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LOWER AND UPPER BOUNDS FOR THE SPLITTING OF SEPARATRICES OF THE PENDULUM UNDER A FAST QUASIPERIODIC FORCING

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Abstract. Quasiperiodic perturbations with two frequencies $(1/\varepsilon, \gamma/\varepsilon)$ of a pendulum are considered, where γ is the golden mean number. We study the splitting of the three-dimensional invariant manifolds associated to a two-dimensional invariant torus in a neighbourhood of the saddle point of the pendulum. Provided that some of the Fourier coefficients of the perturbation (the ones associated to Fibonacci numbers) are separated from zero, it is proved that the invariant manifolds split for ε small enough. The value of the splitting, that turns out to be $\mathrm{O}\left(\exp\left(-\operatorname{const}/\sqrt{\varepsilon}\right)\right)$, is correctly predicted by the Melnikov function.

1. Introduction

The rapidly (and periodically) forced pendulum has been widely used as a model for the motion near a resonance of Hamiltonian systems with two degrees of freedom. As is well known in several situations [DS92], [Gel93], the separatrices of the perturbed system do not coincide, giving rise to the so-called splitting of separatrices, which seems to be the main cause of stochastic behaviour in Hamiltonian systems.

In this announcement we consider a quasiperiodic high-frequency perturbation of the pendulum (it can be regarded as a model near a resonance of a Hamiltonian system with three degrees of freedom), described by the Hamiltonian function

$$\frac{\omega \cdot I}{\varepsilon} + h(x, y, \theta, \varepsilon), \tag{1}$$

where

$$\omega \cdot I = \omega_1 I_1 + \omega_2 I_2, \qquad h(x, y, \theta, \varepsilon) = \frac{y^2}{2} + \cos x + \varepsilon^p m(\theta_1, \theta_2) \cos x,$$

with symplectic form $\mathrm{d}x \wedge \mathrm{d}y + \mathrm{d}\theta_1 \wedge \mathrm{d}I_1 + \mathrm{d}\theta_2 \wedge \mathrm{d}I_2$. We assume that ε is a small positive parameter and that p is a positive parameter. Mainly due to a technical limitation imposed by the Extension Theorem (Theorem 2), we will restrict ourselves to the case $p > 3$. We also assume that the frequency is of the form ω/ε for

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$\omega = (1, \gamma)$, where $\gamma = (1 + \sqrt{5})/2$ is the golden mean. The equations of motion related to Hamiltonian (1) are:

$$
\begin{aligned}
\dot{x} &= y, & \dot{y} &= (1 + \varepsilon^p m(\theta_1, \theta_2)) \sin x, \\
\dot{\theta}_1 &= \frac{1}{\varepsilon}, & \dot{I}_1 &= -\varepsilon^p \cos x \frac{\partial m}{\partial \theta_1}(\theta_1, \theta_2), \\
\dot{\theta}_2 &= \frac{\gamma}{\varepsilon}, & \dot{I}_2 &= -\varepsilon^p \cos x \frac{\partial m}{\partial \theta_2}(\theta_1, \theta_2).
\end{aligned} \tag{2}
$$

The function m is assumed to be a 2π-periodic function of two variables θ_1 and θ_2. Thus it can be represented as a Fourier series:

$$m(\theta_1, \theta_2) = \sum_{k_1, k_2} m_{k_1 k_2} \, \mathrm{e}^{i(k_1\theta_1 + k_2\theta_2)}.$$

We assume that, for some positive numbers r_1 and r_2,

$$\sup_{k_1, k_2} \left| m_{k_1 k_2} \, \mathrm{e}^{r_1|k_1| + r_2|k_2|} \right| < \infty, \tag{3}$$

and that there are positive numbers a and k_0 such that

$$|m_{k_1 k_2}| \ge a \, \mathrm{e}^{-r_1|k_1| - r_2|k_2|} \tag{4}$$

for all k_1, k_2 such that $|k_1| = F_{n+1}$ and $|k_2| = F_n$, where F_n and F_{n+1} are Fibonacci numbers, which are defined by the following recurrent formula:

$$F_0 = 1, \qquad F_1 = 1, \qquad F_{n+1} = F_n + F_{n-1}, \quad n \ge 1. \tag{5}$$

We call the corresponding terms in the perturbation *resonant* or *Fibonacci terms*.

For example, the function

$$m(\theta_1, \theta_2) = \frac{\cos\theta_1 \cos\theta_2}{(\cosh r_1 - \cos\theta_1)(\cosh r_2 - \cos\theta_2)}$$

satisfies these conditions.

The upper bound (3) implies that the function m is analytic on the strip $\{|\Im\theta_1| < r_1\} \times \{|\Im\theta_2| < r_2\}$. Equation (4) implies that this function cannot be continued analytically onto a larger strip. Let us select $\alpha \in (0, 1]$. Estimate (3) implies that

$$|m(\theta_1, \theta_2)| \le K\varepsilon^{-2\alpha} \tag{6}$$

on the strip

$$|\Im\theta_1| \le r_1 - \varepsilon^\alpha, \quad |\Im\theta_2| \le r_2 - \varepsilon^\alpha.$$

Formula (4) implies that the upper bound (6) cannot be improved. It will be seen that the value of the splitting depends essentially on the width of these strips. Moreover, formula (4) will allow us to estimate in the next section the size of the Melnikov function in terms of the separatrix of the unperturbed pendulum. (See Remark 4.) The function m under consideration has a singularity "of the second order", in the sense that the upper bound (6) for the maximum of the modulus is quadratic with respect to the inverse of the distance to the boundary of the strip. In a similar way the case of a singularity of any "order" q can be considered. In this case $m_{k_1 k_2}$ should be replaced by $m_{k_1 k_2}/\,|k|^{q-2}$ in (3) and (4).

The Hamiltonian (1) can be regarded as a singular perturbation of the pendulum

$$h_0 = \frac{y^2}{2} + \cos x. \tag{7}$$

The unperturbed system has a saddle point $(0,0)$ and a homoclinic trajectory given by

$$x_0(t) = 4\arctan(\mathrm{e}^t), \qquad y_0(t) = \dot{x}_0(t).$$

The complete system (2) has a whiskered torus $\mathcal{T}$: $(0,0,\theta_1,\theta_2)$. The whiskers are $3D$-hypersurfaces in the $4D$-dimensional extended phase space (x,y,θ_1,θ_2). These invariant manifolds are close to the unperturbed pendulum separatrix.

Our *main result* (Theorem 3) is that, if condition (4) is verified, then for $p > 3$ and small $\varepsilon > 0$ the invariant manifolds split, and the value of the splitting (i.e., the distance function between these invariant manifolds) is correctly predicted by the Melnikov function, which is $\mathrm{O}(\mathrm{e}^{-\mathrm{const}/\sqrt{\varepsilon}})$.

Remark 1. Our model (1) is based on a previous work by C. Simó [Sim94], where Neishtadt's Averaging Theorem [Nei84] was generalized to quasiperiodic systems, giving rise to upper estimates of the splitting which are exponentially small with respect to the parameter of perturbation ε. Related upper estimates can be found in [CG94], [Gal94], [BCG95], [Ben96]. In contrast to these results, Theorem 3 provides *both* lower and upper bounds for our model.

Remark 2. As an example in [DGJS96b] shows, the splitting can be of the order of some power of ε if the function m is not analytic. This makes a first qualitative difference between periodic and quasiperiodic perturbations. Indeed, in the periodic case, only the C^1 dependence with respect to θ of the perturbed Hamiltonian is needed to prove that the splitting is $\mathrm{O}(\mathrm{e}^{-c/\varepsilon})$, where c is the width of the analyticity strip of the unperturbed separatrix. (In both cases, the analyticity of the unperturbed system is essential.)

Remark 3. In the case of an entire function m, we think that the method used in the present paper can be modified in order to improve the estimate of the error and to prove that the Melnikov function gives the actual asymptotics at least when the resonant terms decrease not much faster than $1/k!$.

2. The Melnikov function

As is well known, the Melnikov function

$$M(\theta_1,\theta_2;\varepsilon) = \int_{-\infty}^{\infty} \{h_0, h\}(x_0(t), y_0(t), \theta_1 + t/\varepsilon, \theta_2 + \gamma t/\varepsilon)\, dt \tag{8}$$

gives a first order approximation of the difference between the values of the unperturbed pendulum energy h_0 on the stable and unstable manifolds. Using the Fourier series of $m(\theta_1,\theta_2)$, one can compute the Fourier coefficients of $M(\theta_1,\theta_2)$ as

$$M_{k_1k_2}(\varepsilon) = -\frac{2\pi i \varepsilon^p (k_1+\gamma k_2)^2}{\varepsilon^2 \sinh\left(\pi(k_1+\gamma k_2)/(2\varepsilon)\right)} \cdot m_{k_1k_2}.$$

In order to bound the Melnikov function, it is important to know for each fixed ε which are the indices (k_1,k_2) corresponding to the biggest Fourier coefficient $M_{k_1k_2}(\varepsilon)$. From the expression above, $M_{k_1k_2}(\varepsilon)$ is a product of two factors. For ε fixed and small enough, the first factor of $M_{k_1k_2}(\varepsilon)$ behaves as $\varepsilon^{p-2}(k_1+\gamma k_2)^2 \times \mathrm{e}^{-\pi(k_1+\gamma k_2)/(2\varepsilon)}$, and it turns out that it becomes bigger for small $k_1+\gamma k_2$, i.e., just for the resonant terms where the second factor $m_{k_1k_2}$ decreases with respect to (k_1, k_2) according to the behaviour (3) and (4). Here we have the main difference between the quasiperiodic case and the periodic one: for a periodic perturbation, the first Fourier coefficients $M_{\pm 1}(\varepsilon)$ of the Melnikov function $M(\theta;\varepsilon)$ give generically the main contribution to the Melnikov function $M(\theta;\varepsilon)$, since $M_k(\varepsilon) = \mathrm{O}(\mathrm{e}^{-|k|/\varepsilon})$. However, in the quasiperiodic case the biggest Fourier coefficient depends strongly on the exponent $k\cdot(1,\gamma) = k_1 + \gamma k_2$ of the Fourier coefficient, i.e., on the rational approximations of γ.

For $\gamma = (1+\sqrt{5})/2$, it is very well known that its best approximation by rational numbers is given by the quotient of successive Fibonacci numbers (5). Indeed, it is easy to check that for large values of n one has the following approximation of γ by Fibonacci numbers:

$$F_n - \gamma F_{n-1} = (-1)^n \frac{C_F}{F_{n-1}} + \mathrm{O}\left(\frac{1}{F_{n-1}^3}\right), \qquad C_F = \frac{1}{\gamma+\gamma^{-1}},$$

whereas for the other integers one has the following result.

Lemma 1. *If $N \in \mathbb{N}$ is not a Fibonacci number, then for all integers k*

$$|k - \gamma N| > \frac{\gamma C_F}{N}.$$

Using this lemma one can see that the indices (k_1, k_2) corresponding to the leading Fourier coefficients $M_{k_1k_2}(\varepsilon)$ depend on ε. In fact, the largest terms correspond to $(k_1, k_2) = \pm\left(F_{n(\varepsilon)+1}, -F_{n(\varepsilon)}\right)$, where $F_{n(\varepsilon)}$ is the Fibonacci number closest to $F^*(\varepsilon) = \sqrt{\phi_0/\varepsilon}$, where $\phi_0 = \pi/(2(\gamma+\gamma^{-1})(r_1\gamma + r_2))$. Except for a small neighbourhood of $\varepsilon = \varepsilon^*\gamma^{-n}$, with ε^* given in (10), there is a unique Fibonacci number closest to $F^*(\varepsilon)$, and then only the two corresponding terms dominate in the Fourier series. Studying the size of this term (which also depends on r_1 and r_2), one can observe that it is $\mathrm{O}(\mathrm{e}^{-c/\sqrt{\varepsilon}})$. With a more detailed analysis, one can see that a better estimate is provided by taking c to be not a constant function but a bounded oscillating one:

$$c(\delta) = C_0 \cosh\left(\frac{\delta - \delta_0}{2}\right) \quad \text{for} \quad \delta \in [\delta_0 - \log\gamma, \delta_0 + \log\gamma], \tag{9}$$

where

$$C_0 = \sqrt{\frac{2\pi(\gamma r_1 + r_2)}{\gamma+\gamma^{-1}}}, \qquad \delta_0 = \log\varepsilon^*, \quad \varepsilon^* = \frac{\pi(\gamma+\gamma^{-1})}{2\gamma^2(r_1\gamma + r_2)}, \tag{10}$$

continued periodically onto the whole real axis. In this way, the function c is piecewise-analytic, continuous, and $2\log\gamma$-periodic. This is summarized in the following lemma.

Lemma 2 (Properties of the Melnikov function)**.** *The Melnikov function defined by (8) is a 2π-periodic function of θ_1 and θ_2, such that*

1) *$M(\theta_1 - T/\varepsilon, \theta_2 - \gamma T/\varepsilon; \varepsilon)$ is analytic in the product of strips*

$$\{|\Im\theta_1| < r_1\} \times \{|\Im\theta_2| < r_2\} \times \{|\Im T| < \pi/2\};$$

2) *the maximum of the modulus of the Melnikov function taken on real arguments, $\max_{(\theta_1,\theta_2)\in\mathbb{T}^2}|M(\theta_1,\theta_2)|$, can be bounded from above and from below*

by terms of the form

$$\mathrm{const}\,\varepsilon^{p-1}\exp\left(-\frac{c(\log\varepsilon)}{\sqrt{\varepsilon}}\right) \tag{11}$$

with different ε-independent constants, where the function c in the exponent is defined by (9)*;*

3) *for a fixed small ε only four terms (at most) dominate in the Fourier series for the Melnikov function and the rest can be estimated from above by* $\mathrm{O}(\mathrm{e}^{-C_1/\sqrt{\varepsilon}})$*, where the constant* $C_1 > \max c(\delta) = C_0\cosh(\log\sqrt{\gamma})$.

Since we have established that for most small values of ε only the terms with $(k_1, k_2) = \pm(F_{n(\varepsilon)+1}, -F_{n(\varepsilon)})$ are important, the Melnikov function is essentially

$$M(\theta_1,\theta_2;\varepsilon) \approx 2\left|M_{F_{n(\varepsilon)+1},-F_{n(\varepsilon)}}\right|\sin\left(F_{n(\varepsilon)+1}\theta_1 - F_{n(\varepsilon)}\theta_2 + \varphi(\varepsilon)\right).$$

The zeros of the Melnikov function correspond to homoclinic trajectories. The above formula implies that the zeros of the Melnikov function form two lines on the torus. As already noticed by C. Simó [Sim94], the averaged slopes of those lines approach γ when $\varepsilon \to 0$.

Remark 4. We note that the Melnikov function is not invariant with respect to canonical changes of variables. After a change, e.g., after a step of the classical averaging procedure, a lot of nonzero harmonics, which were not present in the original system, can appear. If in the original system the Fibonacci terms were not big enough, these new harmonics may give larger contribution to the splitting. This idea was used in [Sim94] to detect the splitting for a system with only four perturbing terms.

Remark 5. The hypothesis that ω in the frequency vector is just $(1,\gamma)$ can be relaxed. The generalization of the present result to the case when γ is a quadratic number is straightforward, with a similar expression (11) for the size of the Melnikov function. The case in which $\omega = (\omega_1, \omega_2)$, with the ratio ω_1/ω_2 being of constant type (the continued fraction expansion has bounded coefficients), but not quadratic, can be similarly analyzed, but in this case $c(\delta)$ is no longer a periodic function. In some sense one can say, properly speaking, that there are no asymptotics. But it seems that there still exist upper and lower bounds, with the factor $\sqrt{\varepsilon}$ in the denominator of the exponential term. The case of two frequencies whose ratio ω_1/ω_2 is not of constant type, as well as the case of more than two perturbing frequencies, is more complicated.

In the following sections we sketch the method used to justify that the prediction given by the Melnikov function is correct. The method used here is a generalization to the quasiperiodic case of the method used in [Laz84], [DS92], [Gel93].

3. Normal form and local manifolds

The first step is to give a description of the dynamics near the $2D$-dimensional invariant torus $\mathcal{T}$. So, we will show the existence of a convergent normal form in a neighbourhood of $\mathcal{T}$.

As we have seen during the analysis of the Melnikov function, the size of the splitting depends essentially on the widths of the analyticity strip (r_1, r_2) of the angular variables θ_1, θ_2, as well as on the width of the analyticity strip of the separatrix $(x_0(t), y_0(t))$. Therefore, to detect the splitting in the quasiperiodic case the loss of domain in the angular variables must be very small (i.e., $\mathrm{O}(\varepsilon^\alpha)$, where α depends on the Diophantine properties of the frequencies). This makes another difference with the periodic case, where the size of the splitting does *not* depend on the width of the analyticity strip of the angular variable θ, but only on the width of the analyticity strip of the separatrix $(x_0(t), y_0(t))$. When dealing with the frequencies $(1,\gamma)$ one needs a reduction of $\mathrm{O}(\sqrt{\varepsilon})$ at most. Hence, during the proof of the convergence of the normal form one has to bound carefully the loss of domain (with respect to the angular variables) in order to achieve such a small reduction.

Finally, we want to stress that if the amount of reduction is something bigger, one can only produce upper bounds for the splitting of separatrices.

Theorem 1 (Normal Form Theorem). *Let $\varepsilon \in (0, \varepsilon_0)$. In a neighbourhood of the hyperbolic torus $\mathcal{T}$ there is a canonical change of variables $(x, y) \to (X, Y)$, which depends 2π-periodically on θ_1 and θ_2, such that the Hamiltonian* (1) *takes the form*

$$H(XY, \varepsilon) = H_0(XY) + \varepsilon^{p-1} H_1(XY, \varepsilon),$$

where H_0 is the normal form Hamiltonian for the unperturbed pendulum. Moreover, the change of variables has the form

$$\begin{aligned} x &= x^{(0)}(X, Y) + \varepsilon^{p-1} x^{(1)}(X, Y, \theta_1, \theta_2; \varepsilon), \\ y &= y^{(0)}(X, Y) + \varepsilon^{p-1} y^{(1)}(X, Y, \theta_1, \theta_2; \varepsilon), \end{aligned} \tag{12}$$

where $(x^{(0)}, y^{(0)})$ are normal form coordinates for the unperturbed pendulum.

The functions H_0, H_1, $x^{(0)}$, $y^{(0)}$, $x^{(1)}$, and $y^{(1)}$ are analytic and uniformly bounded in the complex domain defined by

$$|X|^2 + |Y|^2 < r_0^2, \qquad |\Im\theta_1| < r_1 - \sqrt{\varepsilon}, \qquad |\Im\theta_2| < r_2 - \sqrt{\varepsilon},$$

for r_1 and r_2 in (3) *and some positive constant $r_0 > 0$.*

The proof of this theorem can be found in [DGJS96a].

The Normal Form Theorem provides a convenient parametrization for the local invariant manifolds. Let λ be $H'(0, \varepsilon)$. Then,

$$\begin{aligned} x &= x^s(T, \theta_1, \theta_2) \equiv x(0,\, \mathrm{e}^{-\lambda T}, \theta_1, \theta_2), \\ y &= y^s(T, \theta_1, \theta_2) \equiv y(0,\, \mathrm{e}^{-\lambda T}, \theta_1, \theta_2), \end{aligned} \qquad \text{for } T \geq T_0, \tag{13}$$

and

$$\begin{aligned} x &= x^u(T, \theta_1, \theta_2) \equiv x(\mathrm{e}^{\lambda T}, 0, \theta_1, \theta_2), \\ y &= y^u(T, \theta_1, \theta_2) \equiv y(\mathrm{e}^{\lambda T}, 0, \theta_1, \theta_2), \end{aligned} \qquad \text{for } T \leq -T_0, \tag{14}$$

where we have used the change (12). Theorem 1 also implies that, in the domains above,

$$\begin{aligned} \left|x^\beta(T, \theta_1, \theta_2) - x_0(T)\right| &\leq C\varepsilon^{p-1}, \\ \left|y^\beta(T, \theta_1, \theta_2) - y_0(T)\right| &\leq C\varepsilon^{p-1}, \end{aligned} \qquad \text{for } \beta = s, u.$$

4. Extension Theorem

The Normal Form Theorem provides a local approximation for the unstable manifold in terms of the unperturbed separatrix, which is $\mathrm{O}\left(\varepsilon^{p-1}\right)$. The following theorem extends this local approximation for solutions of system (2) to a global one. Since the unperturbed separatrix $(x_0(T), y_0(T))$ has a singularity on $T = \pm\pi/2$, we will restrict ourselves to $|\Im T| \leq \pi/2 - \sqrt{\varepsilon}$, i.e., up to a distance to the singularity $T = \pm\pi/2$ of the same order as the loss of domain in the angular variables. Besides, the extension time $t+T$ will be chosen big enough in order that the unperturbed separatrix reaches again the domain of convergence of the normal form. This procedure follows the same ideas as in the Extension Theorem of [DS92], and its complete proof can be also found in [DGJS96a].

Theorem 2 (Extension Theorem)**.** *Assume $p>2$. Then, there exists $\varepsilon_0>0$ such that the following extension property holds:*

For any positive constants C and T_0 there exists a constant C_1, such that for any $\varepsilon\in(0,\varepsilon_0)$, every solution of system (2) that satisfies the initial conditions

$$|x(t_0)-x_0(t_0+T)|\le C\varepsilon^{p-1},\quad |y(t_0)-y_0(t_0+T)|\le C\varepsilon^{p-1},$$

$$|\Im\theta_1(t_0)|\le r_1-\sqrt{\varepsilon},\quad |\Im\theta_2(t_0)|\le r_2-\sqrt{\varepsilon},$$

for some $T\in\mathbb{C}$, $t_0\in\mathbb{R}$ with

$$|\Im T|\le\pi/2-\sqrt{\varepsilon},\qquad -T_0\le t_0+\Re T<0,$$

can be extended for $-T_0\le t+\Re T\le T_0$ satisfying

$$|x(t)-x_0(t+T)|\le C_1\varepsilon^{p-2},\quad |y(t)-y_0(t+T)|\le C_1\varepsilon^{p-2}.$$

In particular, Theorem 2 can be applied to the local invariant unstable manifold given in (14). As we will see in Lemmas 3 and 4, the above approximation of these invariant manifolds in such a complex domain will allow us to derive suitable bounds of the error on the real axis to detect the splitting. Before closing this section let us note that, as a direct consequence of the Extension Theorem, the difference of unperturbed energies along the invariant manifolds can also be estimated.

Corollary 1. *The following estimate holds:*

$$h_0(x^u,y^u)-h_0(x^s,y^s)=M(\theta_1-T/\varepsilon,\theta_2-\gamma T/\varepsilon)+\mathrm{O}\left(\varepsilon^{2(p-2)}\right),$$

where h_0 is evaluated on the invariant manifolds corresponding to T,θ_1,θ_2,

$$\Re T\in(T_0-R,T_0),\quad |\Im T|\le\pi/2-\sqrt{\varepsilon},\quad |\Im\theta_k|\le r_k-\sqrt{\varepsilon},\quad k=1,2, \tag{15}$$

for any positive constants T_0 and R, $R<T_0$.

5. First return

By Theorem 1, the local unstable invariant manifold is ε^{p-1}-close to the unperturbed separatrix. By Theorem 2, it can be continued for $-T_0\le t+\Re T\le T_0$, provided that the parameters (θ_1,θ_2,T) belong to the complex domain (15), and it remains ε^{p-2}-close to the unperturbed separatrix. Since this unperturbed homoclinic orbit comes back to the domain of the normal form, the same happens to the unstable manifold, which can be compared with the local stable manifold.

In order to describe the difference between the global unstable manifold and the local stable one, it is convenient to take H and $T=-\log Y/H'(XY)$ as canonical coordinates near the stable separatrix. The equation of the local stable manifold is then $H=0$. In this coordinate system the unstable manifold is the graph of a function H^u: $H=H^u(T,\theta_1,\theta_2)$, which depends 2π-periodically on θ_1 and θ_2, and has zero mean, due to the Hamiltonian character of the perturbation. Using the parametrization provided by the normal form, it turns out that H^u is quasiperiodic in T:

$$H^u(T,\theta_1,\theta_2) = H^u(0,\theta_1 - T/\varepsilon, \theta_2 - \gamma T/\varepsilon).$$

Moreover, by the Extension Theorem and its Corollary 1, H^u is given in first order by the Melnikov function for (T,θ_1,θ_2) in the complex domain (15):

$$H^u(T,\theta_1,\theta_2) = M(\theta_1 - T/\varepsilon, \theta_2 - \gamma T/\varepsilon) + \mathrm{O}\left(\varepsilon^{2p-4}\right). \tag{16}$$

It is important to notice that the term $F = \mathrm{O}\left(\varepsilon^{2p-4}\right)$ in equation (16) is an analytic function in the complex domain (15) which depends 2π-periodically on θ_1 and θ_2, has zero mean, and is quasiperiodic in T. The following general lemma allows us to bound its Fourier coefficients.

Lemma 3. *Let $F(\theta_1 + s/\varepsilon, \theta_2 + \gamma s/\varepsilon)$ be a 2π-periodic function of the variables θ_1, θ_2, analytic in the product of strips $|\Im\theta_1| \le r_1$, $|\Im\theta_2| \le r_2$, and $|\Im s| \le \rho$, and $|F| \le A$ for these values of the variables. Then for all k_1, $k_2 \in \mathbb{Z}$*

$$|F_{k_1 k_2}| \le A\,\mathrm{e}^{-|k_1| r_1 - |k_2| r_2}\,\mathrm{e}^{-\rho|k_1+\gamma k_2|/\varepsilon}.$$

Finally, the next lemma gives the exponentially small upper bound for the function F for real values of the variables.

Lemma 4. *Consider the $(2\log\gamma)$-periodic function $c_{\rho,r_1,r_2}(\delta)$ defined on the interval $[\log\varepsilon^* - \log\gamma, \log\varepsilon^* + \log\gamma]$ by*

$$c_{\rho,r_1,r_2}(\delta) = C_0 \cosh\left(\frac{\delta - \log\varepsilon^*}{2}\right),$$

$$\varepsilon^* = \frac{\rho(\gamma+\gamma^{-1})}{(\gamma r_1 + r_2)\gamma^2}, \qquad C_0 = 2\sqrt{\frac{(\gamma r_1 + r_2)\rho}{\gamma+\gamma^{-1}}},$$

and continued by $2\log\gamma$-periodicity. Let F satisfy the conditions of Lemma 3. If $\gamma = (1+\sqrt{5})/2$ is the golden mean number and the mean value of the function F is zero, then

$$|F(\theta_1,\theta_2)| \le \mathrm{const}\, A \exp\left(-\frac{c_{\rho,r_1,r_2}(\log\varepsilon)}{\sqrt{\varepsilon}}\right) \tag{17}$$

on the real values of its arguments. The constant depends continuously on $r_1 > 0$ and $r_2 > 0$.

Applying these two lemmas to the error function $F = \mathrm{O}\left(\varepsilon^{2p-4}\right)$ in equation (16), we obtain the desired exponentially small estimates. Now we can summarize the above results on the splitting function $H^u(T,\theta_1,\theta_2)$ in the following theorem, which is the main result of this paper.

Theorem 3 (Main Theorem)**.** *There exist positive constants T_0 and R, $R < T_0$, such that in the coordinate system (H,T,θ_1,θ_2) the unstable manifold can be represented as the graph of the function $H = H^u(T,\theta_1,\theta_2;\varepsilon)$, where the function H^u depends 2π-periodically on θ_1 and θ_2. In the domain*

$$\Re T \in (T_0 - R, T_0), \qquad |\Im T| \le \frac{\pi}{2} - \sqrt{\varepsilon},$$

$$|\Im\theta_1| < r_1 - \sqrt{\varepsilon}, \qquad |\Im\theta_2| < r_2 - \sqrt{\varepsilon},$$

this function is analytic and close to the Melnikov function:

$$H^u(T,\theta_1,\theta_2) = M(\theta_1 - T/\varepsilon, \theta_2 - \gamma T/\varepsilon) + \mathrm{O}\left(\varepsilon^{2p-4}\right).$$

Moreover,

$$H^u(T,\theta_1,\theta_2) = H^u(0,\theta_1 - T/\varepsilon, \theta_2 - \gamma T/\varepsilon),$$

and its mean value is zero:

$$\int_{\mathbb{T}^2} H^u(0,\theta_1,\theta_2)\, d\theta_1 d\theta_2 = 0. \tag{18}$$

Furthermore, for $p > 3$ and real T, θ_1 and θ_2,

$$|H^u(T,\theta_1,\theta_2) - M(\theta_1 - T/\varepsilon, \theta_2 - \gamma T/\varepsilon)| \le \mathrm{const}\, \varepsilon^{2p-4} \exp\left(-\frac{c(\log\varepsilon)}{\sqrt{\varepsilon}}\right),$$

where $c(\delta)$ is defined in (9). If condition (4) is fulfilled, then there exists $\varepsilon_0 > 0$ such that, for $0 < \varepsilon < \varepsilon_0$, the maximum of the modulus of the Melnikov function is larger than the right-hand side of the last upper bound.

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