact that the unstable invariant curve can be continued in the real domain indefinitely and that it possesses oscillatory behavior near the point A_1. The existence of an equichordal curve would mean that for some parameter value the oscillation would cease, so that the curve can be closed with an analytic piece passing through A_1.

4. NONEXISTENCE OF HETEROCLINIC AND HOMOCLINIC CONNECTIONS

In this section we will focus on a range of techniques that can be used to prove the nonexistence of heteroclinic or homoclinic connections in "regular" dynamical systems.

4.1. **The complexification and multi-valuedness.** As we have mentioned, the map U admits an extension to the complex domain as a 2-valued map. More precisely, except for the points $(x, y) \in \mathbb{C}^2$ for which $(x-b)^2 + y^2 = 0$, formula (2.1) yields two *analytic* branches of U.

The two branches of U cannot be separated in a natural way for the same reason that the two branches of $\sqrt{\cdot}$ cannot be separated, i.e. each branch is an analytic continuation of the other, and thus only choosing branch cuts can produce single-valued branches. However, the introduction of the branch cuts is a drastic operation that precludes any application of global methods. Thus, both branches of U must be given equal priority.

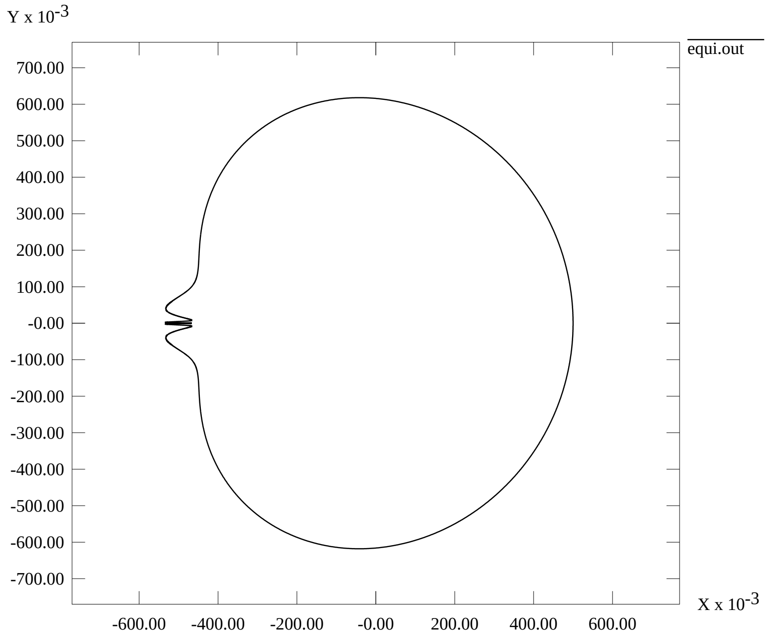

FIGURE 3. The curve $\Gamma(A_2)$ for eccentricity $a = 0.6$.

We note that the restriction of the complexification of U back to the real domain is a 2-valued real map. It is given by the formula

$$U(P) = -P \pm \frac{P - O_2}{||P - O_2||}. \tag{4.1}$$

The computation of $U(P)$ may be described as a two-step process:

1. We select one of the two points of the line PO_2 which are distant by 1 from P.
2. We reflect the point selected in the first step in the origin O.

The process of computing the nth iteration $U^n(P)$ involves 2^n choices of the sign in formula (4.1) and it somewhat resembles a random walk.

4.2. **The single-valued case.** The multi-valued character of U is a source of numerous technical difficulties in our solution of the $\mathcal{EPP}$. Therefore, it will be beneficial to examine a very simple argument, first published by Ushiki in [9], which is applicable if instead of U we consider a single-valued map $F : \mathbb{C}^2 \to \mathbb{C}^2$ such that F^{-1} exists and is single-valued as well. It is not difficult to see that in this situation F is a biholomorphic map.

Theorem 4.1. *Let $F : \mathbb{C}^2 \to \mathbb{C}^2$ be a biholomorphic map. Let us assume that F has two hyperbolic fixed points A_1 and A_2 (not necessarily distinct). The intersection $W^s(A_1) \cap W^u(A_2)$ is at most a countable set.*

In this theorem the sets $W^s(A_1)$ and $W^u(A_2)$ are the stable manifold of A_1 and unstable manifold of A_2, respectively. As in the real case, they are defined as

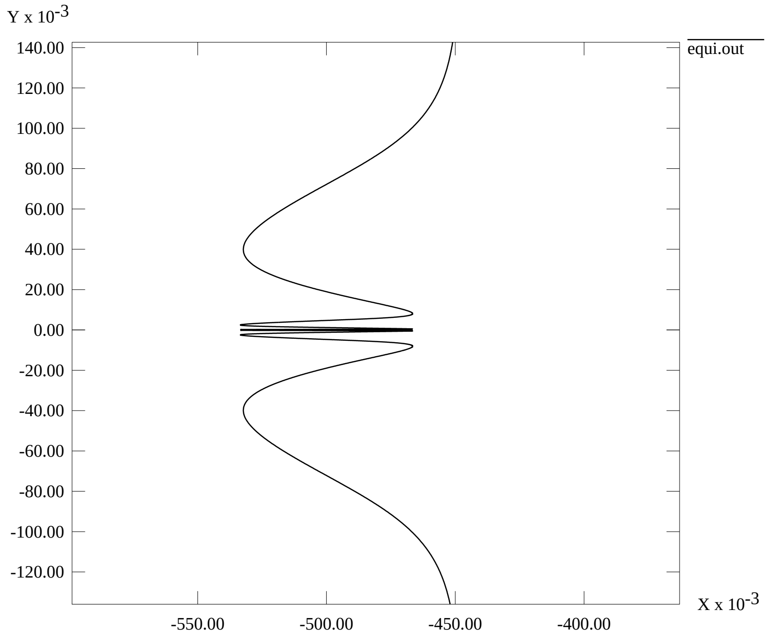

FIGURE 4. The curve $\Gamma(A_2)$ for eccentricity $a = 0.6$ magnified near A_1.

follows:

$$\begin{aligned} W^s(A_1) &= \left\{ P \in \mathbb{C}^2 \mid \lim_{n\to\infty} F^n(P) = A_1 \right\}, \\ W^u(A_2) &= \left\{ P \in \mathbb{C}^2 \mid \lim_{n\to-\infty} F^n(P) = A_2 \right\}. \end{aligned}$$

The fundamental theorem of Hadamard-Perron implies that these sets are embedded copies of $\mathbb{C}$.

We will say that there is a heteroclinic connection between A_1 and A_2 if there is a homeomorphism $\gamma : [0,1] \to \mathbb{C}^2$ such that $\gamma(0) = A_1$, $\gamma(1) = A_2$ and $\gamma(]0,1[) \subseteq W^s(A_1) \cap W^u(A_2)$ (if $A_1 = A_2$, then the term "homoclinic connection" is used). Thus, the result of Ushiki implies nonexistence of heteroclinic and homoclinic connections for biholomorphic maps of $\mathbb{C}^2$.

Two well-known examples of biholomorphic maps of $\mathbb{C}^2$ are

1. $U(x,y) = (1 - ax^2 + y, bx)$, where $b \neq 0$ (the Hénon map);
2. $U(x,y) = (2x - y + k\sin x, x)$, where $k \neq 0$ (the so-called standard map).

The original interest in these examples concerned only the real domain. The result of Ushiki by using complex-analytic methods simplified the proof of chaotic behavior of these mappings for all parameter values.

Remark 4.2. It is not known whether Theorem 4.1 holds for a biholomorphic map $F : \mathbb{C}^n \to \mathbb{C}^n$, $n \geq 3$.

4.3. **A proof of Theorem 4.1.** Let us suppose that $W^s(A_1) \cap W^u(A_2)$ is uncountable. Let us consider the set $X = W^s(A_1) \cup W^u(A_2)$. This set is a connected Riemann surface. Moreover, $F(X) = X$, i.e. $G = F|X$ is an automorphism of X.

Let $\tilde{X}$ be the universal cover of X and let $\tilde{G}$ be the lifting of G to $\tilde{X}$. The map $\tilde{G}$ is clearly an automorphism of $\tilde{X}$. Moreover, $\tilde{G}$ has at least two fixed points in X, one attracting and one repelling. The Uniformization Theorem tells us that $\tilde{X}$ is isomorphic either to $\mathbb{C}$, $\mathbb{D}$ or $\hat{\mathbb{C}}$ (Riemann sphere). The first two surfaces do not admit automorphisms with two nonelliptic fixed points, and thus $\tilde{X}$ is isomorphic to the Riemann sphere. Moreover, $\tilde{G}$ is conjugate to a multiplication by a number. In addition $X = \tilde{X}$. But $X \subseteq \mathbb{C}^2$. We obtain a contradiction by applying Liouville's theorem, as $\hat{\mathbb{C}}$ cannot be embedded into $\mathbb{C}^2$.

4.4. **Remarks on Theorem 4.1 and $\mathcal{EPP}$.** The proof we have just presented is based on the study of the map $F|X$ where the set $X = W^s(A_1) \cup W^u(A_2)$. When there is a heteroclinic or homoclinic connection, X becomes a connected Riemann surface, while $F|X$ is its automorphism. This observation outlines our strategy for handling heteroclinic and homoclinic connections for multi-valued mappings. The main complication is that $F|X$ is no longer an automorphism but a multi-valued mapping itself. Therefore, a straightforward application of the Uniformization Theorem needs to be replaced by a more involved argument.

4.5. **The invariant manifolds of multi-valued maps.** The local invariant manifold theory can be applied to the branches of a multi-valued map. For instance, U has the principal branch U_+ defined by the principal branch of $\sqrt{\cdot}$. This branch, when considered in the complex domain, is nonsingular at A_1 and A_2. Therefore, there exist local invariant manifolds $W^s_{loc}(A_1)$ and $W^u_{loc}(A_2)$, this time considered as subsets of $\mathbb{C}^2$. These manifolds are analytically embedded disks.

The maps $U_+|W^s(A_1)$ and $U_+|W^u(A_2)$ may be analytically linearized, according to a result dating back to Poincaré. Thus, there exist functions $\psi_1 : W^s(A_1) \to \mathbb{C}$ and $\psi_2 : W^u(A_2) \to \mathbb{C}$, biholomorphic in a neighborhood of A_1 and A_2 respectively, such that for $i = 1, 2$ we have

$$\psi_i \circ U_+ = \lambda_i \cdot \psi_i \tag{4.2}$$

where $\lambda_i = (1 \mp a)/(1 \pm a)$ are the eigenvalues of $DU_+(A_i)$ corresponding to the vertical direction. The functions ψ_i, $i = 1, 2$ will be called the *local linearizing parameters.*

The global invariant manifolds $W^s(A_1)$ and $W^u(A_2)$ cannot be simply constructed via formulas (3.1). The formulas themselves could be made sense of by allowing *all* images or preimages under U. However, the resulting sets would not have a nice local structure.

It is possible to define two Riemann surfaces, which we will denote by $W^s(A_1)$ and $W^u(A_2)$, in a way that will make them much more useful to answer the question about the existence of heteroclinic or homoclinic connections. The construction will be performed in two stages. The first stage consists in constructing the *unbranched* invariant manifolds ${}_0W^s(A_1)$ and ${}_0W^u(A_2)$, which will possibly have countable numbers of punctures. The Riemann surfaces $W^s(A_1)$ and $W^u(A_2)$ are obtained by simply filling in the punctures.

Let us first define the unbranched stable manifold ${}_0W(A_1)$. Informally, it consists of all germs of orbits $(y_n)_{n=0}^{\infty}$ of the multi-valued map U such that for sufficiently large n the point $y_n \in W^s_{loc}(A_1)$ and $y_{n+1} = U_+(y_n)$. We are about to give a more detailed definition. The reader should consult Figure 5 for a graphic illustration.

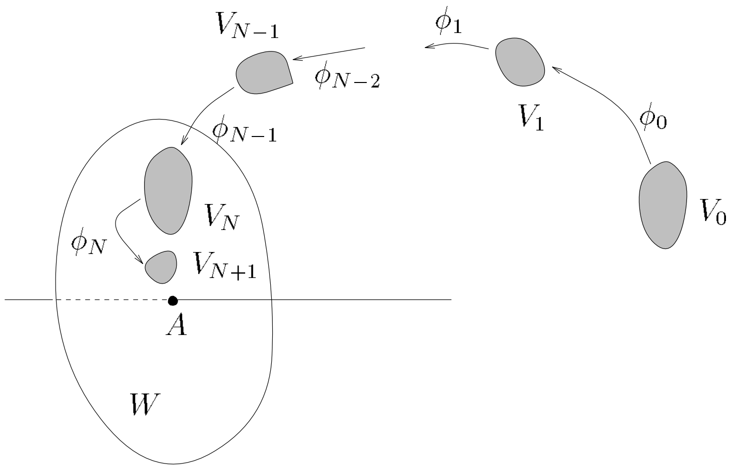

FIGURE 5. The unbranched stable manifold of $A = A_1$ constructed from $W = W^s_{loc}(A_1)$.

Definition 4.3. The unbranched stable manifold of the point $A = A_1$ is a Riemann surface ${}_0W^s(A)$ which as a set consists of sequences of germs $(m_n)_{n=0}^{\infty}$, where each m_n is a germ of a curve V_n at a point y_n. We will require the following additional properties:

1. for every $n \geq 0$ there is a unique regular local branch ϕ_n of the relation U such that $\phi_n(V_n) = V_{n+1}$ and $\phi_n(y_n) = y_{n+1}$;
2. for sufficiently large n we have $V_n \subseteq W^s_{loc}(A)$ and $\phi_n = F$.

The branch points, resulting in punctures, appear when one of the curves V_n intersects the branch manifold of U, i.e. the set of those (x, y) for which $(x-b)^2 + y^2 = 0$, or $x - b = \pm iy$. One also has to consider possible branch points at infinity. No singularity more complicated than a branch point can be generated, due to the fact that U is algebraic.

The branch points appear a finite number at a time, i.e. if we put an upper bound of n in the above construction, then only a finite number of branch points are involved. Due to this property, we may recursively fill in the punctures and obtain a Riemann surface $W^s(A_1)$.

The construction of the unstable manifold is carried out in an analogous way, by considering the germs of trajectories $(y_n)_{n=-\infty}^{0}$ such that for sufficiently large negative n we have $y_n \in W^u_{loc}(A_2)$ and $y_{n+1} = U_+(y_n)$.

4.6. **The connection surface.** When there exists a heteroclinic connection, a third Riemann surface, called the *connection surface* and denoted by $\mathcal{H}$, can be constructed by considering the germs of the double-sided trajectories $(y_n)_{n=-\infty}^{\infty}$ of U such that for sufficiently large n we have $y_n \in W^s_{loc}(A_1)$, $y_{n+1} = U_+(y_n)$ and for sufficently large negative n we have $y_n \in W^u_{loc}(A_2)$, $y_{n+1} = U_+(y_n)$ (cf. Figure 6). Again, with a small effort we may fill in the punctures.

The two mappings which truncate the double-sided sequences $(y_n)_{n=-\infty}^{\infty}$ to the one-sided sequences $(y_n)_{n=0}^{\infty}$ and $(y_n)_{n=-\infty}^{0}$ induce two holomorphic mappings $p_1 : \mathcal{H} \to W^s(A_1)$ and $p_2 : \mathcal{H} \to W^u(A_2)$, which we will call *the projections.* We note that in the single-valued case these mappings are injective and allow us to write

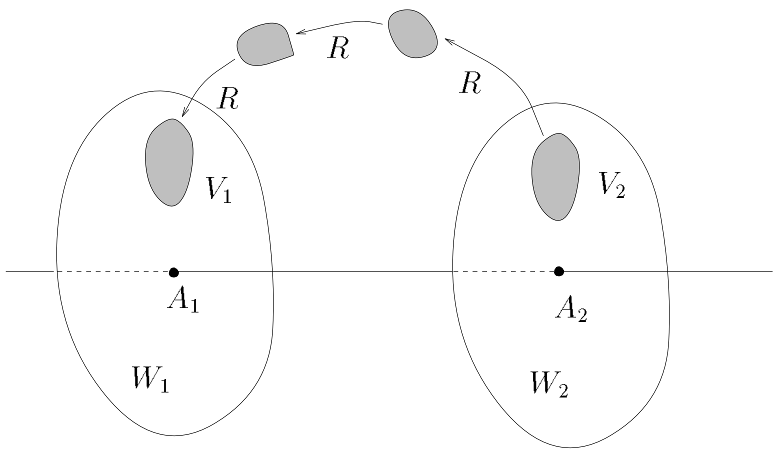

FIGURE 6. The connection surface.

$\mathcal{H} = W^s(A_1) \cap W^u(A_2)$. In the multi-valued situation, we only have the following diagram of holomorphic mappings:

$$\begin{tikzcd} & \mathcal{H} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ W^s(A_1) & & W^u(A_2) \end{tikzcd} \tag{4.3}$$

4.7. **The shift maps, projections and linearizing parameters.** The shift (to the left) on the sequences $(y_n)_{n=0}^{\infty}$ induces a (single-valued) map $\sigma_1 : W^s(A_1) \to W^s(A_1)$. Without difficulty we verify that σ_1 is analytic. In a similar way, the shift to the right induces an analytic map $\sigma_2 : W^u(A_2) \to W^u(A_2)$.

When there is a heteroclinic or homoclinic connection, the shift to the left on double-sided sequences induces a biholomorphic map $\sigma : \mathcal{H} \to \mathcal{H}$.

The linearizing parameters allow extensions to $W^s(A_1)$ and $W^u(A_2)$. Indeed, we may naturally consider $W^s_{loc}(A_1) \subset W^s(A_1)$ and $W^u_{loc}(A_2) \subset W^u(A_2)$. For instance, any point $y_0 \in W^s_{loc}(A_1)$ is identified with the unique sequence $(y_n)_{n=0}^{\infty}$ such that for all $n \geq 1$ we have $y_{n+1} = U_+(y_n)$. Subsequently, we define

$$\begin{aligned} \psi_1(m) &= \lim_{n\to\infty} \lambda_1^{-n} \psi_1(\sigma_1^n(m)), \\ \psi_2(m) &= \lim_{n\to\infty} \lambda_2^{n} \psi_2(\sigma_2^n(m)), \end{aligned}$$

where $m \in W^s(A_1)$ and $m \in W^u(A_2)$ respectively. The functions $\psi_1 : W^s(A_1) \to \mathbb{C}$ and $\psi_2 : W^u(A_2) \to \mathbb{C}$ defined in this way are analytic, surjective and satisfy the equations

$$\begin{aligned} \psi_1 \circ \sigma_1 &= \lambda_1 \cdot \psi_1, \\ \psi_2 \circ \sigma_2 &= \lambda_2^{-1} \cdot \psi_2. \end{aligned}$$

The following diagram summarizes the relationships between various objects that we have constructed. In particular, the two parallelograms contained in it commute:

$$\begin{tikzcd}
 & & & \mathcal{H} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[dll, shift right, "\sigma^{-1}"'] & \\
 & \mathcal{H} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[urr, shift right, "\sigma"'] & W^s(A_1) & & W^u(A_2) \arrow[dll, "\sigma_2"] \\
W^s(A_1) \arrow[urr, "\sigma_1"] & & W^u(A_2) & &
\end{tikzcd} \tag{4.4}$$

4.8. **The shadow maps.** There are three maps $Sh_1 : W^s(A_1) \to \hat{\mathbb{C}}^2$, $Sh_2 : W^u(A_2) \to \hat{\mathbb{C}}^2$ and $Sh : \mathcal{H} \to \hat{\mathbb{C}}^2$ which will be called the *shadow maps*. Each of them is induced by mapping a sequence (y_n) to y_0. The images lie in $\hat{\mathbb{C}}^2$ as certain branch points map to ∞. The images of the shadow maps are complicated subsets of $\hat{\mathbb{C}}^2$ which are roughly the same as the result of application of the formulas (3.1). Roughly speaking, the Riemann surfaces $W^s(A_1)$, $W^u(A_2)$ and $\mathcal{H}$ are desingularizations of the subsets $Sh_1(W^s(A_1))$, $Sh_2(W^u(A_2))$ and $Sh(\mathcal{H})$ of $\hat{\mathbb{C}}^2$. We leave it to the reader to define the last three sets more directly by using trajectories.

4.9. **The classification of components.** Our main effort will be to show that there is an algebraic curve $V \subseteq \hat{\mathbb{C}}^2$ such that

$$Sh_1(W^s(A_1)) \cup Sh_2(W^u(A_2)) \subseteq V. \tag{4.5}$$

In view of Chow's Theorem, it is sufficient to find a compact analytic variety V with the above property, as then this is automatically an algebraic variety. The implication for the $\mathcal{EPP}$ would be that the equichordal curve, if it existed, would be algebraic. Showing nonexistence of an algebraic equichordal curve proves to be a relatively easy task, and thus our strategy leads to the solution of the $\mathcal{EPP}$. For a detailed argument the reader should consult [6].

The connected components of the connection surface $\mathcal{H}$ are permuted by the automorphism σ. Thus, $\mathcal{H}$ splits into a union of cyclically permuted components, where infinite cycles are allowed. A connected component M of $\mathcal{H}$ will be called *elliptic*, *parabolic* or *hyperbolic*, depending on whether the universal covering space of M is isomorphic to $\hat{\mathbb{C}}$ (Riemann sphere), $\mathbb{C}$ or $\mathbb{D}$ (Poincaré disk). Each type of component is analyzed by a separate argument.

4.10. **Elliptic and parabolic components.** The following result admits an easy proof:

Lemma 4.4. *There are no elliptic connected components of $\mathcal{H}$.*

A bit longer argument leads to the following classification of parabolic components:

Theorem 4.5. *If M is a parabolic connected component of $\mathcal{H}$, then M is a part of a cycle of length 1, i.e. $\sigma(M) = M$. Furthermore, $W^s(A_1)$ and $W^u(A_2)$ are* *isomorphic to $\mathbb{C}$, and $\mathcal{H}$ is isomorphic to $\mathbb{C}_*$ (cylinder). There exists a unique algebraic curve V of genus 0 such that*

$$W^s_{loc}(A_1) \cup W^u_{loc}(A_2) \subseteq V. \tag{4.6}$$

Moreover, $Sh_1(W^s(A_1)) \cup Sh_2(W^u(A_2)) \subseteq V$.

Thus V is a rational variety isomorphic to $\hat{\mathbb{C}} = \mathbb{P}_1$ (the projective space of dimension 1). This situation corresponds to the result stated in Theorem 4.1. It is easy to check that $U|V$ has a single-valued branch conjugate to multiplication by a number, just as in Theorem 4.1.

4.11. **Hyperbolic components.** The most subtle point of our solution of the $\mathcal{EPP}$ is an analysis of the hyperbolic components of the connection surface $\mathcal{H}$. We refer the reader to [6] for details. We note that the analysis uses theorems of Fatou and Riesz concerning the boundary behavior of complex functions defined on the unit disk.

4.12. **The invariant parameter and compactness.** The function

$$\psi = (\psi_1 \circ p_1) \cdot (\psi_2 \circ p_2)$$

is well defined and analytic on $\mathcal{H}$ and it has the property $\psi \circ \sigma = \psi$, due to the *resonance condition* $\lambda_1 \lambda_2 = 1$. Thus, it is natural to call ψ the *invariant parameter*.

The following class of connected components of $\mathcal{H}$ will play a critical role in our solution of the $\mathcal{EPP}$:

Definition 4.6. A connected component M of $\mathcal{H}$ is called *regular* iff the invariant parameter $\psi|M$ is constant.

By $\mathcal{H}_{reg}$ we denote the union of all regular components. Our next major goal is to prove that $\mathcal{H}_{reg} \neq \emptyset$. It will be accomplished by variational methods.

4.13. **The extreme property of A_1 and A_2.** At the heart of our method there is a variational method. It is based on the observation that A_1 and A_2 have a special extreme property. We proceed to describe this property in the case of A_1 in detail. Let $\lambda_1 = (1-a)/(1+a)$ be the eigenvalue of the linearization of U at A_1 along the vertical direction. Let us study the sequences of points $(P_n)_{n=0}^{\infty}$ with the property that $U(P_n) = P_{n+1}$ for all $n \geq 0$ and $\|P_n - A_1\| \leq K\lambda_1^n$, where K is a constant. It can be shown that for sufficiently large n we have $U = U_+$ where U_+ is the principal branch of U. In other words, the only way to approach A_1 with the rate λ_1 is to follow the principal branch of U. Other sequences P_n, obtained by a different choice of the branches of U, may still have the property $\lim_{n\to\infty} P_n = A_1$. However, they will approach A_1 at a rate slower than λ_1.

The extreme property eventually produces compactness of the regular components of $\mathcal{H}$. This fact is crucial in our proof.

Appendix A. The result of Shäfke and Volkmer

The result of Shäfke and Volkmer [7] addresses the problem of quantifying the oscillatory behavior of the trajectories of U. We will formulate this result in the notation used in this paper.

Theorem A.1. *Let $P_n = T^n(P_0)$ for all $n \geq 0$, where $P_0 = (-b, 1/2) \in C$. Let $P_n = (x_n, y_n)$. Then $\lim_{n\to\infty} x_n = -(1 + h(a))/2$, where*

$$h(a) = \omega e^{-\frac{\pi^2}{2a}} \left[1 + \frac{\pi^2}{24} a + O(a^2)\right], \tag{A.1}$$

and $1.359 \leq \omega \leq 1.361$.

We note the fact that P_0 belongs to the equichordal curve C, should one exist. This is a consequence of the reflectional symmetries of C. Moreover, if C exists, then $h(a) = 0$. Therefore, the asymptotics of $h(a)$ given in the above theorem implies nonexistence of equichordal curves for sufficiently small a. This result and the analyticity of h imply that there may be only a finite number of values of a for which there is an equichordal curve.

Appendix B. A historical sketch

B.1. **The first result.** Fujiwara [3] showed that there are no convex curves with three equichordal points. This result should be considered elementary.

B.2. **The case of large eccentricities.** The progress on the $\mathcal{EPP}$ was marked by results which gradually decrease the range of eccentricities for which this theorem holds. For instance, it is not too difficult to see that there are no equichordal curves for $a \in (1/2, 1)$ [2]. As we decrease the lower limit, nonexistence becomes gradually more difficult to prove. A typical result of this kind is formulated as

Theorem B.1. *There are no equichordal curves with eccentricities $a \in (\epsilon, 1]$.*

The basis for the results in this direction is a result of G. A. Dirac (1952), according to which the equichordal curve, if it exists, lies inside the set $B(O_1, 1/2 + b) \cup B(O_2, 1/2 + b)$ and outside of the set $B(O_1, 1/2 - b) \cup B(O_2, 1/2 - b)$. It has also been known since the 1920's [8] that the equichordal curve would have to be symmetric with respect to the reflection in the line O_1O_2 as well as in the bisector of the segment O_1O_2. Let Z be the point defined by the property that Z lies on the perpendicular to the line O_1O_2 at O_2 and that $\|Z - O_2\| = 1/2$. The symmetries imply that if C is an equichordal curve, then $Z \in C$. The strategy, exemplified by [5] is to consider the sequence of points $T^n(Z)$, where $T = T_1 \circ T_2$ (T^n denotes the n-fold composition). It proves that this sequence simultaneously converges to the line O_1O_2 and oscillates. Thus, it will fail to satisfy the Dirac bounds.

B.3. **The case of small eccentricities.** The asymptotic analysis of the $\mathcal{EPP}$ was initiated by E. Wirsing [10] in 1958 and it was continued by R. Schäfke and H. Volkmer in [7]. In this approach, one studies the asymptotic problem as $a \to 0$. Wirsing discovered that the problem belongs to the category of perturbation theory "beyond all orders", the first problem of this kind known to us and rigorously demonstrated not to be solvable by a power series in powers of a. The result of Schäfke and Volkmer proves the absence of equichordal curves for sufficiently small a. They claim to have proven that there are no equichordal curves for $a < .03$. However, this result requires a careful analysis of truncation and round-off errors while juggling between a function, its Laplace transform and the Taylor series.

Appendix C. The measurable Equichordal Point Problem

Yet another interpretation of $\mathcal{EPP}$ is contained in the following open problem:

Problem C.1 (The measurable $\mathcal{EPP}$). Let D be a measurable subset of the plane. A point O of the plane is called a *measurable equichordal point* if for every straight line ℓ passing through O the Lebesgue measure of the intersection $\ell \cap D$ is nonzero, finite and it does not depend on ℓ. Is there a set D with two measurable equichordal points?

This problem seems to require methods that are quite different from the ones considered in the solution of the original $\mathcal{EPP}$.

Appendix D. A one-dimensional classification problem

A question arises concerning which Riemann surfaces can occur as a connection surface $\mathcal{H}$ for multi-valued algebraic maps similar to the one that appears in the $\mathcal{EPP}$. This question will be reformulated abstractly in this section.

Let M be a noncompact Riemann surface and let $\sigma : M \to M$ be an automorphism such that the cyclic group $\Gamma = \{\sigma^n : n \in \mathbb{Z}\}$ acts on M freely and discretely and co-compactly, i.e. the quotient C/Γ is compact. Moreover, let $\psi : M \to \mathbb{C}$ be a holomorphic function on M which satisfies the functional equation

$$\psi \circ \sigma = \mu\psi$$

where $\mu \in \mathbb{C}_*$ and $|\mu| \neq 1$.

Problem D.1. Let g be the genus of the compact Riemann surface M/Γ. Is it possible that $g \geq 2$?

This question is motivated by the question whether the connection surface $\mathcal{H}$ may contain a component M such that $Sh(M)$ is contained in an algebraic curve of genus ≥ 2.

Recently J. Mather has given a positive answer to this question (verbal communication). In this section we will explain briefly how his example is constructed.

Let us assume that there exists a nonconstant map $\phi : S \to \mathbb{C}_*/\Lambda$, where $\Lambda = \{\mu^j : j \in \mathbb{Z}\}$ is a multiplicative subgroup of $\mathbb{C}_*$ acting on $\mathbb{C}_*$ by multiplication and S is a Riemann surface of genus g. The fiber product $M \subseteq S \times \mathbb{C}_*$ defined as the set of all pairs (z, w) such that $\phi(z) = \pi(w)$ where $\pi : \mathbb{C}_* \to \mathbb{C}_*/\Lambda$ is the natural projection, admits an automorphism $\sigma : M \to M$ mapping (z, w) to $(z, \mu w)$. It is clear that the pair (M, σ) has the desired property. Indeed, M has genus $\geq g$. However, M may or may not be connected. Nevertheless, we may select a connected component of M mapped into itself by some power of σ, and obtain a connected example in this way.

We note that it is not difficult to construct the map ϕ. We need to consider a multi-valued function on $\mathbb{C}/\Lambda'$, for instance, $\sqrt{\wp(z)}$ where $\wp$ is the Weierstrass function. The corresponding Riemann surface N and the branched covering $\phi : N \to \mathbb{C}/\Lambda'$ satisfy the requirements of our construction. The particular choice $\sqrt{\wp(z)}$ yields N of genus 2.

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