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NONLINEAR PARABOLIC PROBLEMS ON MANIFOLDS, AND A NONEXISTENCE RESULT FOR THE NONCOMPACT YAMABE PROBLEM

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ABSTRACT. We study the Cauchy problem for the semilinear parabolic equations $\Delta u - Ru + u^p - u_t = 0$ on $\mathbf{M}^n \times (0, \infty)$ with initial value $u_0 \geq 0$, where $\mathbf{M}^n$ is a Riemannian manifold including the ones with nonnegative Ricci curvature. In the Euclidean case and when $R = 0$, it is well known that $1 + \frac{2}{n}$ is the critical exponent, i.e., if $p > 1 + \frac{2}{n}$ and u_0 is smaller than a small Gaussian, then the Cauchy problem has global positive solutions, and if $p < 1 + \frac{2}{n}$, then all positive solutions blow up in finite time. In this paper, we show that on certain Riemannian manifolds, the above equation with certain conditions on R also has a critical exponent. More importantly, we reveal an explicit relation between the size of the critical exponent and geometric properties such as the growth rate of geodesic balls. To achieve the results we introduce a new estimate for related heat kernels.

As an application, we show that the well-known noncompact Yamabe problem (of prescribing constant positive scalar curvature) on a manifold with nonnegative Ricci curvature cannot be solved if the existing scalar curvature decays "too fast" and the volume of geodesic balls does not increase "fast enough". We also find some complete manifolds with positive scalar curvature, which are conformal to complete manifolds with positive constant and with zero scalar curvatures. This is a new phenomenon which does not happen in the compact case.

1. INTRODUCTION

We shall announce new results in the study of the global existence and blow up of the following semilinear parabolic Cauchy problem:

$$(1.1) \qquad \begin{cases} Hu \equiv H_0 u + u^p = \Delta u - Ru - \partial_t u + u^p = 0 & \text{in} \quad \mathbf{M}^n \times (0, \infty), \\ u(x, 0) = u_0(x) \geq 0 \quad \text{in} \quad \mathbf{M}^n, \end{cases}$$

where $\mathbf{M}^n$ with $n \geq 3$ is a noncompact complete Riemannian manifold, Δ is the Laplace-Beltrami operator, and $R = R(x)$ is a bounded function.

Throughout the paper we make the following assumptions.

(i) There are positive constants b, C, and K such that

$$(1.2) \qquad |B(x, r)| \leq Cr^n; \quad |B(x, 2r)| \leq C2^n |B(x, r)|, \ r > 0; \quad \text{Ricci} \geq -K.$$

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(ii) G, the fundamental solution of the linear operator $H_0 = \Delta - R - \partial_t$ in (1.1), has global Gaussian upper bound, i.e.,

$$0 \le G(x,t;y,s) \le \frac{C}{|B(x,(t-s)^{1/2})|} e^{-b\frac{d(x,y)^2}{t-s}}, \tag{1.3}$$

for all $x, y \in \mathbf{M}^n$ and all $t > s$.

(iii) When $d(x,y)^2 \le t-s$, G satisfies

$$G(x,t;y,s) \ge \min\left\{ \frac{1}{C|B(x,(t-s)^{1/2})|}, \frac{1}{C|B(y,(t-s)^{1/2})|} \right\}. \tag{1.3$'$}$$

If one assumes that the Ricci curvature of $\mathbf{M}^n$ is nonnegative and $R = 0$, it is well known that that above properties hold (see [LY]). More general conditions on $\mathbf{M}^n$ (without the nonnegativity of Ricci) that support (1.2), (1.3), and (1.3$'$) can be found in [G]. When R is not identically zero, (1.3) and (1.3$'$) pose certain restrictions on R as indicated in Corollary 2.3 below.

Equation (1.1) with $R = 0$ contains the following important special cases. When the Riemannian metric is just the Euclidean metric, (1.1) becomes the semilinear parabolic equation which has been studied by many authors. In the paper [Fu], Fujita proved the following results:

(a) when $1 < p < 1 + \frac{2}{n}$ and $u_0 > 0$, problem (1.1) possesses no global positive solutions;

(b) when $p > 1 + \frac{2}{n}$ and u_0 is smaller than a small Gaussian, then (1.1) has global positive solutions. So $1 + \frac{2}{n}$ is the critical exponent.

For a rich literature on the subsequent development of the topic, we refer the reader to the survey paper [Le].

In recent years many authors have undertaken the research on semilinear elliptic operators on manifolds, including the well-known Yamabe problem (see [Sc] and [Yau]). In contrast, not much has been done for their semilinear parabolic counterpart. It looks more imperative to fill this gap when one takes into account the tremendous literature about the heat kernel of a complete Riemannian manifold, which is still a linear theory. The study of Ricci flows also leads to semilinear parabolic problems (see [H] and [Sh]). We shall need some new techniques to study when blow up of solutions occur, and when global positive solutions exist for equation (1.1). In fact the prevailing methods in treating semilinear problems are variational and comparison method. The variational approach seems hard to apply when the manifold is noncompact. The comparison method also faces some difficulties on curved manifolds because the intrinsic operators are not of constant coefficients.

The method we are using is based on some new inequalities involving the heat kernels. We are able to reveal an explicit relation between the size of the critical exponent and geometric properties of the manifold such as the growth rate of geodesic balls (see Theorem B).

It is interesting to note that Theorem B immediately leads to a nonexistence result for the well-known noncompact Yamabe problem of prescribing positive constant scalar curvatures, i.e., whether the following elliptic problem has positive solutions on $\mathbf{M}^n$ (see Theorem C):

$$\Delta u - \frac{n-2}{4(n-1)} R u + u^{(n+2)/(n-2)} = 0. \tag{1.4}$$

This problem has been posed by J. Kazdan [K] and S. T. Yau [Yau]. The compact version of the problem was proposed by Yamabe [Yam], proved by Trudinger [Tr] and Aubin [Au] in some cases, and eventually proved by R. Schoen [Sc] completely. In the noncompact case, Aviles and McOwen [AM] obtained some existence result for the problem of prescribing constant negative scalar curvature. However, as far as we know, there has been no result on the problem of prescribing constant positive scalar curvature, which was termed more difficult in [AM].

When $\mathbf{M}^n$ is $\mathbf{R}^n$ with the Euclidean metric, then problem (1.4) becomes $\Delta u + u^{(n+2)/(n-2)} = 0$, which does have positive solutions (see [Ni])

$$u_\lambda(x) = \frac{[n(n-2)\lambda^2]^{(n-2)/4}}{(\lambda^2 + |x|^2)^{(n-2)/2}}, \qquad \lambda > 0.$$

So it is reasonable to expect that (1.4) has a solution. However, Theorem C below asserts that unlike the compact Yamabe problem, problem (1.4) cannot be solved in general. In fact, if the existing scalar curvature decays "too fast" and the volume of geodesic balls does not increase "fast enough", then (1.4) has no solution.

Before stating the main results explicitly we list some notation.

Definition 1.1. A function $u = u(x,t)$ such that $u \in L^2_{loc}(\mathbf{M}^n \times (0,\infty))$ is called a solution of (1.1) if

$$u(x,t) = \int_{\mathbf{M}^n} G(x,t;y,0)u_0(y)\,dy + \int_0^t \int_{\mathbf{M}^n} G(x,t;y,s)u^p(y,s)\,dy\,ds$$

for all $(x,t) \in \mathbf{M}^n \times (0,\infty)$.

$G = G(x,t;y,s)$ will denote the fundamental solution of the linear operator H_0 in (1.1). For any $c > 0$, we write

$$G_c(x,t;y,s) = \begin{cases} \frac{1}{|B(x,(t-s)^{1/2})|}\exp(-c\frac{d(x,y)^2}{t-s}), & t > s, \\ 0, \quad t < s. \end{cases}$$

Let u_0 be a positive function in $L^\infty(\mathbf{M}^n)$ and $a > 0$; we write

$$h_a(x,t) = \int_{\mathbf{M}^n} G_a(x,t;y,0)u_0(y)\,dy; \tag{1.5}$$

$$h(x,t) = \int_{\mathbf{M}^n} G(x,t;y,0)u_0(y)\,dy. \tag{1.6}$$

Given $V = V(x,t)$ and $c > 0$, we introduce the notation

$$\begin{aligned} N_{c,\infty}(V) \equiv \sup\nolimits_{x\in\mathbf{M}^n,t>0} &\int_{-\infty}^{\infty}\int_{\mathbf{M}^n} |V(y,s)|G_c(x,t;y,s)\,dy\,ds \\ &+ \sup\nolimits_{y\in\mathbf{M}^n,s>0}\int_{\infty}^{\infty}\int_{\mathbf{M}^n} |V(x,t)|G_c(x,t;y,s)\,dx\,dt. \end{aligned} \tag{1.7}$$

We note that $N_{c,\infty}(V)$ may be infinite for some V. However, the fact that $N_{c,\infty}(V)$ is finite for some specific functions will play a key role in the proof of the theorems.

2. Main results

The main results of the paper are the next three theorems.

Theorem A. *Suppose* $R = 0$ *and* (1.2), (1.3), *and* (1.3′) *hold; the critical exponent of* (1.1) *is* $p^* = 1 + \frac{1}{s^*}$, *where*

$$s^* = \sup\{s \mid \lim_{t\to\infty} \sup t^s \|W(\cdot,t)\|_{L^\infty} < \infty,$$
$$\text{for a nonnegative and nontrivial } W \text{ such that } H_0 W = 0\}.$$

Remark 2.1. Theorem A, which establishes the existence of the critical exponent, can be proved by following the proof of Theorem 1 in [Me]. However, this theorem does not provide any estimate of the size of the exponent. It does not even tell whether p^* is closer to 1 or ∞. Our main concern is therefore to find an explicit relation between the critical exponent and geometrical properties of the manifold. This is done in the next

Theorem B. *Let* $\mathbf{M}^n$ *be any Riemannian manifold and* $R = R(x)$ *any bounded functions such that* (1.2), (1.3), *and* (1.3′) *hold; then the following conclusions are true.*

(a) Suppose, for $t > s \geq 0$,

$$\sup_{x\in\mathbf{M}^n, t} \int_{r_0}^\infty \int_{\mathbf{M}^n} \frac{G_c(x,t;y,s)}{|B(y,s^{1/2})|^{p-1}}\, dy\, ds + \sup_{y\in\mathbf{M}^n, s} \int_{r_0}^\infty \int_{\mathbf{M}^n} \frac{G_c(x,t;y,s)}{|B(x,t^{1/2})|^{p-1}}\, dx\, dt < \infty$$

for some $r_0 > 0$ *and a suitable* $c > 0$; *then* (1.1) *has global positive solutions for some* $u_0 \geq 0$.

(b) Suppose for some $x_0 \in \mathbf{M}^n$,

$$\lim_{r\to\infty} \inf \frac{|B(x_0,r)|}{r^\alpha} < \infty;$$

then (1.1) *has no global positive solutions for any* $p < 1 + \frac{2}{\alpha}$ *and any* $u_0 \geq 0$.

To exemplify the conditions in Theorem B, we provide two corollaries.

Corollary 2.1. *Under the same assumptions as in Theorem B, suppose*

$$\int_{r_0}^\infty \sup_{x\in\mathbf{M}^n} \frac{1}{|B(x,r^{1/2})|^{p-1}}\, dr < \infty$$

for some $r_0 > 0$; *then* (1.1) *has global positive solutions for some* $u_0 \geq 0$. *In particular, if, for an* $\alpha > 0$, $\inf_{x\in\mathbf{M}^n} |B(x,r)| \geq Cr^\alpha$ *when* r *is sufficiently large, then for* $p > 1 + \frac{2}{\alpha}$, (1.1) *has global positive solutions for some* u_0.

In the next corollary, we show that if $\mathbf{M}^n$ has bounded geometry in the sense of E. B. Davies (see p. 172 in [D]), which means the existence of a function $b(r)$ and $c > 0$ such that

$$c^{-1} b(r) \leq |B(x,r)| \leq c b(r) \tag{2.1}$$

for all $x \in \mathbf{M}^n$ and $r > 0$, then the critical exponent of (1.1) can be explicitly determined.

Corollary 2.2. *Suppose, in addition to the assumptions in Theorem B,* (2.1) *holds; then* p^*, *the critical exponent of* (1.1), *is given by* $p^* = 1 + \frac{2}{\alpha^*}$, *where*

$$\alpha^* = \inf\left\{\alpha > 0 \,\middle|\, \lim_{r\to\infty} \inf \frac{|B(x,r)|}{r^\alpha} < \infty,\ x \in \mathbf{M}^n\right\}.$$

Remark 2.2. By (2.1), the above number α^* is independent of the choice of $x \in \mathbf{M}^n$.

Remark 2.3. Under the assumptions of part (a) of Theorem B, (1.1) has global positive solutions if $u_0 \geq 0$ satisfies

$$u_0 \in C^2(\mathbf{M}^n), \qquad \lim_{d(x,0)\to\infty} u_0(x) = 0, \quad \text{and} \quad \|u_0\|_{L^\infty(\mathbf{M}^n)} + \|u_0\|_{L^1(\mathbf{M}^n)} \leq b_0,$$

where b_0 is a sufficiently small number and 0 is a point in $\mathbf{M}^n$. This result is new even in the Euclidean case.

Next we turn our attention to the noncompact Yamabe problem of prescribing positive scalar curvatures.

Theorem C. *Suppose* (1.2), (1.3), *and* (1.3′) *hold and*

$$\liminf_{r\to\infty} \frac{|B(x_0, r)|}{r^\alpha} < \infty \tag{2.2}$$

for some x_0 *and* $\alpha < (n-2)/2$; *then the Yamabe problem* (1.4) *has no solutions.*

It is important to know what conditions should be imposed on the Ricci curvature and the scalar curvature R so that the conditions of Theorem C be met. To this end we have

Corollary 2.3. *(a) Suppose the Ricci curvature of* $\mathbf{M}^n$ *is nonnegative, and for a suitable* $c > 0$, $N_{c,\infty}(R)$ *is sufficiently small; then* (1.2), (1.3), *and* (1.3′) *hold. Hence the Yamabe problem has no solutions if* (2.2) *holds.*

(b) In particular, suppose $\mathbf{M}^n$ *is a manifold with nonnegative Ricci curvature and* $|B(x,r)| \sim r^\alpha$ *for* $2 < \alpha < (n-2)/2$ *and large* $r > 0$. *Then if*

$$0 \leq R(x) \leq \epsilon/(1 + d^{2+\delta}(x, 0))$$

for a sufficiently small ϵ *and a* $\delta > 0$, *then the Yamabe problem* (1.4) *has no solution. Here 0 is a point in* $\mathbf{M}^n$. *An example is* $\mathbf{M}^9 = \mathbf{R}^3 \times S^1 \times \cdots \times S^1$ *with the metric tensor being the direct product of those usual ones on* $\mathbf{R}^3$ *and* S^1.

Remark 2.4. In the above example, we can just change the metric in $\mathbf{R}^3$ suitably so that the scalar curvature of $\mathbf{M}^9$ be positive but (1.4) still have no solution. Indeed, by part (b) of the corollary, any metric g' on $\mathbf{R}^3$ satisfying the next three conditions can be a choice: (i) $|B(x', r)| \sim r^3$, $x' \in \mathbf{R}^3$; (ii) Ricci is nonnegative; (iii) the scalar curvature $R' \leq \epsilon/(1 + d^{2+\delta}(x', 0))$ for $x' \in \mathbf{R}^3$, $\delta > 0$, and a small $\epsilon > 0$.

Remark 2.5. We would like to point out another fundamental difference between the noncompact Yamabe problem and the compact one. In the compact case, no manifold with positive scalar curvature is conformal to a manifold with zero scalar curvature. This is reflected from the fact that the equation $\Delta u - Ru = 0$ with $R > 0$ has no positive solution on compact manifolds. However, this is not the case for noncompact manifolds. The following is a complete noncompact manifold which is conformal to a complete manifold of constant positive scalar curvature, and to a complete manifold with zero scalar curvature. Let $\mathbf{M} = S^3 \times \mathbf{R}^1$ with the metric being the direct product of the usual ones on S^3 and $\mathbf{R}^1$. Then $R = 6$, $n = 4$ and hence equation (1.4) becomes

$$\Delta u - u + u^3 = 0,$$

which has a solution $u = 1$. At the same time the equation

$$\Delta u - u = 0$$

has a positive solution

$$u(x_1, x_2) = e^{x_2} + e^{-x_2} \geq 1,$$

where $x_1 \in S^3$ and $x_2 \in \mathbf{R}^1$. Clearly the above solution generates a complete metric of zero scalar curvature since it is bounded away from zero. Similar phenomenon can be shown for $\mathbf{M} \times \mathbf{R}^k$ where $\mathbf{M}$ is any compact manifold with positive scalar curvature (see [Zhang5]).

Let us briefly discuss the method we are going to adopt. We will use the Schauder fixed point theorem to achieve existence. This requires some new estimates involving the heat kernel on $\mathbf{M}^n$, such as, for $0 < a < b$ and $s < \tau < t$,

(i)

$$G_a(x, t; z, \tau) G_b(z, \tau; y, s) \leq C[G_c(x, t; z, \tau) + G_c(z, \tau; y, s)] G_a(x, t; y, s),$$

(ii)

$$G_b(x, t; z, \tau) G_a(z, \tau; y, s) \leq C[G_c(x, t; z, \tau) + G_c(z, \tau; y, s)] G_a(x, t; y, s).$$

(iii) Suppose $0 < a < b$; there exist positive constants $C_{a,b}$ and c depending only on a and b such that, for all $t > s \geq 0$,

$$\int_s^t \int_{\mathbf{M}^n} G_a(x, t; z, \tau)\, |V(z, \tau)|\, G_b(z, \tau; y, s)\, dz\, d\tau \leq C_{a,b} N_{c,\infty}(V) G_a(x, t; y, s);$$

$$\int_s^t \int_{\mathbf{M}^n} G_b(x, t; z, \tau)\, |V(z, \tau)|\, G_a(z, \tau; y, s)\, dz\, d\tau \leq C_{a,b} N_{c,\infty}(V) G_a(x, t; y, s).$$

To prove Theorem C, the key idea is to obtain global bounds (1.3) and (1.3$'$) for the fundamental solution of H_0, which involves certain conditions on the scalar curvature. Then we show that the parabolic problem (1.1) with $p = \frac{n+2}{n-2}$ has no global positive solutions under the assumptions of Theorem C. Since every positive solution of the Yamabe equation (1.4) is a global positive solution of the parabolic problem (1.1), the former cannot exist either.

Details of the proof can be found in the paper [Zhang5].

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