ELECTRONIC RESEARCH ANNOUNCEMENTS
OF THE AMERICAN MATHEMATICAL SOCIETY
Volume 3, Pages 114–118 (October 29, 1997)
S 1079-6762(97)00035-8

INVARIANT SETS WITH ZERO MEASURE AND FULL HAUSDORFF DIMENSION

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(Communicated by Svetlana Katok)

Abstract. For a subshift of finite type and a fixed Hölder continuous function, the zero measure invariant set of points where the Birkhoff averages do not exist is either empty or carries *full* Hausdorff dimension. Similar statements hold for conformal repellers and two-dimensional horseshoes, and the set of points where the pointwise dimensions, local entropies, Lyapunov exponents, and Birkhoff averages do not exist simultaneously.

Introduction

Let f be a continuous map on a compact topological space X. For each continuous function $g\colon X \to \mathbb{R}$, we define the *irregular set for the Birkhoff averages of* g by

$$\mathcal{B}(g) = \left\{ x \in X : \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} g(f^k x) \text{ does not exist} \right\}.$$

Note that the set $\mathcal{B}(g)$ is f-invariant. By Birkhoff's Ergodic Theorem, $\mu(\mathcal{B}(g)) = 0$ for every f-invariant measure μ on X.

The irregular sets $\mathcal{B}(g)$ are usually considered of little interest in ergodic theory and have rarely been considered in the literature. As a rule they are *a priori* discarded. Here we announce results of [3] showing that in a number of situations ubiquitous in dynamics, the set $\mathcal{B}(g)$ is either empty or carries *full* topological entropy as well as *full* Hausdorff dimension.

The results presented here follow from stronger statements proved in [3]. We shall illustrate our methods of proof, which strongly rely on multifractal analysis.

Subshifts of finite type

Let σ be the shift map on $\{1, \dots, p\}^{\mathbb{N}}$ with the standard topology. Let A be a $p\times p$ matrix whose every entry a_{ij} is either 0 or 1. Let $\Sigma \subset \{1, \dots, p\}^{\mathbb{N}}$ be the compact σ-invariant subset composed of the sequences $(i_0 i_1 \cdots)$ such that $a_{i_n i_{n+1}} = 1$ for every $n \geq 0$. The map $\sigma|\Sigma$ is called the *subshift of finite type* with transfer matrix A. We recall that $\sigma|\Sigma$ is topologically mixing if and only if there is a positive

Received by the editors September 2, 1997.
1991 *Mathematics Subject Classification.* Primary 58F15, 58F11.
Luis Barreira was partially supported by the projects PRAXIS XXI, 2/2.1/MAT/199/94 and JNICT, PBIC/C/MAT/2140/95. Jörg Schmeling was supported by the Leopoldina-Forderpreis.

integer k such all entries of A^k are positive. The topological entropy of $\sigma|\Sigma$ is $h(\sigma) = \log \rho(A)$, where $\rho(A)$ denotes the spectral radius of A.

We say that two functions g_1 and g_2 are *cohomologous* if $g_1 - g_2 = \psi - \psi \circ \sigma + c$, for some $\psi \colon \Sigma \to \mathbb{R}$ continuous and $c \in \mathbb{R}$. If g_1 and g_2 are cohomologous, then $\mathcal{B}(g_1) = \mathcal{B}(g_2)$. In particular, if g is cohomologous to 0, then $\mathcal{B}(g)$ is the empty set.

Theorem 1. *For a topologically mixing subshift of finite type, the Hölder continuous functions $g_1, \ldots, g_m$ are non-cohomologous to 0 if and only if*

$$h(\sigma|\mathcal{B}(g_1) \cap \cdots \cap \mathcal{B}(g_m)) = h(\sigma).$$

For a topologically mixing subshift of finite type, the family of Hölder continuous functions non-cohomologous to 0 is dense in the space of continuous functions with respect to the supremum norm [3].

Set $\mathcal{B} = \bigcup_g \mathcal{B}(g)$. By Theorem 1, $h(\sigma|\mathcal{B}) = h(\sigma)$ for a topologically mixing subshift of finite type. This formula was first established by Pesin and Pitskel' [4] in the case of the Bernoulli shift on two symbols. Their methods of proof are different from ours; moreover, it is not clear if their proof can be generalized to arbitrary subshifts of finite type.

Repellers

Let f be a C^1 expanding map of a manifold M, and $J \subset M$ a repeller of f. This means that there are constants $c > 0$ and $\beta > 1$ such that $\|d_x f^n u\| \geq c\beta^n \|u\|$ for every $x \in J$, $u \in T_x M$, and $n \geq 1$, and that $J = \bigcap_{n \geq 0} f^{-n} V$ for some open neighborhood V of J. The map f is called *conformal* if $d_x f$ is a multiple of an isometry at every point $x \in M$. Examples of conformal expanding maps include one-dimensional Markov maps and holomorphic maps.

Let μ be a probability measure on J. Each Markov partition of a repeller J has associated a one-sided subshift of finite type $\sigma|\Sigma$, and a coding map $\chi \colon \Sigma \to J$ for the repeller. We define the *irregular set for the local entropies of* μ by

$$\mathcal{H}_f(\mu) = \left\{ \chi(x) \in J : \lim_{n \to \infty} -\frac{\log \mu(\chi(C_n(x)))}{n} \text{ does not exist} \right\},$$

where $C_n(x)$ is the cylinder set of length n containing x. We define also the *irregular set for the Lyapunov exponents of* f by

$$\mathcal{L}_f = \left\{ x \in J : \lim_{n \to \infty} \frac{1}{n} \log \|d_x f^n\| \text{ does not exist} \right\},$$

and the *irregular set for the pointwise dimensions of* μ by

$$\mathcal{D}(\mu) = \left\{ x \in J : \lim_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} \text{ does not exist} \right\},$$

where $B(x,r)$ is the ball of radius r centered at x. By Kingman's Subadditive Ergodic Theorem, $\mu(\mathcal{L}_f) = 0$ for every f-invariant measure μ on J. Schmeling and Troubetzkoy [6] proved that $\mu(\mathcal{D}(\mu)) = 0$ for every measure μ invariant under an expanding map.

We write $a(x) = \|d_x f\|$ for each $x \in M$, and denote by $\dim_H J$ the Hausdorff dimension of J. For a repeller J of a conformal $C^{1+\varepsilon}$ expanding map f, the equilibrium measure m_D of $-\dim_H J \cdot \log a$ on J is the unique f-invariant measure of maximal dimension. Let m_E be the measure of maximal entropy, and $G(f|J)$ the family of Gibbs measures on J with a Hölder continuous potential.

Theorem 2. *For a compact repeller of a topologically mixing $C^{1+\varepsilon}$ conformal expanding map, for some $\varepsilon > 0$, and $\mu \in G(f|J)$, the three measures μ, m_D, and m_E are distinct if and only if*

$$h(f|\mathcal{D}(\mu) \cap \mathcal{H}_f(\mu) \cap \mathcal{L}_f) = h(f|J) \quad \textit{and} \quad \dim_H(\mathcal{D}(\mu) \cap \mathcal{H}_f(\mu) \cap \mathcal{L}_f) = \dim_H J.$$

Horseshoes

Let f be a C^1 diffeomorphism of a manifold M, and $\Lambda \subset M$ a hyperbolic set for f. This means that there is a continuous df-invariant splitting $T_\Lambda M = E^s \oplus E^u$, and constants $c > 0$ and $\lambda \in (0,1)$ such that if $x \in \Lambda$ and $n \geq 0$ then $\|d_x f^n v\| \leq c\lambda^n \|v\|$ for every $v \in E^s_x$, and $\|d_x f^{-n} v\| \leq c\lambda^n \|v\|$ for every $v \in E^u_x$.

Pesin and we [2], [1] showed that $\mu(\mathcal{D}(\mu)) = 0$ for every hyperbolic measure μ invariant under a $C^{1+\varepsilon}$ diffeomorphism. Let $\mathcal{M}_D$ be the set of f-invariant measures μ on Λ such that $\dim_H \mu = \dim_H \Lambda$. Note that $\mathcal{M}_D$ may be empty.

Theorem 3. *For a compact locally maximal saddle-type hyperbolic set for a topologically mixing $C^{1+\varepsilon}$ surface diffeomorphism, for some $\varepsilon > 0$, and $\mu \in G(f|\Lambda)$, we have $\mu \neq m_E$ and $\mu \notin \mathcal{M}_D$ if and only if*

$$h(f|\mathcal{D}(\mu) \cap \mathcal{H}_f(\mu)) = h(f|\Lambda) \quad \textit{and} \quad \dim_H(\mathcal{D}(\mu) \cap \mathcal{H}_f(\mu)) = \dim_H \Lambda.$$

Proofs

The above theorems follow from stronger statements proved in [3]. Here we provide a proof of Theorem 1 in the case $m = 1$. It contains all the main ingredients of our methods of proof.

Let g be a Hölder continuous function on Σ. By the multifractal analysis of Gibbs measures on subshifts of finite type by Pesin and Weiss [5], given $\varepsilon > 0$, there are ergodic measures μ_1 and μ_2 such that $h_{\mu_i}(\sigma) > h(\sigma) - \varepsilon$ for $i = 1$, 2, and

$$\int_\Sigma g \, d\mu_1 \neq \int_\Sigma g \, d\mu_2.$$

Choose $\delta \in (0, \varepsilon)$ such that $|\int_\Sigma g \, d\mu_1 - \int_\Sigma g \, d\mu_2| > 4\delta$.

For $i = 1$, 2 and $\ell \geq 1$, let Γ_i^ℓ be the set of points $x \in \Sigma$ such that if $n \geq \ell$ then

$$\left| \frac{1}{n} \sum_{j=0}^{n} g(f^j x) - \int_\Sigma g \, d\mu_i \right| < \delta.$$

Set $p_s = s \pmod 2$, and let ℓ_s be an increasing sequence of positive integers such that $\mu_{p_s}(\Gamma_{p_s}^{\ell_s}) > 1 - \varepsilon/2^s$ for each integer $s \geq 1$.

Let k be a positive integer such that all entries of A^k are positive, where A is the transfer matrix of Σ. We define inductively the increasing sequences of positive integers n_s and m_s by $m_1 = n_1 = \ell_1$,

$$m_s = (n_{s-1} + k + \ell_{s+1})! \quad \text{and} \quad n_s = n_{s-1} + k + m_s.$$

We define families of cylinder sets by $\mathfrak{C}_s = \{C_{m_s}(x) : x \in \Gamma_{p_s}^{\ell_s}\}$, $\mathfrak{D}_1 = \mathfrak{C}_1$, and

$$\mathfrak{D}_s = \{\underline{C} C \overline{C} : \underline{C} \in \mathfrak{D}_{s-1}, \, \overline{C} \in \mathfrak{C}_s, \text{ and } |C| = k\}.$$

Set

$$\Lambda = \bigcap_{s \geq 1} \bigcup_{C \in \mathfrak{D}_s} C,$$

and define a measure μ on Λ by $\mu(C) = \mu_1(C)$ if $C \in \mathfrak{D}_1$, and by $\mu(\underline{C}C\overline{C}) = \mu(\underline{C})\mu_{p_s}(\overline{C})$ if $\underline{C}C\overline{C} \in \mathfrak{D}_s$ for some $s > 1$. We extend μ to Σ by $\mu(A) = \mu(A \cap \Lambda)$ for each measurable set $A \subset \Sigma$. If $s > 1$ and $\underline{C} \in \mathfrak{D}_{s-1}$, then

$$\mu\left(\bigcup_{\overline{C}\in\mathfrak{D}_s} \underline{C} \cap \overline{C}\right) \ge \mu(\underline{C})\left(1 - \frac{\varepsilon}{2^s}\right),$$

and hence, if $\varepsilon < 2$ then

$$\mu(\Lambda) \ge \prod_{s=1}^{\infty}\left(1 - \frac{\varepsilon}{2^s}\right) > 0.$$

Let $x \in C \in \mathfrak{D}_s$. Note that $\sigma^{|C|-m_s}x \in \Gamma_{p_s}^{\ell_s}$ and $|C|/m_s \to 1$ as $s \to \infty$. Hence, if s is sufficiently large then

$$\begin{aligned}
&\left|\frac{1}{|C|}\sum_{j=0}^{|C|} g(\sigma^j x) - \int_\Sigma g\, d\mu_{p_s}\right| \\
&\quad \le \left|\frac{1}{m_s}\sum_{j=0}^{m_s} g(\sigma^{|C|-m_s+j}x) - \int_\Sigma g\, d\mu_{p_s}\right| \cdot \frac{m_s \sum_{j=0}^{|C|} g(\sigma^j x)}{|C|\sum_{j=0}^{m_s} g(\sigma^{|C|-m_s+j}x)} \\
&\quad\quad + \left|1 - \frac{m_s \sum_{j=0}^{|C|} g(\sigma^j x)}{|C|\sum_{j=0}^{m_s} g(\sigma^{|C|-m_s+j}x)}\right| \cdot \left|\int_\Sigma g\, d\mu_{p_s}\right| \\
&\quad < 2\delta.
\end{aligned}$$

By the choice of δ, we have $\mathcal{B}(g) \supset \Lambda$.

Now let $x \in \Lambda$. We will prove that if q is sufficiently large then

$$-\frac{\log \mu(C_q(x))}{q} \ge h(\sigma) - \eta$$

for some $\eta \in (0, 2\varepsilon)$. This implies that

$$h(\sigma|\mathcal{B}(g)) \ge h(\sigma|\Lambda) \ge h_{\mu|\Lambda}(\sigma) \ge h(\sigma) - 2\varepsilon,$$

and since ε is arbitrary, $h(\sigma|\mathcal{B}(g)) = h(\sigma)$.

We proceed by induction on q. Choose an integer s_q such that $|C^{s_q}| \le q < |C^{s_q+1}|$, where $\mathfrak{D}_{s_q+1} \ni C^{s_q+1} \subset C_q(x) \subset C^{s_q} \in \mathfrak{D}_{s_q}$. Assume that

$$|C^{s_q}| \le q \le |C^{s_q}| + k + \ell_{s_q+1}. \tag{1}$$

Then $q/|C^{s_q}| \to 1$ as $q \to \infty$. By Breiman's Theorem and induction, if q is sufficiently large then

$$-\frac{\log \mu(C_q(x))}{q} \ge -\frac{\log \mu(C^{s_q})}{|C^{s_q}|} \cdot \frac{|C^{s_q}|}{q} \ge h(\sigma) - \eta,$$

for some $\eta \in (0, 2\varepsilon)$. When (1) does not hold, we have $\mu(C_q(x)) = \mu(C^{s_q})\mu_{p_{s_q+1}}(C')$, where $C_q(x) = C^{s_q}CC'$ such that C' contains an element of $\mathfrak{C}_{s_q+1}$, $|C'| > \ell_{s_q+1}$, and $|C| = k$. By Breiman's Theorem and induction, if q is sufficiently large then

$$\begin{aligned}
-\frac{\log \mu(C_q(x))}{q} &= \left(-\frac{\log \mu(C^{s_q})}{|C^{s_q}|} \cdot |C^{s_q}| - \frac{\log \mu_{p_{s_q+1}}(C')}{|C'|} \cdot |C'|\right) \frac{1}{|C^{s_q}| + k + |C'|} \\
&\ge h(\sigma) - \eta,
\end{aligned}$$

for some $\eta \in (0, 2\varepsilon)$. This completes the proof.

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