ELECTRONIC RESEARCH ANNOUNCEMENTS
OF THE AMERICAN MATHEMATICAL SOCIETY
Volume 3, Pages 119–120 (November 4, 1997)
S 1079-6762(97)00034-6

QUANTIZATION OF POISSON STRUCTURES ON $\mathbf{R}^2$

DMITRY TAMARKIN

(Communicated by Alexandre Kirillov)

Abstract. An 'isomorphism' between the 'moduli space' of star products on $\mathbf{R}^2$ and the 'moduli space' of all formal Poisson structures on $\mathbf{R}^2$ is established.

The problem of quantization of Poisson structures has been posed in [1]. It is well known that any Poisson structure on a two-dimensional manifold is quantizable. In this paper we establish an 'isomorphism' between the 'moduli space' of star products on $\mathbf{R}^2$ and the 'moduli space' of all formal Poisson structures on $\mathbf{R}^2$ by construction of a map from Poisson structures to star products. Certainly, this isomorphism follows from the Kontsevich formality conjecture [2]. Most likely, our map can be used as a first step in constructing an L_∞-quasiisomorphism in the formality conjecture for $\mathbf{R}^2$. The author would like to thank Boris Tsygan and Paul Bressler for the attention and helpful suggestions.

The set of all star-products $\mathbf{S}$ is acted upon by the group $\mathcal{D} \dot{\times} \mathrm{Diffeo}\mathbf{R}^2$, where $\mathcal{D}$ is the group of operators of the form $1 + hD_1 + h^2D_2 + \cdots$ with D_k to be arbitrary differential operators. The set of all formal Poisson structures $\mathbf{P}$ consists of formal series in h with bivector fields as the coefficients. Formal Poisson structures are acted upon by the group $\mathrm{Diffeo}\,\mathbf{R}^2 \dot{\times} \exp(h\mathrm{Vect}[[h]])$, where Vect is the Lie algebra of vector fields on $\mathbf{R}^2$. These actions define equivalence relations. We want to have a pair of maps $f_1 : \mathbf{S} \to \mathbf{P}$ and $f_2 : \mathbf{P} \to \mathbf{S}$ such that

$$f_1 \circ f_2(x) \sim x, \quad f_2 \circ f_1(x) \sim x,$$

$$x \sim y \to f_{1,2}x \sim f_{1,2}y. \tag{1}$$

By a map from $\mathbf{S}$ we mean a differential expression in terms of the coefficients of the bidifferential operators corresponding to the star products. Maps from $\mathbf{P}$ are defined similarly.

We can replace $\mathbf{S}$ by a subspace. Let P, Q be a nondegenerate pair of (real) polarizations of $\mathbf{R}^2$. Define a subset $\mathbf{S}_{P,Q}$ of $\mathbf{S}$ in the following way: $m \in \mathbf{S}_{P,Q}$ iff $m(f,g) = fg$ if f is constant along P or g is constant along Q.

Proposition 1. *Let x, y be a nondegenerate coordinate system on $\mathbf{R}^2$ such that x is constant along Q and y is constant along P. Then there exists a unique map $\mathbf{S} \to \mathcal{D} : m \mapsto U(m) = 1 + hV(m)$ such that*

$$\begin{aligned} &1) \quad m_{P,Q}(m) = U^{-1}(m(Uf, Ug)) \in \mathbf{S}_{P,Q}, \\ &2) \quad Ux = x,\ Uy = y,\ U1 = 1. \end{aligned} \tag{2}$$

Received by the editors September 2, 1997.
1991 *Mathematics Subject Classification.* Primary 81Sxx.

U is uniquely defined by the condition $U(x^{*m} * y^{*n}) = x^m y^n$ (where star denotes the star product m).

We denote by $m_{P,Q} : \mathbf{S} \to \mathbf{S}_{P,Q}$ the map which sends m to $m_{P,Q}(m)$. Further, x, y will mean the same as in Proposition 1. Thus, it is enough to find maps $p_1 : \mathbf{S}_{P,Q} \to \mathbf{P}$ and $p_2 : \mathbf{P} \to \mathbf{S}_{P,Q}$ with the same properties as f_1, f_2 have. Indeed, put

$$f_2 = i \circ p_2, \qquad f_1 = p_1 \circ m_{P,Q} \tag{3}$$

(here $i : \mathbf{S}_{P,Q} \to \mathbf{S}$ is the inclusion).

The following theorem gives an explicit construction for p_2 which appears to be a bijective map so that we can put $p_1 = p_2^{-1}$. Denote by $\mathcal{C}_P$ (resp. $\mathcal{C}_Q$) the space of functions, constant along Q (resp. P). Denote by $\mathcal{V}_P$ (resp. $\mathcal{V}_Q$) the space of vector fields preserving the polarizations and tangent to P (resp. Q). Denote by $\mathcal{D}_P$ the subalgebra of the algebra of differential operators consisting of operators D such that $D(\mathcal{C}_Q) \subset \mathcal{C}_Q$ and $D(fg) = fD(g)$ if $f \in \mathcal{C}_P$. Denote by $\mathcal{D}_Q$ the same algebra where P and Q are interchanged. In the coordinates x, y we have $\mathcal{C}_P = \{f(x)\}$, $\mathcal{V}_P = \{f(x)\partial_x\}$, $\mathcal{D}_P = \sum f_i(x)\partial_x^i$ and the same things with P replaced by Q and x replaced by y. Denote by $\overline{\mathcal{D}_P}$ (resp. $\overline{\mathcal{D}_Q}$) the subring of $\mathcal{D}_P$ (resp. $\mathcal{D}_Q$) consisting of operators which annihilate constant functions.

Note that the space of bivector fields is isomorphic to $\mathcal{V}_P \otimes_{\mathbf{R}} \mathcal{V}_Q$. Let $\mathcal{D}_{P,k}$ be the space of maps $Vp^{\otimes k} \to \overline{\mathcal{D}_P}$ (which are differential operators in terms of the coefficients).

Theorem 1. *a) There exists a unique sequence $c_k \in \mathcal{D}_{P,k} \otimes \mathcal{D}_{Q,k}, k = 0, 1, 2, \dots$, $c_k = \sum_i a_k^i \otimes b_k^i$, $c_0(X, Y) = 1 \otimes 1$, such that for any bivector field $\Psi = \sum_i X_i \wedge Y_i$, $X_i \in \mathcal{V}_P, Y_i \in \mathcal{V}_Q$, the formula*

$$\begin{aligned} m(\Psi, P, Q, f, g) = fg + \sum_{k, i_1, \dots, i_{k+1}} & h^{k+1} L_{X_{i_1}} \{a_k^n(X_{i_2}, X_{i_3}, \dots, X_{i_{k+1}})f\} \\ & \times L_{Y_{i_1}} \{b_k^n(Y_{i_2}, Y_{i_3}, \dots, Y_{i_{k+1}})g\} \\ = \sum_k m_k(f, g). \end{aligned} \tag{4}$$

gives a star-product.

b) Put $p_2 : \mathbf{P} \to \mathbf{S}_{P,Q} : \Psi \to m(\Psi, P, Q, \cdot, \cdot)$. Put $p_1 = p_2^{-1}$. Then p_1 and p_2 provide an isomorphism of $\mathbf{P}$ and S by (3).

References

[1] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer, Deformation theory and quantization, I, II, Ann. Phys. 11 (1978), 61–151. MR **58:**14737a, MR **58:**14737b

[2] M. Kontsevich, Formality conjecture, preprint, to appear in Proc. of Summer School on Deformation Quantization in Ascona.

Department of Mathematics, Pennsylvania State University, 218 McAllister Building, University Park, PA 16802

E-mail address: tamarkin@math.psu.edu